MULTIPLICATION BY BLASCHKE PRODUCTS AND STABILITY OF IDEALS IN LIPSCHITZ ALGEBRAS

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1. Introduction.

Let $D$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $T$ its boundary. For $\alpha \in (0, +\infty)$ denote by $\Lambda^\alpha$ the classical Lipschitz-Zygmund space of smooth functions on the circle:

$$\Lambda^\alpha \overset{\text{def}}{=} \{ f \in C(T) : \|A^m f\|_\infty = O(|h|\alpha), \; h \in \mathbb{R} \},$$

where $m$ is any integer with $m > \alpha$, $\| \cdot \|_\infty$ is the usual $L^\infty$ norm, and $A^m_h$ stands for the $m$th order difference operator. (Recall that the $A^m_h$'s are defined by induction: $A^k_h = A^k_h A^{k-1}_h$,

$$(A^1_h f)(\zeta) \overset{\text{def}}{=} f(e^{ih\zeta}) - f(\zeta), \quad \zeta \in T.$$)

Further, let $\Lambda^\alpha_A$ be the analytic subspace of $\Lambda^\alpha$: $\Lambda^\alpha_A \overset{\text{def}}{=} \Lambda^\alpha \cap H^\infty$, where as usual $H^\infty$ stands for the algebra of bounded analytic functions on $D$. An equivalent definition ([Z], vol. 1) is as follows:

$$\Lambda^\alpha_A = \{ f \in H^\infty : f^{(m)}(z) = O((1 - |z|)^{\alpha - m}), \; z \in D \};$$

here $m$ is again an integer such that $m > \alpha$, and $f^{(m)}$ is the $m$th order derivative of $f$.

This paper is devoted to a certain subtle point concerning the multiplicative structure of functions in $\Lambda^\alpha_A$.

Suppose $f \in \Lambda^\alpha_A$ and $\theta$ is an inner function (i.e. $\theta \in H^\infty$ and $\lim_{r \to 1^-} |\theta(r\zeta)| = 1$ for almost all $\zeta \in T$). Assume that

$$f \theta \in \Lambda^\alpha_A.$$ (1)

It is known (see [Shi 1], [Shi 2] or Theorem B below) that in the case $0 < \alpha < 1$ (1) implies

$$f \theta^k \in \Lambda^\alpha_A \quad \text{for all} \; k \in \mathbb{N},$$ (2)

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where N is the set of positive integers. That seems to be rather natural. However, the situation changes radically if $\alpha$, the order of smoothness, becomes $> 1$. The implication (1) $\Rightarrow$ (2) is still valid for singular (cf. [G], chapter II) inner functions $\theta$, yet it turns out that for any $\alpha > 1$ one can find a function $f, \theta \in \Lambda^s_\alpha$, and a Blaschke product $B$ such that $fB \in \Lambda^s_\alpha$ but $fB^2 \not\in \Lambda^s_\alpha$. This surprising phenomenon was discovered by N. A. Shirokov [Shi 1,2]. In fact, his ingenious construction provides a function $f$ in $A^\infty \triangleq \bigcap_{\alpha > 0} \Lambda^s_\alpha$ and a Blaschke product $B$ such that $fB \in A^\infty$, but the modulus of continuity of $(fB^2)'$ need not satisfy any prescribed estimate.

Thus, the passage from $fB$ to $fB^2$ is sometimes accompanied with a great loss of smoothness. This striking result displays a subtle distinction between the cases $0 < \alpha < 1$ and $\alpha > 1$, as far as factorization of $\Lambda^s_\alpha$ functions is concerned. (The Zygmund classes $A^k$, $k \in \mathbb{N}$, will not be considered in this paper, so we do not mention the case $\alpha = 1$.)

On the other hand, it has been proved that if $n < \alpha < n + 1$ $(n \in \mathbb{N})$, then the inclusion

$$(3) \quad fB^{n+1} \in \Lambda^s_\alpha$$

does imply (and is, therefore, equivalent to)

$$(4) \quad fB^k \in \Lambda^s_\alpha \quad \text{for all } k \in \mathbb{N},$$

$f$ being a function in $\Lambda^s_\alpha$ and $B$ a Blaschke product. This fact is again a consequence of Shirokov's results [Shi 1,2]; it is also contained in Theorem B, due to the author, which we cite in Section 2 below. If one replaces (3) by a weaker condition

$$(5) \quad fB^n \in \Lambda^s_\alpha,$$

it turns out that, generally speaking, (5) is no longer sufficient for (4) to hold. It should be noted though that (5) implies $fB \in \Lambda^s_\alpha, \ldots, fB^{n-1} \in \Lambda^s_\alpha$. This is because $\Lambda^s_\alpha$ possesses the following "division property" (cf. [H], [S]): whenever $f \in \Lambda^s_\alpha, \theta$ is inner and $f/\theta \in H^\infty$, it follows that $f/\theta \in \Lambda^s_\alpha$.

A natural problem in this context is: given $n \in \mathbb{N}$ and $\alpha \in (n, n + 1)$, describe the Blaschke products $B$ in terms of their zeros, for which the implication (5) $\Rightarrow$ (4) does hold with an arbitrary $f \in \Lambda^s_\alpha$.

Before proceeding with the solution, we introduce some notation. For $\theta$ an inner function, set $I^s(\theta) \triangleq \Lambda^s \cap \theta H^\infty$, so that $I^s(\theta)$ is a closed ideal in the algebra $\Lambda^s_\alpha$. For $g \in H^\infty$, let $T_g$ denote the multiplication map defined by $T_g f = fg$.

Obviously, the above problem is equivalent to characterizing the $B$'s for which

$$(6) \quad T_B I^s(B^n) \subset \Lambda^s_\alpha,$$
where $\alpha > 1$, $\alpha \notin \mathbb{N}$, $n = \lfloor \alpha \rfloor$. (Here $\lfloor \alpha \rfloor$ denotes the integral part of $\alpha$.)

Thus we are actually concerned with a certain stability property of the ideal $I^\alpha(B^n)$ with respect to multiplication by “its own” Blaschke product $B$ and/or by its powers $B^k$. Note that, in view of the above discussion, (6) is equivalent to

$$T_B I^\alpha(B^n) \subset \Lambda^\alpha_A \quad \text{for all } k \in \mathbb{N}. \tag{7}$$

If $I^\alpha(B^n)$ is nontrivial and satisfies (6) or (7), $B$ will be called stable. (Perhaps the term “$\alpha$-stable” would sound more natural, but we shall soon see that this property does not depend on $\alpha$.)

Assuming in addition that the zeros $\{z_j\}$ of $B$ form an interpolating sequence for $\Lambda^\alpha_A$ (for a precise definition see Section 2 below), we now provide a complete characterization of all such $B$’s that are stable.

**Theorem 1.** Let $n \in \mathbb{N}$, $n < \alpha < n + 1$, and let $B$ be a $\Lambda^\alpha_A$-interpolating Blaschke product with zeros $\{z_j\}_{j=1}^\infty$. The following are equivalent.

(i) $B$ is stable.

(ii) $\inf_{k \in \mathbb{N}(\setminus j)} |z_j - z_k| = O(1 - |z_j|)$, $j \in \mathbb{N}$.

In fact, we prove a more general assertion. Given two exponents $\alpha$ and $\beta$ such that $n < \beta \leq \alpha < n + 1$, a Blaschke product $B$ will be termed $(\alpha, \beta)$-stable iff $I^\alpha(B^n) \neq \{0\}$ and

$$T_B I^\alpha(B^n) \subset \Lambda^\beta_A.$$

In other words, $B$ is $(\alpha, \beta)$-stable iff for any $f \in \Lambda^\alpha_A$ (5) implies $f B^n + 1 \in \Lambda^\beta_A$ (and hence $f B^k \in \Lambda^\beta_A$ for all $k \in \mathbb{N}$). For the sake of completeness, we note that for $n < \alpha < n + 1$ the set $T_B I^\alpha(B^n)$ is always contained in $\Lambda^\alpha_A$ with $0 < \gamma \leq n$ and is never contained (unless $I^\alpha(B^n) = \{0\}$) in $\Lambda^\gamma_A$ with $\gamma > \alpha$. That is why we assume $\beta \in (n, \alpha]$.

Of course, $(\alpha, \alpha)$-stability is just “stability” as defined above, and so Theorem 1 is a special case of the next fact.

**Theorem 2.** Let $n \in \mathbb{N}$, $n < \beta \leq \alpha < n + 1$, and let $B$ be a $\Lambda^\alpha_A$-interpolating Blaschke product with zeros $\{z_j\}_{j=1}^\infty$. The following are equivalent.

(i) $B$ is $(\alpha, \beta)$-stable.

(ii) $\inf_{k \in \mathbb{N}(\setminus j)} |z_j - z_k| = O((1 - |z_j|)^{(\beta - n)/(\alpha - n)})$, $j \in \mathbb{N}$.

The rest of the paper is organized as follows. In Section 2 we cite a few results that will be used in the sequel; we also specify the notion of a “$\Lambda^\alpha_A$-interpolating Blaschke product” that occurs in Theorems 1 and 2. Section 3 contains the proof of Theorem 2. In Section 4 we give an application of Theorem 2 to embedding
theorems for star-invariant subspaces of the Hardy classes $H^p$. Finally, Section 5 contains some examples and remarks.

2. Preliminaries.

Let $\alpha \in (0, +\infty)$, $\alpha \notin \mathbb{N}$, $n = [\alpha]$. Suppose $f \in A_\alpha^2$. It is well known (and easily shown) that for $\zeta_1, \zeta_2 \in \text{clos } D \overset{\text{def}}{=} \{ |z| \leq 1 \}$ and for $s = 0, 1, \ldots, n$

\[
|f^{(s)}(\zeta_1) - \sum_{m=s}^{n} \frac{f^{(m)}(\zeta_2)}{(m-s)!}(\zeta_1 - \zeta_2)^{m-s}| \leq C|\zeta_1 - \zeta_2|^{|\alpha - s|},
\]

where $C$ is a constant independent of $\zeta_1$ and $\zeta_2$. (Note that in the case $0 < \alpha < 1$ we have $n = 0$, and so (8) reduces to the usual Lipschitz condition of order $\alpha$.)

A closed subset $E$ of clos $D$ is said to be $A_\alpha^2$-interpolating if any interpolation problem

\[
f|E = \varphi_0, f'|E = \varphi_1, \ldots, f^{(n)}|E = \varphi_n
\]

has a solution $f \in A_\alpha^2$, provided that the data $\varphi_s$: $E \to \mathbb{C}$ satisfy the necessary conditions stated above:

\[
|\varphi_s(\zeta_1) - \sum_{m=s}^{n} \frac{\varphi_m(\zeta_2)}{(m-s)!}(\zeta_1 - \zeta_2)^{m-s}| \leq C|\zeta_1 - \zeta_2|^{|\alpha - s|} \quad (\zeta_1, \zeta_2 \in E)
\]

for $s = 0, 1, \ldots, n$.

The following characterization of $A_\alpha^2$-interpolating sets is due to E. M. Dyn'kin [Dyn].

**Theorem A.** Let $\alpha \in (0, +\infty)$, $\alpha \notin \mathbb{N}$. A closed set $E$, $E \subset \text{clos } D$, is $A_\alpha^2$-interpolating if and only if the two conditions hold:

(a) \[\inf \{ \rho(\zeta_1, \zeta_2): \zeta_1, \zeta_2 \in E \cap D, \zeta_1 \neq \zeta_2 \} > 0,\]

where $\rho(\zeta_1, \zeta_2) \overset{\text{def}}{=} |\zeta_1 - \zeta_2| / |1 - \bar{\zeta}_1 \zeta_2|$

(b) There is a constant $c > 0$ such that for any arc $I$, $I \subset T$, we have

\[
\sup_{\zeta \in I} \text{dist} (\zeta, E) \geq c|I|,
\]

where as usual $\text{dist} (\zeta, E) \overset{\text{def}}{=} \inf_{z \in E} |\zeta - z|$ and $|I|$ is the length of $I$.

Of course, condition (a) means that the set $E \cap D$ is countable and its points $\{z_j\}$ form a separated sequence with respect to the pseudohyperbolic distance $\rho(\cdot, \cdot)$. Moreover, (a) and (b) together imply [Dyn] that this sequence is, in fact, uniformly separated (or $H^\infty$-interpolating), i.e.
\begin{equation}
\inf_j \prod_{k \neq j} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| > 0.
\end{equation}

It should be noted also that the class of \( A^* \)-interpolating sets does not actually depend on \( \alpha \).

Given a Blaschke product

\[ B(z) = B(z_j)(z) = \prod_{j=1}^{\infty} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z} \]

with pairwise distinct zeros (set \( \bar{z}_j/|z_j| \) if \( z_j = 0 \), we call it \( A^* \)-interpolating if the closure of its zeros, \( \text{clos} \{ z_j \} \), is a \( A^* \)-interpolating set. As mentioned above, for such \( B \)'s we have (10); on the other hand, the set \( E = \text{clos} \{ z_j \} \) must satisfy the Beurling-Carleson condition

\[ \int_T^{\log \text{dist} (\zeta, E) |d\zeta| > -\infty} \]

(i.e. the non-uniqueness condition for \( A^* \) [C]), whence the ideals \( I^* (B^*) \) are nontrivial for all \( k \in \mathbb{N} \).

As another auxiliary result we cite the next Theorem B, due to the author [D 1]. (The most essential part of it is also contained in a previous paper [D 2].) In order to state it, we introduce the following notation: given \( \theta \in H^\infty \) and \( \varepsilon > 0 \), let \( \Omega(\theta, \varepsilon) \) be \( \{ z \in D : |\theta(z)| < \varepsilon \} \).

**Theorem B.** Let \( \alpha \in (0, +\infty) \), \( m \in \mathbb{N} \), \( m > \alpha \). Suppose \( f \in A^* \) and \( \theta \) is an inner function. The following statements are equivalent.

(i) \( f/\theta^m \in A^* \).

(ii) \( f/\theta^m \in A^* \).

(iii) \( f/\theta^k \in A^* \) \( \forall k \in \mathbb{Z} \).

(iv) For some \( \varepsilon \in (0, 1) \) (or, equivalently, for any \( \varepsilon \in (0, 1) \)), we have

\begin{equation}
(11) \quad f(z) = O((1 - |z|^\alpha) \quad \text{as} \quad |z| \to 1, \ z \in \Omega(\theta, \varepsilon). \end{equation}

It is this last quantitative condition that will be used as a multiplication criterion.

Finally, the following lemma will be needed (cf. [G], Chapter X, Lemma 1.4).

**Lemma C.** Let \( B \) be an interpolating (i.e. \( H^\infty \)-interpolating) Blaschke product with zeros \( \{ z_j \} \) such that the infimum occurring in (10) equals \( \delta \). Then there exist \( \lambda = \lambda(\delta), 0 < \lambda < 1, \) and \( \varepsilon = \varepsilon(\delta), 0 < \varepsilon < 1, \) such that

\begin{equation}
(12) \quad \Omega(B, \varepsilon) \subseteq \bigcup_j \{ z \in D : \rho(z, z_j) < \lambda \}. \end{equation}
(Recall that the non-euclidean metric $\rho(\cdot, \cdot)$ is defined by $\rho(z, w) = |z - w|/|1 - \bar{z}w|$).

3. Proof of Theorem 2.

For the reader's convenience, we reproduce the theorem itself and then proceed with the proof.

Given a sequence $\{z_j\}_{j=1}^{\infty} \subset D$, we set $d_j \overset{\text{def}}{=} \inf_{k \in \mathbb{N} \setminus \{j\}} |z_j - z_k|$.

**Theorem 2.** Let $n \in \mathbb{N}$, $n < \beta \leq \alpha < n + 1$, and let $B$ be a $\Lambda^*_A$-interpolating Blaschke product with zeros $\{z_j\}_{j=1}^{\infty}$. The following are equivalent.

(i) $B$ is $(\alpha, \beta)$-stable (see sect. 1).

(ii) $d_j = O((1 - |z_j|)^{(\beta - n)/(\alpha - n)})$, $j \in \mathbb{N}$.

**Proof.** (i) $\Rightarrow$ (ii). Let $E = \text{clos} \{z_j\}$. Define the interpolation data $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ to be zero on $E$, and let $\varphi_n : E \to \mathbb{C}$ be defined by

$$\varphi_n(z_j) = d_j^{\alpha - n} (j \in \mathbb{N}), \quad \varphi_n | E \cap \mathbb{T} = 0.$$ 

This done, conditions (9) are easily verified. Indeed, the $\varphi_s$'s being continuous on $E$, it suffices to check that

$$|\varphi_s(z_j) - \sum_{m=s}^{n} \frac{\varphi_m(z_k)}{(m - s)!} (z_j - z_k)^{m-s}| \leq C|z_j - z_k|^{\alpha - s}$$

for $s = 0, 1, \ldots, n$, where $z_j$ and $z_k$ are two distinct zeros of $B$, and $C$ is a constant.

**Case 1.** $s = 0, 1, \ldots, n - 1$. The left-hand side in (13) equals

$$\left| \frac{\varphi_n(z_k)}{(n - s)!} (z_j - z_k)^{n-s} \right| = \frac{1}{(n - s)!} d_k^{\alpha - n} |z_j - z_k|^{\alpha - s} \leq |z_j - z_k|^\alpha, $$

where we have used the obvious inequalities

$$1/(n - s)! \leq 1 \quad \text{and} \quad d_k \leq |z_j - z_k|.$$

**Case 2.** $s = n$. The left-hand side in (13) equals

$$|\varphi_n(z_j) - \varphi_n(z_k)| = |d_j^{\alpha - n} - d_k^{\alpha - n}| \leq d_j^{\alpha - n} + d_k^{\alpha - n} \leq 2|z_j - z_k|^\alpha,$$

because both $d_j$ and $d_k$ are $\leq |z_j - z_k|$.

Thus (13) is established (with $C = 2$), and so is (9). Recalling that $E$ is a $\Lambda^*_A$-interpolating set, one can find a function $f \in \Lambda^*_A$ such that

$f | E = f' | E = \ldots = f^{(n-1)} | E = 0, \quad f^{(n)} | E = \varphi_n.$

Hence for all $j \in \mathbb{N}$ we have
(14) \[ f(z_j) = f'(z_j) = \ldots = f^{(n-1)}(z_j) = 0, \quad f^{(n)}(z_j) = d_j^{\alpha-n}. \]

Thus, in each of \( z_j \)'s \( f \) has a zero of multiplicity \( n \), whence \( f \in I^s(B^n) \).

The \((\alpha, \beta)\)-stability of \( B \) now implies \( fB \in A^\alpha_\beta \), which is equivalent to

\[ (fB)^{(n+1)}(z) = O((1 - |z|)^{\beta-n-1}), \quad z \in D. \]

(See the Introduction for the definition of \( A^\alpha_\beta \) in terms of derivatives.) In particular,

\[ (fB)^{(n+1)}(z_j) = O((1 - |z_j|)^{\beta-n-1}). \]

Further, the Leibniz formula says

\[ (fB)^{(n+1)}(z_j) = \sum_{m=0}^{n+1} \binom{n+1}{m} f^{(m)}(z_j) B^{(n+1-m)}(z_j). \]

Clearly, the only non-zero summand here is the one arising for \( m = n \) (recall (14) and the obvious fact that \( B(z_j) = 0 \)). Therefore,

\[ (fB)^{(n+1)}(z_j) = (n+1) f^{(n)}(z_j) B'(z_j) = (n+1) d_j^{\alpha-n} B'(z_j). \]

Now (15) yields

\[ d_j^{\alpha-n} |B'(z_j)| \leq \text{const} \cdot (1 - |z_j|)^{\beta-n-1}. \]

Multiplying both sides by \( 1 - |z_j| \) and noting that \( \inf_j |B'(z_j)|(1 - |z_j|) > 0 \) (this is but a well-known restatement of (10)), we get

\[ d_j^{\alpha-n} \leq \text{const} \cdot (1 - |z_j|)^{\beta-n}, \]

which clearly coincides with (ii).

(ii) \( \Rightarrow \) (i). Let \( f \in I^s(B^n) \). Since a \( A^\alpha_\beta \)-interpolating Blaschke product is also \( H^\infty \)-interpolating, we have (10). Denote the left-hand side of (10) by \( \delta \) and let \( \lambda = \lambda(\delta) \) and \( \varepsilon = \varepsilon(\delta) \) be the same as in Lemma C. Our plan is to use (ii) in order to derive condition (11) with \( \alpha \) replaced by \( \beta \) and \( \theta \) replaced by \( B \). This done, an application of Theorem B will complete the proof.

As mentioned in Section 2 above, \( f \) satisfies (8) where now we set \( s = 0 \):

\[ |f(\zeta_1) - \sum_{m=0}^{n} \frac{f^{(m)}(\zeta_2)}{m!} (\zeta_1 - \zeta_2)^m| \leq C|\zeta_1 - \zeta_2|^\alpha; \]

here \( \zeta_1 \) and \( \zeta_2 \) are arbitrary points in \( \text{clos} \, D \) and \( C \) a positive constant. (A direct way of verifying (16) is to observe that the left-hand side equals
\[
\left| \int_{\xi_1}^{t_1} \int_{\xi_1}^{t_1} \int_{\xi_1}^{t_{n-1}} \int_{\xi_1}^{t_n} f^{(n+1)}(t_{n+1}) \, dt_{n+1} \right|
\]

and to make the obvious estimates on the integrals.

First we let \( z_j \) and \( z_k \) be two distinct zeros of \( B \) and apply (16) with \( \xi_1 = z_k \), \( \xi_2 = z_j \). Noting that

\[
(17) \quad f(z_j) = f'(z_j) = \ldots = f^{(n-1)}(z_j) = 0 \quad \forall j \in \mathbb{N}
\]

(recall that \( f \) is divisible by \( B^n \)), we get

\[
\frac{1}{n!} |f^{(n)}(z_j)||z_j - z_k|^n \leq C|z_j - z_k|^n,
\]

whence

\[
|f^{(n)}(z_j)| \leq Cn! |z_j - z_k|^{n-n}.
\]

Since \( k \) was an arbitrary number is \( \mathbb{N} \setminus \{j\} \), it follows that

\[
(18) \quad |f^{(n)}(z_j)| \leq \text{const} \cdot d_j^{n-n}.
\]

Rewriting (ii) as

\[
d_j^{n-n} \leq \text{const} \cdot (1 - |z_j|)^{\beta-n}
\]

and substituting this in (18), we obtain

\[
(19) \quad |f^{(n)}(z_j)| \leq \text{const} \cdot (1 - |z_j|)^{\beta-n}, \quad j \in \mathbb{N}.
\]

Now suppose \( z \in \Omega(B, c) \). In view of (12) there is a \( j \in \mathbb{N} \) such that \( \rho(z, z_j) < \lambda \). Another application of (16) (this time we set \( \xi_1 = z, \xi_2 = z_j \)) gives

\[
|f(z) - \frac{1}{n!} f^{(n)}(z_j)(z - z_j)^n| \leq C|z - z_j|^n,
\]

where we have once again used (17). Hence

\[
|f(z)| \leq \frac{1}{n!} |f^{(n)}(z_j)||z - z_j|^n + C|z - z_j|^n \leq \text{const} \cdot (1 - |z_j|)^{\beta-n}|z - z_j|^n + C|z - z_j|^n.
\]

(The last inequality relies on (19).)

It is not hard to see (cf. [G], Chapter I, Section 1) that if \( \rho(z, z_j) < \lambda < 1 \) then there are positive constants \( c_1 = c_1(\lambda) \) and \( c_2 = c_2(\lambda) \) such that
\[ 1 - |z_j| \leq c_1(1 - |z|) \]

and

\[ |z - z_j| \leq c_2(1 - |z|). \]

Combining these inequalities with (20) we get

\[ |f(z)| \leq C_1 (1 - |z|)^\theta + C_2 (1 - |z|)^\alpha \leq C_3 (1 - |z|)^\theta, \]

where \( C_1, C_2 \) and \( C_3 \) are some new constants.

Thus, for an arbitrary \( f \in L^\alpha (B^n) \) we have established the estimate

\[ f(z) = O((1 - |z|)^\theta), \quad z \in \Omega(B, e), \]

which coincides with (11) up to the obvious replacements.

Conditions (iii) and (iv) in Theorem B being equivalent, we conclude that

\[ fB^k \in A^\theta \quad \forall k \in \mathbb{Z}. \]

In particular, \( fB \in A^\theta_A \). Therefore \( B \) is \((\alpha, \beta)\)-stable, as required.

4. Embedding theorems for star-invariant subspaces.

For \( p > 0 \), let \( H^p \) denote the classical Hardy space (see [G] or [K]) in the unit disk, \( D \). For \( p \in [1, + \infty) \) and \( \theta \) an inner function, let \( K^p_\theta \) stand for the corresponding star-invariant subspace:

\[ K^p_\theta \overset{\text{def}}{=} H^p \cap \theta \overline{H}^p_0, \]

where \( H^p_0 \overset{\text{def}}{=} \{ f \in H^p : f(0) = 0 \} \) and the bar denotes complex conjugation. The term “star-invariant” here means “invariant under the backward shift operator”.

(It is a matter of common knowledge that the totality of \( K^p_\theta \), as \( \theta \) ranges over all inner functions, coincides with the family of all closed star-invariant subspaces in \( H^p, p \in [1, + \infty) \).

In the case \( p \in (0, 1) \) we set

\[ K^p_\theta \overset{\text{def}}{=} \text{clos}_{H^p} K^\infty_\theta; \]

here \( \text{clos}_{H^p} \) denotes the closure with respect to the \( H^p \) metric.

This section deals with some embedding theorems of the form \( T_f K^p_\theta \subset H^q \), where \( p \) and \( q \) are positive exponents satisfying

\[ 0 < \max (1, p) < q < + \infty, \]

\( f \) is a function holomorphic in \( D \) and smooth up to the boundary, and \( T_f \) is the multiplication map defined by \( T_f g = f g \).

The following proposition was proved by the author in [D 1] along with Theorem B (see Section 2 above).
THEOREM B'. Let $p$ and $q$ satisfy (21). Set $\alpha = 1/p - 1/q$, and let $m$ be an integer for which $mp > 1$. Given a function $f$, $f \in \Lambda_A^*$, and an inner function $\theta$, each of the conditions (i) – (iv) in Theorem B is equivalent to

$$T_f K_{\theta^m}^p \subset H^q.$$  

In fact, from the proof [D1] one sees that the implication (ii) $\Rightarrow$ (v) holds when $\theta^m$ is replaced by an arbitrary inner function $\theta_1$, i.e. under the above assumptions on $p$, $q$, $\alpha$ and $f$

$$f \theta_1 \in \Lambda_A^* \Rightarrow T_f K_{\theta_1}^p \subset H^q.$$  

Our next result is

THEOREM 3. Let $0 < p < 1 < q < +\infty$, $\alpha \overset{\text{def}}{=} 1/p - 1/q$, and suppose there is a positive integer $n$ for which $n < \alpha < 1/p < n + 1$. Suppose further that $B$ is an $\Lambda_A^*$-interpolating Blaschke product with zeros $\{z_j\}$. If

$$\sup_j \frac{d_j}{1 - |z_j|^2} = +\infty$$  

then there exists an $f$, $f \in \Lambda_A^*$, such that

$$T_f K_{B^n}^p \subset H^q$$  

but

$$T_f K_{B^{n+1}}^p \notin H^q.$$  

Before proceeding with the proof, we remark that $K_{B^n}$ coincides with the $H^p$-closed linear span of the family of rational fractions

$$\left\{ \frac{1}{(1 - \bar{z_j}z)^k} : j \in \mathbb{N}, \ k = 1, 2, \ldots, n \right\}.$$  

Thus, when we enlarge this family by letting in addition $k = n + 1$, the effect may be fatal (i.e., the corresponding embedding theorem may become false). It should be noted that in the case where $1 < p < q < +\infty$ such a phenomenon does not occur.

**Proof.** By Theorem 1, (23) means that $B$ is not stable, i.e. $T_B F^p (B^n) \notin \Lambda_A^*$. This in turn implies the existence of an $f$, $f \in \Lambda_A^*$, such that $fB^n \in \Lambda_A^*$ but $fB^{n+1} \notin \Lambda_A^*$. Applying (22) with $\theta_1 = B^n$, we obtain (24). Applying Theorem B' with $\theta = B$, $m = n + 1$, we arrive at (25).

The following generalization of Theorem 3 can be derived in a similar fashion with recourse to Theorem 2.
Theorem 4. Let $0 < p < 1 < r \leq q < +\infty$, $\alpha \triangleq 1/p - 1/q$, $\beta \triangleq 1/p - 1/r$. Suppose that for some $n, n \in \mathbb{N}$, we have $n < \beta < p^{-1} < n + 1$. If $B$ is a $A^\alpha_\lambda$-interpolating Blaschke product with zeros $\{z_j\}$ for which
\[
\sup_j d_j (1 - |z_j|)^{-(\beta - n)/(\alpha - n)} = +\infty
\]
then there exists an $f, f \in A^\alpha_\lambda$, such that (24) holds but $T_f K_{B^{n+1}}^\alpha \not\subset H^\beta$.

5. Remarks and examples.

1. In connection with embedding theorems for the $K_\theta^p$ spaces we cite [Co 1,2,3], [TV] and [D 2,3] where some partial information can be found on the embeddings $K_\theta^p \subset B^p(\mu)$ or $K_\theta^p \subset B^q(\mu)$, $\mu$ being a suitable measure on clos $D$ and $\theta$ an inner function. A complete characterization of the pairs $(\theta, \mu)$ for which the embedding holds still seems to be unknown.

2. Suppose that the sequence $\{z_j\} \subset D$ is $A^\alpha_\lambda$-interpolating and satisfies, in addition, the following regularity conditions:
\[
d_j = |z_j - z_{j+1}|, \quad \sup_j \frac{1 - |z_j|}{1 - |z_{j+1}|} < +\infty.
\]
Under these assumptions we are able to prove the converse of Theorem 3: if $B = B_{\{z_j\}}$ is stable then $T_f K_{B^{n}}^\alpha \subset H^q$ implies $T_f K_{B^{n+1}}^\alpha \subset H^q$. A similar supplement to Theorem 4 can be provided.

3. We proceed by giving a few examples.

(a) Let $\{z_j\}$ be a sequence in $D$ tending to 1 nontangentially (i.e. $\sup_j |1 - z_j|/ (1 - |z_j|) < +\infty$) such that $\inf_{j \neq k} \rho(z_j, z_k) > 0$. The arising Blaschke product $B = B_{\{z_j\}}$ is easily shown to be stable.

(b) Suppose $n < \beta < \alpha < n + 1, n \in \mathbb{N}$. Fix $\gamma \geq 1$ and let $\{z_j\}$ be defined by
\[
|1 - z_j| = 2^{-j}, \quad 1 - |z_j| = 2^{-\gamma j}, \quad \text{Im } z_j > 0.
\]
It is not hard to see that $c_1 \cdot 2^{-j} \leq d_j \leq c_2 \cdot 2^{-j}$ (here $c_1$ and $c_2$ are absolute constants), and so condition (ii) in Theorem 2 holds iff $\gamma \leq (\alpha - n)/(\beta - n)$. In particular, taking $\gamma = (\alpha - n)/(\beta - n)$ one obtains a Blaschke product that is $(\alpha, \beta)$-stable but not $(\alpha, \beta_1)$-stable whenever $\beta < \beta_1 \leq \alpha$.

(c) Furthermore, consider the “super-tangential” sequence $\{z_j\}$ defined by
\[
|1 - z_j| = 2^{-j}, \quad 1 - |z_j| = 2^{-2j}, \quad \text{Im } z_j > 0.
\]
Clearly, condition (ii) in Theorem 2 is never fulfilled and so, for any values of \( \alpha \) and \( \beta \), the Blaschke product \( B_{\{z_j\}} \) fails to be \( (\alpha, \beta) \)-stable.

In order to make sure that the sequences constructed in (a), (b) and (c) above are \( \mathcal{A}_4 \)-interpolating, one may use either Theorem A or, still better, the following proposition [Kot]: if

\[
\{z_j\} \subset \mathbb{D}, \quad \lim_{j \to \infty} z_j = 1, \quad |z_j - 1| \geq |z_{j+1} - 1| \quad \text{and} \quad \sup_{j \neq k} \frac{|1 - z_j||1 - z_k|}{|z_j - z_k|^2} < +\infty,
\]

then \( \{z_j\} \cup \{1\} \) is a \( \mathcal{A}_4 \)-interpolating set.

4. Results obtained in [D1], [D2] imply that if \( B = B_{\{z_j\}} \) is not stable then it must necessarily be sparse, i.e.

\[
\sup_{j} \prod_{k \neq j} \left| \frac{z_k - z_j}{1 - \overline{z_k}z_j} \right| = 1.
\]

On the other hand, there are sparse Blaschke products that are stable; e.g. let \( z_j = 1 - (j!)^{-2} \), \( B = B_{\{z_j\}} \).

REFERENCES


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