ZERO SETS OF FUNCTIONS IN
HARMONIC HARDY SPACES

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1. Introduction.

For $0 < p < \infty$ we let $h^p$ be the space of all real-valued harmonic functions in the
open unit disc $U$ which satisfy the growth condition

\[(1.1) \quad \|u\|_p = \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty, \]

and we let $H^p$, as usual, be the Hardy space of all holomorphic $f$ in $U$ which satisfy
(1.1) with $f$ in place of $u$.

It is well known that the zero sets $Z(f)$ of $H^p$-functions $f$ are the same for all $p$: If $f \in H^p$ for some $p$ and $f \not\equiv 0$ then $Z(f)$ satisfies the Blaschke condition; conversely, if a set $S \subset U$ satisfies the Blaschke condition then $S = Z(f)$ for some bounded holomorphic $f$. In this note we consider the analogous question for $h^p$ and find the following answer:

THEOREM. If $0 < p < q < \infty$ then there is a set $S \subset U$ such that

(a) $S = Z(u)$ for some $u \in h^p$, but
(b) if $v \in h^q$ and $Z(v) \supset S$ then $v \equiv 0$.

This is reminiscent of the situation in several complex variables where the holomorphic $H^p$-spaces have different zero sets for different values of $p$ [6, p. 145].

The case $p = 1$ can be settled right away, and with the same $S$ for all $q > 1$. Let $S$ be the circle of radius $1/2$ centered at $1/2$, but without the point $1$. Then $S = Z(u)$ if

\[(1.2) \quad u(\lambda) = \text{Re} [\lambda/(1 - \lambda)]. \]

Since $2u + 1 > 0$ in $U$, and positive harmonic functions are in $h^1$, we have $u \in h^1$. Assume now that $v \in h^q$ for some $q > 1$. Then $v = \text{Re} f$ for some $f \in H^q$, $|f|^q$ has

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a harmonic majorant in \( U \), hence also in the disc \( D \) bounded by \( \overline{S} \). Therefore \( f \in H^p(D) \), and if \( \text{Re} \, f = 0 \) at every point of \( S \), it follows from the Poisson integral in \( D \) that \( \text{Re} \, f \equiv 0 \) in \( D \), hence also in \( U \).

The general case of the theorem is a corollary of the more precise Theorems A and B stated below in terms of the spaces \( \text{Re} \, H^p \) (the real parts of \( H^p \)-functions) and \( G^p \), where \( u \in G^p \), by definition, if \( u \) is harmonic in \( U \) and

\[
\sup_{\lambda \in U} (1 - |\lambda|)^{1/p} |u(\lambda)| < \infty.
\]

These spaces are related by the inclusions

\[
\text{Re} \, H^p \subset h^p \subset G^p \quad (0 < p < \infty).
\]

That \( \text{Re} \, H^p \subset h^p \) is trivial. When \( 1 < p < \infty \), the classical theorem of Marcel Riesz shows that \( \text{Re} \, H^p = h^p \). But \( h^1 \) is larger than \( \text{Re} \, H^1 \) (even though \( h^1 \subset \text{Re} \, H^p \) for all \( p < 1 \)), and if \( 0 < p < 1 \) there are functions in \( h^p \) which lie in no \( \text{Re} \, H^q \); see [2; Chap. 4], for example. Another difference between \( h^p \) and \( \text{Re} \, H^p \) (when \( 0 < p < 1 \)) is that \( \text{Re} \, H^p \) is separable whereas \( h^p \) is not. This, and other aspects of \( h^p \), are described in [7].

The difficulty in proving \( h^p \subset G^p \) occurs when \( p < 1 \), since the Poisson integral in then not available. The first proof, given by Hardy and Littlewood [5; Th. 1] was based on some elementary but rather complicated lemmas. Fefferman and Stein simplified this by finding a fairly easy proof of the inequality

\[
|u(\lambda)|^p \leq \frac{K}{m(D)} \int_D |u|^p \, dm
\]

in which \( K = K(p) < \infty \), \( m \) is plane Lebesgue measure, and \( D \) is any disc with center \( \lambda \) in which \( u \) is harmonic. (See [3; p. 172], [4; p. 121]. The inequality is also a consequence of [5; Th. 5].) To apply (1.5), pick \( \lambda \in U \), \( |\lambda| = 1 - \varepsilon > 1/2 \), let \( D \) have center \( \lambda \), radius \( \varepsilon \), enlarge the domain of integration to the annulus \( \{ \lambda: 1 - 2\varepsilon < |\lambda| < 1 \} \), and read off that

\[
|u(\lambda)|^p \leq \frac{4K}{\varepsilon} \|u\|_{L^p}^p.
\]

On the other hand, \( G^p \) is larger than \( h^p \) for all \( p \). This follows, for example from [1; Th. 5], which implies: If \( \Psi: [0, 1) \rightarrow [1, \infty) \) satisfies \( \psi(r) \uparrow \infty \) as \( r \uparrow 1 \), then there is a holomorphic \( f \) in \( U \) such that \( |f(\lambda)| < \psi(|\lambda|) \) for all \( \lambda \in U \), but

\[
\min_{\theta} |f(r_j e^{i\theta})| \uparrow \infty
\]

for some sequence \( r_j \uparrow 1 \). Take \( \psi(r) = (1 - r)^{-1/p} \), let \( f = u + iv \). Then \( u \) and \( v \) are in \( G^p \), but (1.7) shows that at least one of them is not in \( h^p \).
The preceding discussion shows that Theorems A and B really give more information than the one that we stated in this Introduction.

2. Main results.

From now on $p$ and $q$ are fixed, $0 < p < q < \infty$. Choose $\gamma$ and $\delta$ so that

$$p/q < \gamma < \delta < 1.$$  

Choose $\alpha$, $0 < \alpha < \pi/2$, so that, setting $s = \sin \alpha, c = \cos \alpha$, we have

$$s^2 < \gamma/2p,$$

and then put

$$\beta = 1/pcs.$$  

Let $\Omega$ be the strip consisting of all $z = x + iy$ such that

$$|cy - sx| < c\delta\pi/2.$$  

For $n = 0, \pm 1, \pm 2, \ldots$, put

$$E_n = \{z \in \Omega: x = n\pi/\beta\}, E = \bigcup_{n=-\infty}^{\infty} E_n.$$  

The function

$$\Phi(z) = \frac{\exp(z/c\delta e^{i\alpha}) - 1}{\exp(z/c\delta e^{i\alpha}) + 1}$$

maps $\Omega$ conformally onto $U$. We define

$$S = \Phi(E)$$

and can now state our results.

**Theorem A.** $S = Z(u_0)$ for some $u_0 \in \text{Re } H^p$.

**Theorem B.** If $v \in G^q$ and $Z(v) \ni S$ then $v \equiv 0$.

3. Proof of Theorem A.

Define $f$ in $\Omega$ by

$$f(z) = ie^{-i\beta z},$$

put $u = \text{Re } f, u_0 = u \circ \Phi^{-1}$. Then

$$u(z) = e^{\beta y} \sin \beta x$$

so that $E = Z(u)$, hence $S = Z(u \circ \Phi^{-1}) = Z(u_0)$. 

We have to show that $u_0 \in \text{Re } H^p$.
Let $\psi$ be the harmonic function defined by

\[(3.3) \quad \psi(z) = \exp \left( x + \frac{s}{c} y \right) \cdot \cos \left( y - \frac{s}{c} x \right). \]

By (2.4), $\cos \left( y - \frac{s}{c} x \right) > \cos(\delta \pi/2) > 0$ in $\Omega$, so that

\[(3.4) \quad \log \psi(z) > x + \frac{s}{c} y + \log \cos(\delta \pi/2) \]

in $\Omega$. On the other hand, (3.1), (2.3), and (2.4) show that

\[(3.5) \quad p \log |f| = p \beta y = \left( \frac{s}{c} + \frac{c}{s} \right) y < \frac{sy}{c} + x + \frac{c\delta \pi}{2s}. \]

It follows from (3.4) and (3.5) that there is a constant $K < \infty$ such that

\[(3.6) \quad |f|^p < K\psi \quad \text{in} \quad \Omega. \]

If we now put $f_0 = f \circ \Phi^{-1}$, then $K\psi \circ \Phi^{-1}$ is a harmonic majorant of $|f_0|^p$ in $U$. Hence $f_0 \in H^p$. Since $u_0 = \text{Re } f_0$, the proof is complete.

4. Proof of Theorem B.

Suppose now that $v \in G^q$ and $v(\lambda) = 0$ for all $\lambda \in S$. Put

\[(4.1) \quad w = v \circ \Phi \]

where $\Phi : \Omega \rightarrow U$ is given by (2.6). Then $Z(w)$ contains every segment $E_n$ as in (2.5).
We have to conclude that this forces $w \equiv 0$.

Define

\[(4.2) \quad \Omega_\gamma = \{ z : |cy - sx| < c\gamma \pi/2 \}. \]

This is a strip whose closure lies in $\Omega$. We need an upper bound (namely (4.7)) for the growth of $|w(z)|$ as $z \rightarrow \infty$ within $\Omega_\gamma$. This, followed by an application of the reflection principle and an argument of the Phragmén-Lindelöf type, will lead to the desired conclusion.

The map $\Phi(z) = \lambda$ can be written in the form

\[(4.3) \quad \lambda = \frac{e^r - 1}{e^r + 1} \]

where
\[ (4.4) \quad \tau = e^{-i\pi z/c\delta} = \frac{cx + sy}{c\delta} + i \frac{cy - sx}{c\delta}. \]

A simple calculation leads from (4.3) to

\[ (4.5) \quad \frac{1 + \lambda \lambda^{-1}}{1 - \lambda \lambda^{-1}} = \frac{\cosh(\Re \tau)}{\cos(\Im \tau)}. \]

In \( \Omega, |\Im \tau| < \gamma \pi/2 \delta \), so that \( \cos(\Im \tau) \) is bounded below by a positive constant. Another calculation, using (4.4) and (2.4), shows that in \( \Omega \)

\[ (4.6) \quad |\Re \tau| < \frac{|y|}{cs\delta} + \frac{c\pi}{2s}. \]

If we combine these estimates with the fact that \( v \in G^q \), i.e., that \( |v(\lambda)|^q = O((1 - \lambda \lambda^{-1})^{-1}) \), we obtain

\[ (4.7) \quad |w(z)| < K \exp(|y|/cs\delta q) \quad \text{in} \quad \Omega. \]

A look at Fig. 1 will clarify the next step.

The line \( y = (\pi y/2) - (sx/c) \) intersects the real axis at

\[ (4.8) \quad x = x_0 = \frac{c\pi y}{2s} > cs \pi p = \pi/\beta, \]

using (2.2) and (2.3). The inequality \( x_0 > \pi/\beta \) shows that the reflections of \( \Omega \) in the segments \( E_n \) cover the plane. Since \( w(z) = 0 \) for all \( z \in E_n \) and for all \( n \), it follows
that \( w \) extends to a harmonic function in the whole plane, which we still denote by \( w \), and that

\[
(4.9) \quad w\left(\frac{k\pi}{\beta}, y\right) = 0
\]

for all integers \( k \) and all real \( y \). Moreover, the extended function stil satisfies (4.7).

To finish, we apply the Phragmen-Lindelöf technique to the function \( w \) in the strip

\[
(4.10) \quad \Sigma = \{z: 0 \leq x \leq \pi/\beta\}.
\]

Note that \( w = 0 \) on the edges of \( \Sigma \).

Since \( q\delta > p \), there exists \( t \) such that

\[
(4.11) \quad \frac{1}{c\delta q} < t < \frac{1}{csp} = \beta.
\]

For \( \varepsilon > 0 \) define

\[
(4.12) \quad w_\varepsilon(z) = w(z) - \varepsilon(e^{iy} + e^{-iy})\cos\left(t\left(x - \frac{\pi}{2\beta}\right)\right).
\]

The last cosine is \( \geq \cos(tt\pi/2\beta) > 0 \) in \( \Sigma \), because \( t < \beta \).

The first inequality in (4.11), combined with the estimate (4.7), shows now that \( w_\delta(z) < 0 \) for all \( z \in \Sigma \) for which \( |y| \) is sufficiently large. It also follows from (4.12) that \( w_\varepsilon(z) < 0 \) on the edges of \( \Sigma \). Since \( w_\varepsilon \) is harmonic, the maximum principle shows now that \( w_\varepsilon(z) < 0 \) for all \( z \in \Sigma \). Hence, letting \( \varepsilon \downarrow 0 \), \( w(z) \leq 0 \).

The same argument, applied to \( -w \) in place of \( w \), gives \( w(z) \geq 0 \). So \( w(z) = 0 \) for all \( z \in \Sigma \), hence everywhere.

REFERENCES