TRANSITION PROBABILITIES AND TRACE FUNCTIONS
FOR C*-ALGEBRAS

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Abstract.

Transition probabilities are used to show that certain trace functions are lower semi-continuous on the space of closed ideals of a C*-algebra, equipped with the \( \tau_w \)-topology. This generalizes the well-known theorem of Dixmier that the trace functions are lower semi-continuous on the spectrum of a C*-algebra. The result is then applied to characterize C*-algebras of Type I\(_0\) (or Fell C*-algebras) in terms of the existence of a dense ideal of elements for which the trace functions are continuous. The points of continuity of transition probabilities are characterized, and a necessary and sufficient condition is given for the reduced C*-algebra of a second countable \( \tau \)-discrete principal groupoid to be of type I\(_0\).

1. Introduction.

In this paper we use transition probabilities for pure states of a C*-algebra to study trace functions defined on the spectrum and on certain ideal spaces. This approach enables us to work with an arbitrary C*-algebra, and not just with liminal ones. We also extend the results of [6] on the continuity of transition probabilities and illustrate these results in the case of groupoid C*-algebras.

Let \( A \) be a C*-algebra with spectrum \( \hat{A} \) (the space of equivalence classes of irreducible representations of \( A \)). Suppose that \( (\pi_\lambda) \) is a convergent net in \( \hat{A} \) and let \( L \) be the set of limits. Dixmier [12; 3.5.9] showed that if \( \pi \in L \) then

\[
\liminf \text{Tr} \pi_\lambda(a) \geq \text{Tr} \pi(a) \quad (a \in A^+),
\]

where \( A^+ \) is the set of positive elements of \( A \) and \( \text{Tr} \) is the usual trace for positive operators on a Hilbert space (note that values \(+ \infty\) may occur in (1) and in (2) below). In the special case where \( A \) is liminal (that is, when every irreducible representation of \( A \) consists of compact operators), Milicic [20] extended this result as follows:

\[
\liminf \text{Tr} \pi_\lambda(a) \geq \sum_{\pi \in L} \text{Tr} \pi(a) \quad (a \in A^+).
\]

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We refer the reader to [25], [20], [19] for related convergence properties in uniformly liminal C*-algebras and for applications to semisimple Lie groups.

One of the results of this paper (Theorem 2.4) is that (2) is valid for an arbitrary C*-algebra. In fact, we obtain Theorem 2.4 as a corollary of a quite general lower semi-continuity result for trace functions on the ideal space of a C*-algebra (Theorem 2.3). Whilst the main interest of these results may lie in the case where A contains a non-zero postliminal ideal, we note that even an antiliminal C*-algebra may have a dense set of finite dimensional representations in \( \hat{A} \) (for example, the full C*-algebra of the free group on two generators [10] and the rotation algebra [17], [2]). In the rest of Section 2 we use the generalized version of (2) to give alternative proofs of some results of Fell [14] on finite dimensional irreducible representations.

Once the lower semi-continuity of the trace functions has been established, it is natural, by analogy with continuous trace C*-algebras, to look for C*-algebras containing a dense ideal for which the trace functions are continuous, in some appropriate sense. In Section 3, we study three possible extensions of the class of continuous trace C*-algebras. We show that they all coincide, and that they are precisely equal to the class of C*-algebras of Type I\(_0\) studied in [24; 6.1, 6.2]. Since these are precisely the C*-algebras whose spectra satisfy Fell's condition at each point we will refer to them as Fell C*-algebras. It is possible to obtain still larger classes of C*-algebras with dense ideals of elements of 'continuous trace' by introducing 'multiplicity' integers, see [14], [25], [20]. We hope to pursue this subject further, and one of the aims of this paper is to clarify the 'multiplicity one' case. In the rest of Section 3 we study first the relationship between Fell C*-algebras and C*-algebras of generalized continuous trace (GCT C*-algebras), and then the set of separated points in the spectrum of a Fell C*-algebra, when the spectrum is compact. We give an example of a Fell C*-algebra which does not have GCT, providing a new and simpler example of a separable, liminal C*-algebra for which the set of non-separated points is dense in the spectrum.

Transition probabilities for pure states of a C*-algebra play an important rôle in our approach to Section 2. They are also related to Fell's condition in [6]. In Section 4, we continue the study of their continuity properties. We recall from [29] the definition of the transition probability \( \langle \phi, \psi \rangle \) for a pair \((\phi, \psi)\) in \( P(A) \times P(A) \) (where \( P(A) \) is the set of pure states of \( A \)). If the Gelfand-Naimark-Segal representations \( \pi_\phi \) and \( \pi_\psi \) are inequivalent then \( \langle \phi, \psi \rangle = 0 \). On the other hand, if \( \pi_\phi \simeq \pi_\psi \) then there exists an irreducible representations \( \pi \) of \( A \) and unit vectors \( \xi \) and \( \eta \) in the Hilbert space \( H_\pi \) such that \( \phi = \langle \pi(.)\xi, \xi \rangle \) and \( \psi = \langle \pi(.)\eta, \eta \rangle \). In this case \( \langle \phi, \psi \rangle = |\langle \xi, \eta \rangle|^2 \). Let \( T: P(A) \times P(A) \to [0,1] \) be defined by

\[
T(\phi, \psi) = \langle \phi, \psi \rangle \quad (\phi, \psi \in P(A)).
\]
As observed in [6; p. 8, Remark 2], \( T \) is upper semi-continuous for the product \( w^*\)-topology on \( P(A) \times P(A) \) and hence is continuous at points \( (\phi, \psi) \) for which \( \langle \phi, \psi \rangle = 0 \). It is this fact which is the key to our results in Section 2. In Theorem 4.1 we give a complete description of the set of points of continuity for \( T \).

In Theorem 4.2 we consider the restriction \( T_0 \) of \( T \) to \( R(A) \), the subset of \( P(A) \times P(A) \) consisting of those pairs \( (\phi, \psi) \) such that \( \pi_\phi \) and \( \pi_\psi \) are equivalent. It was shown in [6; Theorem 2.3] that \( T_0 \) is continuous (for the product \( w^*\)-topology) if and only if every \( \pi \) in \( \hat{A} \) satisfies Fell's condition. Here we localize this result by showing that if \( (\phi, \psi) \in R(A) \) then \( T_0 \) is continuous at \( (\phi, \psi) \) if and only if either \( \langle \phi, \psi \rangle = 0 \) or \( \pi_\phi \) satisfies Fell's condition.

In Section 5 we investigate a class of \( C^* \)-algebras which illustrate the results of Sections 3 and 4, namely the reduced \( C^* \)-algebras of separated topological equivalence relations. These are a special class of groupoid \( C^* \)-algebras for which the theory can be developed topologically, without recourse to integration [26]. A separated topological equivalence relation \( R \) is equipped with two topologies, one finer than the other. We show that the Fell points in the spectrum of the reduced \( C^* \)-algebra of \( R \) correspond exactly to those points of \( R \) where the two topologies coincide. We conclude, as a consequence, that the \( C^* \)-algebra is a Fell \( C^* \)-algebra if and only if these two topologies are equal. This extends the results of [22], [23] in a special case. (In [22], [23] a characterization is given of continuous trace \( C^* \)-algebras in the class of principal groupoid \( C^* \)-algebras with twists.) The results of Section 5 were obtained jointly with Mark Priest, and we are grateful for his permission to include them in this paper.

In the rest of this section we establish some more notation and prove some preliminary lemmas.

Let \( A \) be a \( C^* \)-algebra with Banach dual \( A^* \) equipped with the \( w^* \)-topology. Let \( S(A) \) denote the state space of \( A \) and for \( \phi \in S(A) \) let \( \{ \pi_\phi, H_\phi, \xi_\phi \} \) be the usual GNS triple associated with \( \phi \) (see Section 5). If \( \pi \) is an irreducible representation of \( A \) we shall adopt the common practice of using the same symbol to denote the corresponding equivalence class in \( \hat{A} \). Thus \( \pi_1 \simeq \pi_2 \) (as irreducible representations) means \( \pi_1 = \pi_2 \) (in \( \hat{A} \)). We shall denote by \( \theta : P(A) \to \hat{A} \) the continuous, open mapping given by \( \phi \to \pi_\phi \) (see [12; 3.4.11]). Recall that pure states \( \phi \) and \( \psi \) are said to be equivalent if \( \pi_\phi \) and \( \pi_\psi \) are equivalent.

By an ideal of \( A \) we shall always mean a two-sided ideal. The set \( \text{Id}(A) \) of all closed ideals of \( A \) can be equipped with strong and weak topologies, \( \tau_s \) and \( \tau_w \). The precise definition and origins of these topologies are described in [3; Section 2]. The main features are that a net \( (I_\alpha) \) is \( \tau_s \)-convergent to \( I \) in \( \text{Id}(A) \) if and only if \( \| a + I_\alpha \| \to \| a + I \| \) for all \( a \in A \), whilst a base for \( \tau_w \) is given by the family of sets of the form

\[
U(F) = \{ I \in \text{Id}(A) : J \nsubseteq I \text{ for all } J \in F \}
\]
where $F$ is a finite set (possibly empty) of closed ideals of $A$. The method of proof of [3; 3.5b)] shows that the functions $I \rightarrow \|a + I\|$ ($a \in A$) are lower semi-continuous on $(\text{Id}(A), \tau_w)$. As a matter of fact it is easy to show that the $\tau_w$-topology is the weakest topology with respect to which all these functions are lower semi-continuous. The space $(\text{Id}(A), \tau_a)$ is compact and Hausdorff whilst $(\text{Id}(A), \tau_w)$ is compact but not usually Hausdorff. The restriction of $\tau_w$ to $\text{Prim}(A)$ (the set of primitive ideals of $A$) coincides with the Jacobson topology. This will be the default topology on $\text{Prim}(A)$. The only topology which we shall use on $\tilde{A}$ will be the Jacobson topology. If $L$ is a non-empty subset of $\text{Prim}(A)$ we will let $\ker L$ be equal to $\bigcap_{P \in L} P$.

A closed ideal $I$ of $A$ is said to be primal if whenever $n \geq 2$ and $J_1, J_2, \ldots, J_n$ are closed ideals of $A$ with zero product then $J_i \subseteq I$ for at least one value of $i$. Primal ideals have been found to arise naturally in the theory of limits of factorial states [5]. We denote by $\text{Primal}(A)$ (respectively $\text{Primal}'(A)$) (respectively $\text{MinPrimal}(A)$) the set of all primal (respectively properly proper primal) (respectively minimal primal) ideals of $A$. Let $\text{Sub}(A)$ denote the $\tau_a$-closure of $\text{MinPrimal}(A)$ in $\text{Primal}'(A)$. $\text{Sub}(A)$ is a natural base-space for a $C^*$-bundle representation of $A$ (see [3], [7] for investigations in the case when $\text{Sub}(A)$ is equal to $\text{MinPrimal}(A)$).

We shall use $\tilde{A}$ to denote $A$ itself (if $A$ is unital) or $A + \mathbb{C}1$ (if $A$ is non-unital and 1 is an adjoined identity).

For a Hilbert space $H$ we denote by $L(H)$ (respectively $\text{LC}(H)$) the $C^*$-algebra of all bounded (respectively compact) linear operators on $H$. If $\xi$ is a unit vector in $H$, the associated vector state $\omega_\xi$ is defined by

$$\omega_\xi(T) = \langle T\xi, \xi \rangle \quad (T \in L(H)).$$

We now prove some lemmas on convergence in the $\tau_w$ and $\tau_a$-topologies. The first extends [3; Proposition 3.2].

**Lemma 1.1.** Let $A$ be a $C^*$-algebra and let $(I_\alpha)$ be a net in $\text{Id}(A)$ which is $\tau_w$-convergent to a proper ideal in $\text{Id}(A)$. Let $L$ be the set of $\tau_w$-limits of $(I_\alpha)$ in $\text{Prim}(A)$ and set $I = \ker L$. For $J \in \text{Id}(A)$ the following are equivalent:

(i) $I_\alpha \rightarrow J (\tau_w)$,

(ii) $J \supseteq I$.

**Proof.** (i) $\Rightarrow$ (ii) If $I_\alpha \rightarrow J (\tau_w)$ then $I_\alpha \rightarrow P (\tau_w)$ for each $P \in \text{Prim}(A/J)$, so $\text{Prim}(A/J) \subseteq L$. Hence $J \supseteq I$.

(ii) $\Rightarrow$ (i) It is sufficient to show that $I_\alpha \rightarrow I (\tau_w)$. To do this it is sufficient to show that whenever $K \in \text{Id}(A)$ with $I \not\supseteq K$ then $I_\alpha$ is eventually in the set

$$X = \{ R \in \text{Id}(A) : R \not\supseteq K \}$$
(since sets of this type form a sub-base for the $\tau_w$-topology). If $I \nsubseteq K$ then there is a $P \in L$ such that $P \nsubseteq K$, so $X$ is a $\tau_w$-open neighbourhood of $P$. Since $I_x \rightarrow P(\tau_w)$, $I_x$ is eventually in $X$, as required.

**Lemma 1.2.** Let $A$ be a $C^*$-algebra and let $(I_x)$ and $(J_x)$ be nets in $\text{Id}(A)$ with $I_x \subseteq J_x$ for each $x$. Let $I, J \in \text{Id}(A)$ and suppose that $I_x \rightarrow I(\tau_w)$ and $J_x \rightarrow J(\tau_w)$. Then $I \subseteq J$.

**Proof.** If $a \in A$ then

$$\|a + J\| \leq \lim \inf \|a + J_x\| \leq \lim \|a + I_x\| = \|a + I\|.$$ 

Hence $I \subseteq J$.

**Corollary 1.3.** Let $A$ be a $C^*$-algebra and let $(I_x)$ be a net in $\text{Id}(A)$ which is $\tau_s$-convergent to a proper ideal $J$ in $\text{Id}(A)$. Let $L$ be the set of $\tau_w$-limits of $(I_x)$ in $\text{Prim}(A)$ and set $I = \ker L$. Then $J = I$.

**Proof.** Since $I_x \rightarrow J(\tau_w), J \supseteq I$ by Lemma 1.1. Since $I_x \rightarrow I(\tau_w)(\text{by Lemma 1.1}), I \supseteq J$ by Lemma 1.2.

Recall that a point in a topological space is a *cluster point* of a net if it is a limit of a subnet of the original net. The following result may be regarded as a generalization of [14; Theorem 2.1].

**Lemma 1.4.** Let $A$ be a $C^*$-algebra and let $(I_x)$ be a net in $\text{Id}(A)$ which is $\tau_w$-convergent to a proper ideal of $A$. The following conditions are equivalent:

(i) $(I_x)$ is $\tau_s$-convergent,

(ii) every $\tau_w$-cluster point of $(I_x)$ is a $\tau_w$-limit,

(iii) every primitive $\tau_w$-cluster point of $(I_x)$ is a $\tau_w$-limit.

**Proof.** Let $L$ be the set of primitive $\tau_w$-limits of $(I_x)$ and set $I = \ker L$.

(i) $\Rightarrow$ (ii) Assuming (i), $I_x \rightarrow I(\tau_s)$ by Corollary 1.3. Suppose that $(I_\beta)$ is any subnet of $(I_x)$ and that $(I_\beta)$ is $\tau_w$-convergent to an ideal $J \in \text{Id}(A)$. Since $I_\beta \rightarrow I(\tau_s)$, it follows from Lemma 1.2 (applied to $(I_\beta)$) that $I \subseteq J$. By Lemma 1.1, $I_x \rightarrow J(\tau_w)$.

(ii) $\Rightarrow$ (iii) This is immediate.

(iii) $\Rightarrow$ (i) Let $(I_\gamma)$ be any subnet of $(I_x)$. The $\tau_s$-compactness of $\text{Id}(A)$ implies that $(I_\gamma)$ has a subnet $(I_\gamma')$ which is $\tau_s$-convergent to some $J \in \text{Id}(A)$. The set of primitive $\tau_w$-limits of $(I_\gamma)$ is exactly $L$, by assumption, so by applying Corollary 1.3 to $(I_\gamma)$ we obtain that $J = I$. It follows that $I_x \rightarrow I(\tau_s)$.

2. The lower semi-continuity of the trace on $\text{Id}(A)$.

In this section we show that the trace-evaluation functions are lower semi-continuous on $(\text{Id}(A), \tau_w)$, Theorem 2.3, and derive some consequences, among
them the generalization of Milicic’s result (see Introduction) to arbitrary C*-algebras, Theorem 2.4.

We need the following well-known lemma, which may be proved by a routine induction on \( M \) (we omit the details). Alternatively it is possible to give an operator-theoretic proof by considering the matrix \( \langle \xi_i, \xi_j \rangle \), as was pointed out to us by J. Spielberg. This latter argument is given in [9] with a reference to an unpublished paper of Haagerup.

**Lemma 2.1.** Given a positive integer \( M \) and \( \delta_0 > 0 \), there exists \( \delta_1 > 0 \) (depending only on \( M \) and \( \delta_0 \)) such that whenever \( 1 \leq m \leq M \) and \( \xi_1, \xi_2, \ldots, \xi_m \) are unit vectors in a Hilbert space \( H \) satisfying

\[
|\langle \xi_i, \xi_j \rangle| < \delta_1 \quad (i \neq j)
\]

then there exists an orthonormal system \( \{ \eta_1, \eta_2, \ldots, \eta_m \} \) in \( H \) such that

\[
\| \eta_i - \xi_i \| < \delta_0 \quad (1 \leq i \leq m).
\]

**Definition.** If \( A \) is a C*-algebra, \( J \) is a closed ideal in \( A \) and \( a \in A^+ \), define \( \text{Tr}_J(a + J) \) to be \( \text{Tr} \pi_J(a + J) \) where \( \pi_J \) is the reduced atomic representation of \( A/J \) [24; 4.3.7]. Equivalently, \( \text{Tr}_J(a + J) = \sum_{\sigma \in (A/J)^\wedge} \text{Tr} \sigma(a + J) \). Define \( f_a : \text{Id}(A) \to [0, \infty] \) by

\[
f_a(J) = \text{Tr}_J(a + J) \quad (J \neq A)
\]
and let \( f_a(A) = 0 \).

**Lemma 2.2.** Let \( A \) be a C*-algebra, and let \( a \in A^+ \). If \( P \in \text{Prim}(A) \) and \( \pi \in \hat{A} \) with \( P = \ker \pi \) then

\[
f_a(P) = \text{Tr} \pi(a).
\]

**Proof.** Since \( f_a(P) \geq \text{Tr} \pi(a) \), we may suppose that \( \text{Tr} \pi(a) < \infty \). In this case \( \pi(a) \) is compact. Let \( \sigma \) be an irreducible representation of \( A/P \) and let \( \Phi \) be the canonical isomorphism of \( \pi(A) \) onto \( A/P \). Then either \( \sigma \circ \Phi \) annihilates \( \text{LC}(H_n) \) or else \( \sigma \circ \Phi \) is equivalent to the identity representation of \( \pi(A) \) [24; 6.1.4]. Thus

\[
f_a(P) = \sum_{\sigma \in (A/P)^\wedge} \text{Tr} \sigma(a + P) = \sum_{\sigma \in (A/P)^\wedge} \text{Tr}(\sigma \circ \Phi)(\pi(a)) = \text{Tr} \pi(a).
\]

It follows from Lemma 2.2 that if \( I \in \text{Id}(A) \) and \( I \neq A \) then

\[
f_a(I) = \sum_{P \in \text{Prim}(A/I)} f_a(P).
\]

We shall use this fact often in the sequel, without further mention.

We now prove the main theorem of this section.
THEOREM 2.2. Let $A$ be a $C^*$-algebra. For each $a \in A^+$ the function $f_a$ is lower semi-continuous on $(\text{Id}(A), \tau_w)$.

PROOF. If $a = 0$ then $f_a = 0$. We may suppose, therefore, that $a \neq 0$. Since $f_a \geq 0$ and $f_a(A) = 0$, $f_a$ is lower semi-continuous at $A$.

Suppose that $I$ is a proper, closed ideal of $A$, that $\alpha \in \mathbb{R}$, and that $f_a(I) > \alpha$. We seek a $\tau_w$-neighbourhood $W$ of $I$ in $\text{Id}(A)$ such that

$$f_a(J) > \alpha \quad (J \in W).$$

There is a finite set $\{\pi_1, \ldots, \pi_n\}$ of inequivalent irreducible representations of $A/I$ such that

$$\sum_{i=1}^{n} \text{Tr} \pi_i(a) > \alpha \quad (1 \leq i \leq n).$$

For each $i = 1, \ldots, n$ there exists an orthonormal set $\{\xi_k^{(i)} : 1 \leq k \leq m_i\}$ in $H_{\pi_i}$ such that

$$\sum_{i=1}^{n} \sum_{k=1}^{m_i} \langle \pi_i(a) \xi_k^{(i)}, \xi_k^{(i)} \rangle = \alpha + \varepsilon$$

for some $\varepsilon > 0$. For $1 \leq i \leq n$ and $1 \leq k \leq m_i$ let

$$\phi_k^{(i)} = \langle \pi_i(.) \xi_k^{(i)}, \xi_k^{(i)} \rangle \in P(A)$$

and let $\lambda_k^{(i)} = \phi_k^{(i)}(a)$. Let $\beta$ and $\varepsilon_1$ be positive numbers such that

$$\beta + \varepsilon_1 M = \frac{\varepsilon}{n}$$

where $M = m_1 + m_2 + \ldots + m_n$. Let $\delta_0 = \beta(2M \|a\|)^{-1}$. From Lemma 2.1 we can obtain $\delta_1$ corresponding to $M$ and $\delta_0$.

Whenever $(k, i) \neq (p, j)$ the transition probability $\langle \phi_k^{(i)}, \phi_p^{(j)} \rangle$ is zero and so $(\phi_k^{(i)}, \phi_p^{(j)})$ is a point of continuity for $T$, see Section 1. So there exists an open neighbourhood $N$ of 0 in $A^*$ such that whenever $(k, i) \neq (p, j)$ and

$$\phi \in (\phi_k^{(i)} + N) \cap P(A), \quad \psi \in (\phi_p^{(j)} + N) \cap P(A)$$

then $\langle \phi, \psi \rangle^{1/2} < \delta_1$.

Let $N_1 = N \cap \{\rho \in A^* : |\rho(a)| < \varepsilon_1\}$ and for $1 \leq i \leq n$ and $1 \leq k \leq m_i$ let

$$U_{k,i} = (\phi_k^{(i)} + N_1) \cap P(A).$$

Then $V_i = \cap_{k=1}^{m_i} \theta(U_{k,i})$ is an open neighbourhood of $\pi_i$ in $\hat{A}$ $(1 \leq i \leq n)$. Since $V_i$ is open, there exists a closed ideal $J_i$ of $A$ such that $V_i = \hat{J}_i(1 \leq i \leq n)$. Let
\[ W = \{ J \in \text{Id}(A) : J \nsubseteq J_i (1 \leq i \leq n) \}, \]

a \( \tau_w \)-open neighbourhood of \( I \) in \( \text{Id}(A) \).

Let \( J \in W \). Since \( J \nsubseteq J_i \) there exists \( \sigma_i \in V_i \) such that \( \sigma_i(J) = \{ 0 \} \) or, equivalently, \( \sigma_i \in (A/J)^\wedge (1 \leq i \leq n) \). Let \( \pi \in \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} \). By re-ordering if necessary, we may suppose that \( \pi = \sigma_1 \sim \sigma_2 \sim \ldots \sim \sigma_r \) and \( \pi \nsubseteq \sigma_s \) for \( s > r \). To establish (1) it suffices, by (2), to show that

\[
(4) \quad \text{Tr}(\pi(a)) > \left( \sum_{i=1}^r \sum_{k=1}^{m_i} \lambda_k^{(i)} \right) - \frac{\varepsilon}{n}
\]

(note that \( \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} \) partitions into at most \( n \) blocks of equivalent representations and that each block contributes a summand like \( \text{Tr}(\pi(a)) \) the the expression \( f_a(J_i) \)).

Since \( \sigma_i \in V_i (1 \leq i \leq r) \) there exist in \( H_\pi \) unit vectors \( \xi_{k,i} (1 \leq k \leq m_i) \) such that if \( \phi_{k,i} = \langle \pi(.), \xi_{k,i} \rangle \) then \( \phi_{k,i} \in U_{k,i} \). Hence if \( (k,i) \neq (p,j) \) then \( \langle \phi_{k,i}, \phi_{p,j} \rangle^{1/2} < \delta_1 \) and so \( \langle \xi_{k,i}, \xi_{p,j} \rangle < \delta_1 \). Since \( m_1 + \ldots + m_r \leq M \), it follows from Lemma 2.1 that there exist orthonormal vectors \( \eta_{k,i} \) in \( H_\pi (1 \leq i \leq r, 1 \leq k \leq m_i) \) such that \( \| \eta_{k,i} - \xi_{k,i} \| < \delta_0 \). Hence

\[
(5) \quad | \sum_{i=1}^r \sum_{k=1}^{m_i} (\langle \pi(a) \eta_{k,i}, \eta_{k,i} \rangle - \phi_{k,i}(a)) | \leq 2\delta_0 \| a \| M = \beta.
\]

Since \( \phi_{k,i} - \phi_{k} \in N_1 \),

\[
(6) \quad | \sum_{i=1}^r \sum_{k=1}^{m_i} (\phi_{k,i}(a) - \lambda_k^{(i)}) | < \varepsilon_1 M.
\]

It follows from (5), (6) and (3) that

\[
\text{Tr} \pi(a) \geq \sum_{i=1}^r \sum_{k=1}^{m_i} \langle \pi(a) \eta_{k,i}, \eta_{k,i} \rangle > \left( \sum_{i=1}^r \sum_{k=1}^{m_i} \lambda_k^{(i)} \right) - \beta - \varepsilon_1 M = \left( \sum_{i=1}^r \sum_{k=1}^{m_i} \lambda_k^{(i)} \right) - \frac{\varepsilon}{n},
\]

as required by (4).

It is easy to show that the \( \tau_w \)-topology is actually the weakest topology with respect to which all the functions \( f_a \) are lower semi-continuous.

The next theorem generalizes [20; Lemma 8] from the liminal to the general case.
Theorem 2.4. Let $A$ be a $C^*$-algebra and let $(\pi_n)$ be a convergent net in $\hat{A}$. Let $L$ be the set of limits of $(\pi_n)$. Then, in the extended interval $[0, \infty]$,

$$\liminf Tr \pi_n(a) \geq \sum_{\pi \in L} Tr \pi(a) \quad (a \in A^+).$$

Proof. Let $a \in A^+$. Since $L$ is closed there exists an ideal $J$ such that $L = (A/J)^\vee$. Hence

$$(1) \quad f_a(J) = \sum_{\pi \in L} Tr \pi(a).$$

Since $\pi_n \to \pi$ for all $\pi \in L$ we have $\ker \pi_n \to \ker \pi (\tau_w)$ for all $\pi \in L$ and so $\ker \pi_n \to J (\tau_w)$, by Lemma 1.1. By Theorem 2.3 and equation (1):

$$\liminf f_a(\ker \pi_n) \geq \sum_{\pi \in L} Tr \pi(a).$$

By Lemma 2.2

$$f_a (\ker \pi_n) = Tr(\pi_n(a)).$$

The proof of Theorem 2.4 given above does not use Milicic’s special case. Indeed, the whole thrust of our approach has been to show how transition probabilities allow one to work in complete generality. Nevertheless, it is possible to give an alternative proof of Theorem 2.4 by using Milicic’s result. This requires some care since $A$ may not have a non-zero liminal ideal (even if the equivalence classes of finite dimensional, irreducible representations are dense in $\hat{A}$). We outline the argument.

Fix $a \in A^+$ and suppose that the inequality in Theorem 2.4 fails. Then, writing $l = \liminf Tr \pi_n(a)$, we have $0 < l < \infty$ (note that if $l = 0$ then $Tr \pi(a) = 0$ for all $\pi \in L$ by [12; 3.5.9]). There exists a subnet $(\pi_{\beta})_{\beta \in \Gamma}$ of $(\pi_n)$ such that

(i) $\lim Tr \pi_{\beta}(a) = l$,

(ii) $0 < Tr \pi_{\beta}(a) \leq l + 1$ for all $\beta \in \Gamma$.

Let $L_0$ be the set of limits of $(\pi_{\beta})$ and let $J$ be the closed ideal of $A$ such that $(A/J)^\vee$ is the closure of $\{\pi_{\beta} : \beta \in \Gamma\}$. Then $(A/J)^\vee \supseteq L_0 \supseteq L$. For $\pi \in (A/J)^\vee$ we have that $Tr \pi(a) \leq l + 1$ by (ii) and [12; 3.5.9]. Thus $a + J \in I$ where $I$ is the liminal ideal of $A/J$ defined by

$$I = \{x \in A/J : \pi(x) \in LC(H_\pi) \text{ for all } \pi \in (A/J)^\vee\}.$$ 

Working in $(A/J)^\vee$, we have that $\pi_{\beta} \in \hat{I}$ for all $\beta$, and that $\hat{I} \cap L_0$ is non-empty (for otherwise $\pi(a) = 0$ for all $\pi \in L$). Writing $b = a + J$ and applying Milicic’s result to the net $(\pi_{\beta})$ in $\hat{I}$, we obtain
\[ l = \lim \operatorname{Tr} \pi_\beta(b) \geq \sum_{\pi \in L_0} \operatorname{Tr} \pi(b) \]
\[ = \sum_{\pi \in L_0} \operatorname{Tr} \pi(b) \geq \sum_{\pi \in L} \operatorname{Tr} \pi(a) > l, \]
which is a contradiction.

The next result is essentially a combination of Lemma 2.4 and Corollary 1 from [14; Section 10]. The original proofs involve polynomial identities, universal subnets, and spectral theory.

**Corollary 2.5.** Let \( A \) be a \( C^* \)-algebra and let \( n \) be a positive integer. Let \( (\pi_\alpha) \) be a convergent net in \( \hat{A} \) such that \( \dim \pi_\alpha \leq n \) for all \( \alpha \), and let \( L \) be the set of limits of \( (\pi_\alpha) \). Then

(i) \( L \) is a finite set with at most \( n \) elements,
(ii) for each \( \pi \in L \), \( \dim \pi \leq n \),
(iii) \( \sum_{\pi \in L} \dim \pi \leq n \).

**Proof.** It suffices to prove (iii). This follows from Corollary 2.4 by putting \( a = 1 \) (if \( A \) is non-unital we adjoin an identity and note that the canonical homeomorphism of \( \hat{A} \) onto an open subset of \( (\hat{A})^\wedge \) preserves dimension).

The next result is a slight extension of Corollary 3 of [14; Section 10].

**Corollary 2.6.** Let \( A \) be a \( C^* \)-algebra and let \( n \) be a positive integer. Let \( (\pi_\alpha) \) be a convergent net in \( \hat{A} \) and suppose that \( \dim \pi_\alpha = n \) for all \( \alpha \). Let \( L \) be the (necessarily finite) sets of limits of \( (\pi_\alpha) \).

If \( A \) is unital the following are equivalent:

(i) \( \sum_{\pi \in L} \dim \pi = n \),
(ii) \( \operatorname{Tr} \pi_\alpha(a) \rightarrow \sum_{\pi \in L} \operatorname{Tr} \pi(a) \quad (a \in A) \).

If \( A \) is non-unital let \( \sigma \) denote the element of \( (\hat{A})^\wedge \) which annihilates \( A \). Then the following conditions are equivalent:

(i)' \( \sum_{\pi \in L} \dim \pi = n \),
(ii)' \( \operatorname{Tr} \pi_\alpha(a) \rightarrow \sum_{\pi \in L} \operatorname{Tr} \pi(a) \quad (a \in A) \) and \( \pi_\alpha \rightharpoonup \sigma \) in \( (\hat{A})^\wedge \).

**Proof.** Suppose that \( A \) is unital. Suppose also that (i) holds and that \( x \in A \) with \( 0 \leq x \leq 1 \). By applying Theorem 2.4 to \( x \) and \( 1 - x \) we obtain that

\[
\lim \sup \operatorname{Tr} \pi_\alpha(x) \leq \sum_{\pi \in L} \operatorname{Tr} \pi(x) \leq \lim \inf \operatorname{Tr} \pi_\alpha(x).
\]

Then (ii) follows by linearity. Conversely, if (ii) holds we can obtain (i) by putting \( a = 1 \).

Suppose that \( A \) is non-unital. Given that (i)' holds it follows, by applying
Corollary 2.5 (iii) to $(\pi_\sigma)$ (regarded as a net in $(\tilde{A})^\wedge$), that $\pi_\sigma \to \sigma$. The rest of (ii)' is obtained as before. Conversely, if (ii)' holds there is an open neighbourhood $U$ of $\sigma$ in $(\tilde{A})^\wedge$ such that $\pi_\sigma \notin U$ frequently. There exists a closed ideal $J$ of $\tilde{A}$ such that $U = \{\sigma' \in (\tilde{A})^\wedge : \sigma'(J) \neq \{0\}\}$. Since $J \notin A$ there exists $x \in A$ such that $1 - x \in J$. If $\pi \in L$ then, since $\pi_\sigma \to \pi$, $\pi \notin U$ and so $\pi(x) = \pi(1)$. Also, $\pi_\sigma(x) = \pi_\sigma(1)$ frequently. Hence, using (ii)',

$$\sum_{\pi \in L} \dim \pi = \sum_{\pi \in L} \text{Tr} \pi(x) = \lim \text{Tr} \pi_\sigma(x) = n.$$  

The next result is similar to [20; Lema 11]. It explains why we will restrict attention to $\tau_s$-convergent nets in the next section.

**Corollary 2.7.** Let $A$ be a C$^*$-algebra and let $(I_\alpha)$ be a net in $\text{Id} (A)$ converging $(\tau_w)$ to $I \in \text{Id} (A)$. Suppose that there exists a dense subset $S$ of $A^+$ such that

$$\lim f_\alpha (I_\alpha) = f_\alpha (I) < \infty \quad (a \in S).$$

Then $I_\alpha \to I (\tau_s)$.

**Proof.** Let $(I_\beta)$ be any subnet of $(I_\alpha)$. By $\tau_s$-compactness, $(I_\beta)$ has a subnet $(I_\gamma)$ which is $\tau_s$-convergent to some $J \in \text{Id} (A)$. Since $I_\gamma \to I (\tau_w), I \supseteq J$ by Lemma 1.2. For $a \in S$

$$f_\alpha (I) = \lim f_\alpha (I_\gamma) \geq f_\alpha (J) \quad (by \text{ Theorem 2.3}).$$

Hence $f_\alpha (I) = f_\alpha (J)$. If $P \in \text{Prim}(A/J)$ there exists $a \in S$ with $f_\alpha (P) > 0$. Hence $P \supseteq I$ and so $I = J$. It follows that $I_\alpha \to I (\tau_s)$.

**Remark.** If it is further assumed that $S$ above is the positive part of a self-adjoint subalgebra then the method of [20; Lemma 11] can be used to show that the assumption that $I_\alpha \to I (\tau_w)$ is redundant.

3. **Fell C$^*$-algebras.**

In this section we study Fell C$^*$-algebras (C$^*$-algebras of Type I$_0$), viewing them as a natural generalization of the class of continuous trace C$^*$-algebras. We show that they can be characterized in terms of an ideal of 'continuous trace'; we give an example of a Fell C$^*$-algebra which is not a GCT-algebra, and we study the set of separated points in their spectra.

We begin with some definitions:

A positive element $x$ in a C$^*$-algebra $A$ is said to be abelian if rank $\pi(x) \leq 1$ for all $\pi \in \tilde{A}$. If $A$ is generated, as a C$^*$-algebra, by its abelian elements then it is said to be a C$^*$-algebra of Type I$_0$.

C$^*$-algebras of Type I$_0$ were studied in [24; 6.1, 6.2] where it was shown that
they are liminal, and that the class is closed under passage to quotients and to hereditary subalgebras. Continuous trace $C^*$-algebras are of Type $I_0$; in fact a $C^*$-algebra has continuous trace if and only if it is of Type $I_0$ and has Hausdorff spectrum, see below. Part of the interest of $C^*$-algebras of Type $I_0$ is that each postliminal $C^*$-algebra has a canonical composition series of Type $I_0$, whereas its continuous trace composition series are obtained using Zorn's Lemma [24; 6.2.12]. $C^*$-algebras of Type $I_0$ also arise in the study of continuity of transition probabilities [6]. We shall look at this in more detail in the next section.

Let $A$ be a $C^*$-algebra. A point $\pi_0 \in \hat{A}$ satisfies the Fell condition if there exists $a \in A^+$ such that $\pi(a)$ is a 1-dimensional projection for all $\pi$ in some neighborhood of $\pi_0$ in $\hat{A}$. We will call such points Fell points. It is easy to see that if $J$ is an ideal in $A$ and $\pi \in \hat{J} \subseteq \hat{A}$ then $\pi$ is a Fell point in $\hat{J}$ if and only if $\pi$ is a Fell point in $\hat{A}$. Elementary manipulations show that a point $\pi_0$ is a Fell point if and only if there is an abelian element $x$ such that $\pi_0(x) \neq 0$. This shows that a $C^*$-algebra is of Type $I_0$ if and only if each point of its spectrum is a Fell point. Because of this it is convenient to refer to such algebras as Fell $C^*$-algebras, and we shall use this name from now on. For any $C^*$-algebra $A$, the set $F$ of all Fell points in $\hat{A}$ is clearly open. If $A$ is postliminal then it follows from [24; 6.2.11] that $F$ is dense in $\hat{A}$. The corresponding essential closed ideal $J$ of $A$ is the largest Fell ideal of $A$. Since $J$ is liminal $\text{Prim}(J)$ is a $T_1$-space, from which it follows that if $\pi$ is a Fell point in $\hat{A}$ then ker $\pi$ is a minimal primitive ideal of $A$ (see also the proof of Lemma 3.1).

Let $A$ be a $C^*$-algebra and let $P$ be the set of $a \in A^+$ such that the function

$$\pi \mapsto \text{Tr} \pi(a)$$

is finite and continuous on $\hat{A}$. Then the linear span of $P$ is a two-sided ideal of $A$, which is denoted $m(A)$ [12; 4.5.2], and $A$ is said to have continuous trace if $m(A)$ is dense in $A$. A $C^*$-algebra has continuous trace if and only if its spectrum is Hausdorff and each point of the spectrum is a Fell point [12; 4.5.3-4].

The main purpose of this section is to consider possible generalizations of the notion of continuous trace. There are three natural ways to do this, and we begin by describing these, and showing that they are all equivalent.

Following [19] we say that a net $(\pi_a)$ in $\hat{A}$ is properly convergent if it is convergent and every cluster point of $(\pi_a)$ is a limit. It follows from Lemma 1.4 that $(\pi_a)$ is properly convergent in $\hat{A}$ if and only if (ker $\pi_a$) is $\tau_s$-convergent to a proper ideal in $\text{Id}(A)$. (Corollary 2.7 shows that we will need to use the $\tau_s$-topology if we want continuity of the trace functions on a dense ideal (c.f. [14], [20], [25])).

The first possible way of generalizing continuous trace is suggested by Theorem 2.4. Define $S$ to be the set of $a \in A^+$ such that
\[ \text{Tr } \pi_a(a) \to \sum_{a \in L} \text{Tr } \pi(a) < \infty \]

whenever \( (\pi_a) \) is a properly convergent net in \( \hat{A} \) with limit set \( L \). Note that \( S \) can also be defined as the set of \( a \in A^+ \) such that \( f_a(P_a) \to f_a(J) < \infty \) whenever \( (P_a) \) is a net in \( \text{Prime}(A) \) converging \( (\tau_a) \) to a proper closed ideal \( J \). If \( \hat{A} \) is Hausdorff then clearly \( S = P \).

When \( \hat{A} \) is Hausdorff, \( \text{Prime}(A) \) is equal to \( \text{Prime}^+(A) \). This suggests a second possible way of generalizing continuous trace. Define \( U \) to be the set of \( a \in A^+ \) such that \( f_a \) is finite and \( \tau\)-continuous on \( \text{Prime}^+(A) \).

At this stage it might seem that the demands of the second generalization are too ambitious. It is natural to view a C*-algebra as a C*-bundle over \( \text{Sub}(A) \) (see Introduction), so for the third generalization define \( V \) to be set of \( a \in A^+ \) such that \( f_a \) is finite and \( \tau\)-continuous on \( \text{Sub}(A) \).

Our first aim is to show that these three generalizations of continuous trace are actually all the same. We begin with a lemma which extends [20; Theorem 3].

**Lemma 3.1.** Let \( A \) be a C*-algebra and let \( a \) be an element of \( A^+ \) such that \( \text{Tr } \pi(a) \) is finite and bounded for \( \pi \in \hat{A} \). Suppose that \( \pi \in \hat{A} \) and that \( \pi(a) > 0 \). Let \( J \) be a primal ideal contained in \( \text{ker } \pi \). Then \( \{\pi\} \) is open in \( (A/J)^\wedge \).

**Proof.** Note first that if \( \sigma \in \hat{A} \) and \( \sigma \neq \pi \) then there exists an open neighbourhood of \( \pi \) which does not contain \( \sigma \). For, otherwise, \( \pi \in \{\sigma\}^{-} \) and since \( \pi \) (regarded as an irreducible representation of \( \sigma(A) \)) does not annihilate the compact operator \( \sigma(a) \) we have \( \pi \simeq \sigma \) [8; 1.3.4].

Now by [12; 3.5.9] there exists an open neighbourhood \( U \) of \( \pi \) in \( \hat{A} \) such that

\[ \text{Tr } \sigma(a) > \frac{1}{2} \text{Tr } \pi(a) > 0 \quad (\sigma \in U). \]

Suppose that \( \{\pi\} \) is not open in \( (A/J)^\wedge \). By the first paragraph we can find infinitely many points in \( (A/J)^\wedge \cap U \), so by (1), \( f_a(J) = \infty \). But \( f_a \) is bounded on \( \text{Prime}(A) \), by assumption, and \( \text{Prime}(A) \) is dense in \( (\text{Prime}(A), \tau_w) \) [3; 3.1] so Theorem 2.3 implies that \( f_a \) is bounded on \( \text{Prime}(A) \). This gives a contradiction, so \( \{\pi\} \) is open in \( (A/J)^\wedge \).

**Proposition 3.2.** Let \( A \) be a C*-algebra and let \( P, S, U \) and \( V \) be the subsets of \( A^+ \) defined before Lemma 3.1. Then

(i) \( P \subseteq S \), and

(ii) \( S, U, \) and \( V \) are equal.

**Proof.** (i) Suppose that \( a \in P \) and that \( L \) is the set of limits of a (properly) convergent net \( (\pi_a) \) in \( \hat{A} \). Let \( \sigma \in L \). Using Theorem 2.4 we have


\[ \sum_{\pi \in L} \text{Tr} \, \pi(a) \leq \lim \inf \sum_{\pi \in L} \text{Tr} \, \pi_{\pi}(a) = \text{Tr} \, \sigma(a) < \infty. \]

Thus Tr \( \pi(a) = 0 \) for all \( \pi \in L \setminus \{\sigma\} \). It follows that either \( L \) is a singleton or Tr \( \pi(a) = 0 \) for all \( \pi \in L \) and that in both cases

\[ \text{Tr} \, \pi_{\pi}(a) \to \sum_{\pi \in L} \text{Tr} \, \pi(a) < \infty. \]

Hence \( a \in S \).

(ii) It follows at once from the definition of \( U \) and the alternative definition of \( S \) that \( U \subseteq S \).

We now show that \( S \subseteq V \). Let \( a \in S \). Since \( \text{Sub}(A) \subseteq \text{Prim}(A)^s \), by [3; 4.3 b]), \( f_a \) is finite on \( \text{Sub}(A) \). Let \( J \in \text{Sub}(A) \) and let \( \epsilon > 0 \) be given. Since \( a \in S \), there is a \( \tau_s \)-neighbourhood \( M \) of \( J \) such \( |f_a(P) - f_a(J)| < \epsilon/2 \) for all \( P \in \text{Prim}(A) \cap M \). Let \( K \in \text{Sub}(A) \cap M \) and let

\[ N = \{ I \in \text{Id}(A) : f_a(I) > f_a(K) - \epsilon/2 \}. \]

\( N \) is \( \tau_w \)-open, by Theorem 2.3, so \( M \cap N \) is a \( \tau_s \)-neighbourhood of \( K \). Since \( \text{Sub}(A) \subseteq \text{Prim}(A)^s \), \( \text{Prim}(A) \cap M \cap N \) is non-empty. Let \( P \in \text{Prim}(A) \cap M \cap N \). Then

\[ f_a(K) - \epsilon/2 < f_a(P) < f_a(J) + \epsilon/2. \]

Hence \( f_a(K) < f_a(J) + \epsilon \). It follows that \( f_a \) is upper semi-continuous for \( \tau_s \) on \( \text{Sub}(A) \). Since \( f_a \) is also lower semi-continuous for \( \tau_s \) (from Theorem 2.3), \( a \in V \).

Finally, we show that \( V \subseteq U \). Let \( a \in V \). Since \( f_a \) is finite on \( \text{MinPrim}(A) \), \( f_a \) is finite on \( \text{Primal}'(A) \). Suppose that \( J_a \rightarrow J \) (\( \tau_s \)) in \( \text{Primal}'(A) \). It suffices to show that some subnet of \( (f_a(J_a)) \) is convergent to \( f_a(J) \).

For each \( \alpha \) let \( I_{\alpha} \) be a minimal primal ideal contained in \( J_{\alpha} \). We may assume, by \( \tau_s \)-compactness and by passing to a subnet if necessary, that \( (I_{\alpha}) \) is \( \tau_s \)-convergent in \( \text{Primal}(A) \) with limit \( I \), say. Since \( I_{\alpha} \subseteq \text{I}_{\alpha} \) for each \( \alpha \), we have \( I \subseteq J \) and, in particular, \( I \neq A \).

For each \( \alpha \) let \( F_{\alpha} = \{ \pi \in (A/I_{\alpha})^\wedge : \pi(a) \neq 0 \} \). Since \( f_a(I_{\alpha}) \) is finite, the proof of Lemma 3.1 (with \( J \) replaced by \( I_{\alpha} \)) shows that \( \{ \pi \} \) is open in \( (A/I_{\alpha})^\wedge \) for each \( \pi \in F_{\alpha} \).

Let \( G_{\alpha} = \{ \pi \in F_{\alpha} : \pi(J_{\alpha}) \neq \{0\} \} \) and let \( H_{\alpha} = F_{\alpha} \setminus G_{\alpha} = F_{\alpha} \cap (A/J_{\alpha})^\wedge \). Let \( K_{\alpha} = \cap \{ \ker \pi : \pi \in G_{\alpha} \} \) (and if \( G_{\alpha} \) is empty set \( K_{\alpha} = A \)).

Suppose that \( \pi \in H_{\alpha} \). Since \( \{ \pi \} \) is open in \( (A/I_{\alpha})^\wedge \) and \( \pi \notin G_{\alpha} \), we have \( \pi \notin \overline{G_{\alpha}} \).

Thus, if \( G_{\alpha} \) is non-empty, \( H_{\alpha} \cap (A/K_{\alpha})^\wedge \) is empty. Hence

\[ f_a(I_{\alpha}) = \sum_{\pi \in F_{\alpha}} \text{Tr} \, \pi(a) = \sum_{\pi \in G_{\alpha}} \text{Tr} \, \pi(a) + \sum_{\pi \in H_{\alpha}} \text{Tr} \, \pi(a) = f_a(K_{\alpha}) + f_a(J_{\alpha}). \]
We may assume, by $\tau_s$-compactness and by passing to a subnet if necessary, that $(K_\alpha)$ is $\tau_s$-convergent in $\text{Id}(A)$ with limit $K$, say. Since $I_\alpha \subseteq K_\alpha$ for all $\alpha, I \subseteq K$.

Let $\pi \in (A/I)^\wedge$. Then there exists a subnet $(I_\beta)$ of $(I_\alpha)$ and $\pi_\beta \in (A/I_\beta)^\wedge$ such that $\pi_\beta \rightarrow \pi$ (and hence ker $\pi_\beta \rightarrow$ ker $\pi$ in Prim$(A)$).

Suppose, further that $\pi(a) \neq 0$. Then eventually $\pi_\beta(a) \neq 0$, that is, $\pi_\beta \in F_\beta$. Hence there is a subnet $(I_\gamma)$ of $(I_\beta)$ such that either $\pi_\gamma \in G_\gamma$ for all $\gamma$ or $\pi_\gamma \in H_\gamma$ for all $\gamma$. By Lemma 1.2, either ker $\pi \supseteq K$ or ker $\pi \supseteq J$. Hence

\[ f_a(I) = \sum_{\pi \in (A/I)^\wedge} \text{Tr} \pi(a) \leq f_a(J) + f_a(K). \]

Note that since $a \in V$, $f_a(I_a) \rightarrow f_a(I)$.

From (1), (2), and Theorem 2.3 we have

\[ f_a(J) + f_a(K) \geq f_a(I) = \lim f_a(I_\alpha) = \lim (f_a(K_\alpha) + f_a(J_\alpha)) = \lim \sup f_a(K_\alpha) + f_a(J_\alpha) \]

\[ \geq \lim \inf f_a(K_\alpha) + \lim \sup f_a(J_\alpha) \geq \lim \inf f_a(K_\alpha) + \lim \inf f_a(J_\alpha) \geq f_a(K) + f_a(J). \]

Hence $\lim \sup f_a(J_\alpha) = \lim \inf f_a(J_\alpha) = f_a(J)$, as required.

From now on we prefer to work with $U$ rather than $S$ or $V$. Clearly $U + U \subseteq U$ and if $a \in A$ and $aa^* \in U$ then $a^*a \in U$. Also, if $a \in U$, $b \in A$ and $0 \leq b \leq a$ then by applying Theorem 2.3 to $b$ and $a - b$ we see that $b \in U$ (c.f. [12; 4.4.2]). It follows from [12; 4.5.1] that the linear span $X(A)$ of $U$ is a two-sided ideal of $A$ and $X(A)^+ = U$. Our aim is to characterize those algebras for which $X(A)$ is dense in $A$. First we show that Fell C*-algebras belong to this class. This follows immediately from the next Proposition, see Theorem 3.6.

For $\pi, \sigma \in \hat{A}$ let $\pi \sim \sigma$ if $\pi$ and $\sigma$ cannot be separated by disjoint open sets in $\hat{A}$. Set $S(\pi) = \{ \sigma \in \hat{A} : \sigma \sim \pi \}$.

**Proposition 3.3.** Let $A$ be a C*-algebra and let $a \in A^+$ be an abelian element of $A$.

(i) Suppose that $\pi_1$ and $\pi_2$ are distinct elements of $\hat{A}$ such that $\pi_1 \sim \pi_2$. Then at least one of $\pi_1(a)$ and $\pi_2(a)$ is zero.

(ii) $f_a(I) = \|a + I\| (I \in \text{Primal}(A))$.

(iii) $f_a$ is finite and $\tau_s$-continuous on Primal$(A)$. Hence $a \in U$.

**Proof.** (i) There is a net $(\pi_\alpha)$ in $\hat{A}$ such that $\pi_\alpha \rightarrow \pi_1$ and $\pi_\alpha \rightarrow \pi_2$. By passing to a subnet if necessary we may suppose that (ker $\pi_\alpha$) is $\tau_s$-convergent to some ideal $J \in \text{Id}(A)$. Then $J \subseteq \ker \pi_i (i = 1, 2)$. There exists $\pi \in (A/J)^\wedge$ such that $\|a + J\| = \|\pi(a)\|$. Then
\[ \|\pi(a)\| = \|a + J\| = \lim \|\pi_\sigma(a)\| = \lim \text{Tr} \pi_\sigma(a) \]
\[ \geq \sum_{\sigma \in (A/J)^\wedge} \text{Tr} \sigma(a) \quad \text{(by Theorem 2.4)} \]
\[ = \sum_{\sigma \in (A/J)^\wedge} \|\sigma(a)\|. \]

Thus if \( \sigma \in (A/J)^\wedge \) and \( \sigma \not= \pi \) then \( \sigma(a) = 0 \).

(ii) Let \( I \in \text{Primal}(A) \). If \( a \in I \) then
\[ f_a(I) = 0 = \|a + I\|. \]

Suppose that \( a \not\in I \). By (i) there exists a unique \( \pi \in (A/I)^\wedge \) such that \( \pi(a) \not= 0 \). Then
\[ f_a(I) = \text{Tr} \pi(a) = \|\pi(a)\| = \|a + I\|. \]

(iii) This follows from (ii).

**Corollary 3.4.** Let \( A \) be a C*-algebra and suppose that \( \pi \) is a Fell point of \( \hat{A} \). Then \( \pi \) has a Hausdorff open neighbourhood in \( \hat{A} \).

**Proof.** There exists an abelian element \( a \in A^+ \) such that \( \pi(a) \not= 0 \). Then
\[ \{\sigma \in \hat{A} : \sigma(a) \not= 0\} \]
is an open neighbourhood of \( \pi [12; 3.3.2] \) and is Hausdorff by 3.3 (i).

Our aim now is to show that if \( X(A) \) is dense in \( A \) then \( A \) is a Fell C*-algebra.

**Proposition 3.5.** Let \( A \) be a C*-algebra and let \( \pi \in \hat{A} \). Suppose that \( \pi(X(A)) \not= \{0\} \). Then \( \{\pi\} \) is open in \( S(\pi) \).

**Proof.** Suppose that \( \{\pi\} \) is not open in \( S(\pi) \). Then there exists a net \( (\pi_\alpha) \) in \( S(\pi) \) \( \setminus \{\pi\} \) such that \( \pi_\alpha \rightarrow \pi \). For each \( \alpha, J_\alpha = \ker \pi \cap \ker \pi_\alpha \) is primal (because \( \pi_\alpha \in S(\pi) \)). By the \( \tau_s \)-compactness of \( \text{Primal}(A), (J_\alpha) \) has a subnet \( (J_\beta) \) which is \( \tau_s \)-convergent to some \( J \in \text{Primal}(A) \), and \( J \subseteq \ker \pi_\alpha \) using Lemma 1.2.

By Lemma 3.1 there exists an open subset \( V \) of \( \hat{A} \) such that
\[ V \cap (A/J)^\wedge = \{\pi\}. \]

Let \( K \) be the closed, two-sided ideal of \( A \) such that \( \hat{K} = V \). Since \( \pi(K) \not= \{0\} \) and \( \pi(A) \supseteq \text{LC}(H_\pi) \) we have \( \pi(K) \supseteq \text{LC}(H_\pi) \). Let \( a \in U \) such that \( \pi(a) \not= 0 \). Then there exists \( b \in K^+ \) such that \( \pi(b) = \pi(a) \). Let \( c = bab \in K^+ \cap X(A) = K \cap U \). Then \( \pi(c) = \pi(a)^3 \not= 0 \) (since \( \pi(a) \geq 0 \)) and so, since \( c \in U \), \( 0 < \text{Tr} \pi(c) < \infty \). We have
\[ \text{Tr} \pi(c) = f_c(J) \quad \text{(since \( c \in K \))} \]
\[ = \lim f_c(J_\beta) \quad \text{(since \( c \in U \))} \]
\[ \geq \lim \sup (\text{Tr} \pi_\beta(c) + \text{Tr} \pi(c)) \]
\[ \geq 2 \text{Tr} \pi(c) \quad \text{(by [12; 3.5.9])}. \]
This contradiction shows that \( \{ \pi \} \) is open in \( S(\pi) \).

**Theorem 3.6.** Let \( A \) be a C*-algebra and let \( \pi \in \hat{A} \). Then the following conditions are equivalent:

(i) \( \pi \) is a Fell point,

(ii) \( X(A) \not\subset \ker \pi \).

**Proof.** (i) \( \Rightarrow \) (ii) If \( \pi \) is a Fell point then there exists an abelian element \( a \) such that \( \pi(a) \neq 0 \). Proposition 3.3 (iii) implies that \( a \in X(A) \). Hence \( X(A) \not\subset \ker \pi \).

(ii) \( \Rightarrow \) (i) Suppose that (ii) holds. By \([12; 4.4.2(ii)]\) it suffices to find \( c \in A^+ \) such that \( \pi(c) \neq 0 \) and such that the function \( \sigma \rightarrow \Tr \sigma(c) (\sigma \in \hat{A}) \) is finite and continuous at \( \pi \).

By Proposition 3.5 there is a closed two-sided ideal \( K \) of \( A \) such that \( S(\pi) \cap \hat{K} = \{ \pi \} \). Choose \( a \in U \) such that \( \pi(a) \neq 0 \), and choose \( b \in K^+ \) such that \( \pi(a) = \pi(b) \). Then \( c = bab \in K \cap U \) and \( \pi(c) \neq 0 \). Since \( c \in U \), \( \Tr \sigma(c) < \infty \) for all \( \sigma \in \hat{A} \).

Suppose that \( \pi_x \rightarrow \pi \) in \( \hat{A} \) and let \( (\pi_\rho) \) be any subnet of \( (\pi_x) \). By the \( \tau_s \)-compactness of Primal(\( A \)) there exists a subnet \( (\ker \pi_\gamma) \) of \( (\ker \pi_\rho) \) such that \( \ker \pi_\gamma \rightarrow J(\tau_s) \) for some primal ideal \( J \) of \( A \). Lemma 1.2 shows that \( J \subseteq \ker \pi \). Then

\[
\lim \Tr \pi_\gamma(c) = f_c(J) \quad (\text{since } c \in U)
\]

\[
= \Tr \pi(c) \quad (\text{since } c \in K).
\]

Since the subnet \( (\pi_\rho) \) was arbitrary, \( \Tr \pi_x(c) \rightarrow \Tr \pi(c) \).

The next corollary follows immediately from Theorem 3.6

**Corollary 3.7.** Let \( A \) be a C*-algebra. Then the following are equivalent:

(i) \( A \) is a Fell C*-algebra,

(ii) \( X(A) \) is dense in \( A \).

Corollary 3.7 shows that Fell C*-algebras have a right to be thought of as C*-algebras of "generalized continuous trace". But there is already a class of algebras claiming this name for themselves, so we now investigate the connections between these two classes.

A C*-algebra \( A \) is said to have *generalized continuous trace* (GCT or, sometimes, GTC) if the continuous trace ideal \( m(A/I) \) is non-zero for every non-zero quotient \( A/I \) of \( A \) \([12, 4.7.12]\). It is easy to show that GCT algebras are liminal \([12; 4.7.12]\). If \( A \) is liminal and \( \hat{A} \) is compact or if every irreducible representation of \( A \) is finite dimensional then \( A \) has GCT \([13; \S 1], [11; Proposition 13], [1; Example 4.5]\). It is easy, therefore, to give examples of GCT algebras which are not Fell algebras: for example the algebra of sequences of two-by-two complex matrices which tend to a scalar matrix at infinity.
We now give an example of a Fell C*-algebra which does not have GCT (which shows that the two classes don't have much connection). To do this we use the following characterization [12; 4.7.12]: a liminal C*-algebra \( \mathcal{A} \) has GCT if and only if \( \mathcal{A} \) is quasi-separated (a topological space \( T \) is quasi-separated if whenever \( F \) is a non-empty closed subset of \( T \) the interior of the set of separated points of \( F \) is non-empty (or, equivalently, dense)). It is, therefore, sufficient for our example to produce a Fell C*-algebra in which the set of non-separated points is dense in the spectrum.

**Example 3.8.** Let \( H \) be a separable, infinite dimensional Hilbert space and let \(( e_i ) (1 \leq i < \infty )\) be an orthonormal basis for \( H \). For \( i, j \in \{ 1, 2, \ldots \} \) let \( T_{ij} \) be the rank-one operator such that \( T_{ij} e_i = e_j \). For each \( n \in \{ 1, 2, \ldots \} \) let \( C_n \) be the C*-subalgebra of \( \text{LC}(H) \) generated by \( T_{ii} (1 \leq i < \infty ) \) and \( T_{ij} (1 \leq i, j \leq n) \). Then \( C_n \cong M_n(\mathbb{C}) \otimes c_0 \). Let \( B \) be the C*-algebra of continuous functions from \([0, 1]\) into \( \text{LC}(H) \). If \( r \) is a rational number in \((0, 1)\) let \( d(r) \) be the denominator when \( r \) is written as a fraction in its lowest terms and set \( d(0) = d(1) = 1 \). Let \( A \) be the C*-subalgebra of \( B \) consisting of those functions \( f \in B \) such that \( f(r) \in C_{d(r)} \) for each rational \( r \in [0, 1] \).

If \( s \in [0, 1] \) let \( I_s \) be the closed, two-sided ideal of \( A \) consisting of those \( f \in A \) such that \( f(s) = 0 \). If \( s \) is irrational then for each \( n \geq 1 \) there exists a neighbourhood of \( s \) in which each rational number \( r \) satisfies \( d(r) \geq n \). It follows that \( \{ f(s) : f \in A \} \) contains the norm-closure of \( \bigcup_{n=1}^{\infty} C_n \) and hence is equal to \( \text{LC}(H) \).

Thus \( I_s \) is primitive. Otherwise \( A/I_s \cong C_{d(s)} \), so \( \text{Prim}(A/I_s) \) is a countably infinite, discrete space. If \( (s_n) \) is a sequence of irrational numbers in \((0, 1)\) converging to a rational number \( r \) then \( I_{s_n} \rightarrow I_r(\tau_r) \), so \( I_{s_n} \) converges \((\tau_r)\) to each \( P \in \text{Prim}(A/I_r) \), by Lemma 1.1. Thus if \( P \in \text{Prim}(A/I_r) \) \( P \) is not a separated point of \( \text{Prim}(A) \). Since the ideals \( I_r \), for \( r \) rational, have zero intersection it follows that the set of non-separated points is dense in \( \text{Prim}(A) \). Since \( A \) is obviously liminal, \( \text{Prim}(A) \) is homeomorphic to \( \hat{A} \), so \( A \) does not have GCT.

For each \( i \in \{ 1, 2, \ldots \} \) the constant function \( f_i : [0, 1] \rightarrow T_{ii} \) is clearly an abelian element of \( A \). If \( \pi \in \hat{A} \) there exists \( i \) such that \( \pi (f_i) \neq 0 \). Hence the ideal generated by these elements is dense in \( A \). Hence \( A \) is a Fell C*-algebra which does not have GCT.

An example has previously been given of a separable, liminal C*-algebra for which the set of non-separated points is dense in the spectrum [13], but it is more complicated to describe, and it is not easy to see whether it is a Fell C*-algebra.

We now continue to investigate separated points in the spectra of Fell C*-algebras. For \( I, J \in \text{Primal}'(A) \) let \( I \sim J \) if \( I \) and \( J \) cannot be separated by disjoint \( \tau_w \)-open sets in \( \text{Primal}'(A) \).
LEMMA 3.9. Let $A$ be a C*-algebra and let $I, J \in \text{Prim}^\prime(A)$. Then $I \sim J$ if and only if $I \cap J$ is primal.

PROOF. The ideal $I \cap J$ is primal if and only if there exists a net $(P_\alpha)$ in $\text{Prim}(A)$ such that $P_\alpha \to I(\tau_w)$ and $P_\alpha \to J(\tau_w)$ [5; 3.2], [3; 3.2]. If such a net exists then clearly $I \sim J$. Conversely if $I \sim J$ then the denseness of $\text{Prim}(A)$ in $(\text{Prim}(A), \tau_w)$ [3; 3.1] implies that such a net exists.

The next lemma is an immediate consequence of the definition of $\sim$.

LEMMA 3.10. Let $A$ be a C*-algebra Suppose that $(P_\alpha)$ and $(Q_\alpha)$ are nets in $\text{Prim}(A)$ with $P_\alpha \to I(\tau_w)$ and $Q_\alpha \to J(\tau_w)$ for some $I, J \in \text{Prim}^\prime(A)$. If $P_\alpha \sim Q_\alpha$ for each $\alpha$ then $I \sim J$.

LEMMA 3.11. Let $A$ be a C*-algebra. Let $(P_\alpha)$ be a net in $\text{Prim}(A)$ converging $(\tau_w)$ to a primal ideal $I$ and suppose that $(Q_\alpha)$ is a net in $\text{Prim}(A)$, with $Q_\alpha \sim P_\alpha$ for each $\alpha$.

(i) If $I$ is minimal primal and $(Q_\alpha)$ converges $(\tau_w)$ to a primal ideal $J$ then $J \supseteq I$.

(ii) If $I$ is a closed, separated point of $\text{Prim}(A)$, and $\text{Prim}(A)$ is compact then $Q_\alpha \to I$.

PROOF. (i) This follows from Lemmas 3.9 and 3.10.

(ii) Let $(Q_\beta)$ be any subnet of $(Q_\alpha)$. By the compactness of $\text{Prim}(A)$, $(Q_\beta)$ has a convergent subnet $(Q_\gamma)$, converging to some $Q \in \text{Prim}(A)$. Since $I$ is minimal primal [3; 4.5] part (i) implies that $Q \supseteq I$. But $I$ is a closed point of $\text{Prim}(A)$, hence a maximal ideal. Hence $Q = I$, and $Q_\alpha \to I$.

THEOREM 3.12. Let $A$ be a C*-algebra. Suppose that $\text{Prim}(A)$ is compact and that each point of $\text{Prim}(A)$ has a Hausdorff neighbourhood. Let $X$ be the set of separated points of $\text{Prim}(A)$. Then $\text{Prim}(A)$ is a $T_1$-space and $X$ is a dense, open subset of $\text{Prim}(A)$.

PROOF. Since each point has a Hausdorff neighbourhood, $\text{Prim}(A)$ is a $T_1$-space. Suppose that $P$ is a separated point in $\text{Prim}(A)$. Let $(P_\alpha), (Q_\alpha)$ be nets in $\text{Prim}(A)$, with $P_\alpha \sim Q_\alpha$ for each $\alpha$, and $P_\alpha \to P$. By Lemma 3.11 (ii) $Q_\alpha \to P$ also. But $P$ has a Hausdorff neighbourhood, by assumption, so eventually $P_\alpha = Q_\alpha$. Hence $P_\alpha$ is eventually a separated point. It follows that $X$ is open.

Let $P \in \text{Prim}(A)$ and let $I$ be a minimal primal ideal such that $I \subseteq P$. By [3; 3.1] there exists a net $(P_\alpha)$ in $\text{Prim}(A)$ such that $P_\alpha \to I(\tau_w)$. Let $(Q_\alpha)$ be a net in $\text{Prim}(A)$ with $Q_\alpha \sim P_\alpha$ for each $\alpha$. By the $\tau_w$-compactness of $\text{Prim}(A)$ each subnet $(Q_\beta)$ of $(Q_\alpha)$ has a subnet $(Q_\gamma)$ converging $(\tau_w)$ to some $Q$ in $\text{Prim}(A)$, and $Q \supseteq I$, by Lemma 3.11 (i). Since $Q$ has a Hausdorff neighbourhood and $(P_\gamma)$ also converges $(\tau_w)$ to $Q$ it follows that eventually $P_\gamma = Q_\gamma$. Hence $P_\alpha$ is eventually a separated point. Since $P_\alpha \to P$ in $\text{Prim}(A)$, $X$ is dense in $\text{Prim}(A)$.
Remarks: (i) If $A$ is a Fell C*-algebra with compact spectrum then Corollary 3.4 shows that the theorem above applies to $A$. In this case, however the denseness of $X$ in $\text{Prim}(A)$ already follows from the fact that $A$ is a GCT C*-algebra.

(ii) The first paragraph of the proof above actually shows that if $A$ is a C*-algebra with $\text{Prim}(A)$ compact and $T_1$ then the interior of the set of separated points of $\text{Prim}(A)$ is precisely the set of separated points which have a Hausdorff neighbourhood.

4. Points of continuity for transition probabilities.

In this section we extend the results of [6] on continuity questions for the transition probability mapping $T$ (see Section 1) and its restriction $T_0$ to $R(A)$ (the subset of $P(A) \times P(A)$ consisting of those pairs $(\phi, \psi)$ such that $\pi_\phi$ is equivalent to $\pi_\psi$). We equip $P(A) \times P(A)$ with the product $w^*$-topology. The restriction of this topology to $R(A)$ is called the product topology and is denoted $\tau_p$. We shall also be concerned with the quotient topology $\tau_q$ on $R(A)$. This is obtained as follows (see [6]). We define $G(A)$ to be the set of extreme points of the closed unit ball of the Banach dual $A^*$. There is a mapping $q$ of $(G(A), w^*)$ onto $R(A)$ given by

$$q(\phi) = (|\phi|, |\phi^*|) \quad (\phi \in G(A)).$$

The topology $\tau_q$ on $R(A)$ is defined to be the quotient topology induced by $q$ from $(G(A), w^*)$.

In the locally convex, separated space $A^* \times A^*$ the set $P(A) \times P(A)$ is equal to $\partial_e(S(A) \times S(A))$, so it follows from [12; B14] that $P(A) \times P(A)$ is a Baire space. Since $T$ is upper semi-continuous on $P(A) \times P(A)$, the set of points of continuity is a dense $G_\delta$-set [12; B18]. In Theorem 4.1 we describe this set and give an alternative proof of its density. This result extends the observation of [6; p. 8, Remark 2] which was described in Section 1.

Recall that a C*-algebra is said to be elementary if it is isomorphic to the algebra of compact operators $LC(H)$ on some Hilbert space $H$.

**Theorem 4.1.** Let $A$ be a C*-algebra and let $T: P(A) \times P(A) \to [0, 1]$ be defined by

$$T(\phi, \psi) = \langle \phi, \psi \rangle \quad (\phi, \psi \in P(A)).$$

Let

$$E_1 = \{ (\phi, \psi) \in P(A) \times P(A): T(\phi, \psi) = 0 \}$$

and let $E_2$ be the subset of $P(A) \times P(A)$ consisting of those pairs $(\phi, \psi)$ such that there exists a closed ideal $J$ of $A$ such that $J$ is an elementary C*-algebra and
\[ \hat{J} = \{ \pi_{\phi} \}. \text{ Then } E = E_1 \cup E_2 \text{ is dense in } P(A) \times P(A) \text{ and is precisely the set of points at which } T \text{ is continuous.} \]

**Proof.** Let \((\phi, \psi) \in E\). If \((\phi, \psi) \in E_1\) then \(T\) is continuous at \((\phi, \psi)\), [6; p. 8]. Suppose that \((\phi, \psi) \in E_2 \setminus E_1\). Then there exists an elementary closed ideal \(J\) of \(A\) such that \(\hat{J} = \{ \pi_{\phi} \} = \{ \pi_{\psi} \}\). Suppose that \(\phi_{\alpha} \to \phi\) and \(\psi_{\alpha} \to \psi\) in \(P(A)\) in the \(w^*\)-topology. Then, eventually, \(\phi_{\alpha} | J, \psi_{\alpha} | J \in P(J)\) and so

\[ \langle \phi_{\alpha}, \psi_{\alpha} \rangle = \langle \phi_{\alpha} | J, \psi_{\alpha} | J \rangle \to \langle \phi | J, \psi | J \rangle = \langle \phi, \psi \rangle \]

by [6; p. 8, Remark 1]. Thus \(T\) is continuous at \((\phi, \psi)\).

Now suppose that \((\phi, \psi) \in (P(A) \times P(A)) \setminus E\). Note that \(\langle \phi, \psi \rangle \neq 0\) and that \(\phi\) and \(\psi\) are equivalent pure states. We shall complete the proof (and confirm the density of \(E\)) by showing that \((\phi, \psi)\) is the limit of a net in \(E_1\). There are two cases to consider.

**Case 1.** Suppose that \(\{ \pi_{\phi} \}\) is not open in \(\hat{A}\). Let \(\Gamma\) be a base of open neighbourhoods of 0 in \(A^*\). Let \(N \in \Gamma\). Then the canonical image of \((\phi + N) \cap P(A)\) in \(\hat{A}\) is an open neighbourhood of \(\pi_\phi\) in \(\hat{A}\) and so there exists \(\phi_N \in (\phi + N) \cap P(A)\) such that \(\pi_N\) is not equivalent to \(\phi\) or \(\psi\) and hence such that \(\langle \phi_N, \psi \rangle = 0\). Directing \(\Gamma\) by reverse inclusion in the usual way, \((\phi_N, \psi) \to (\phi, \psi)\) in \(P(A) \times P(A)\).

**Case 2.** Suppose that \(\{ \pi_{\phi} \}\) is open in \(\hat{A}\). Then there is a closed ideal \(J\) of \(A\) such that \(\{ \pi_{\phi} \} = \hat{J}\). Since \((\phi, \psi) \notin E\) we must (in the absence of a positive resolution of Naimark’s conjecture) consider the possibility that \(J\) is a simple, antiliminal, inseparable C*-algebra. There exists a unit vector \(\eta \in H_\phi\) such that \(\psi = \omega_\eta \circ \pi_\phi\). Since \(\pi_\phi(A)\) is a prime C*-algebra and \(\pi_\phi(J) \cap LC(H_\phi) = \{0\}\), it follows that \(\pi_\phi(A) \cap LC(H_\phi) = \{0\}\). Let \(\rho\) be the unique state of \(B = \pi_\phi(A) + LC(H_\phi)\) which is zero on \(LC(H_\phi)\) and agrees with \(\omega_\eta\) on \(\pi_\phi(A)\). By Glimm’s vector state theorem [15; Theorem 2], [16; Lemma 9] there is a net \((\eta_\alpha)\) of unit vectors of \(H_\phi\) such that \(\omega_{\eta_\alpha} | B \to \rho\). Then \(\langle \eta_\alpha, \xi_\phi \rangle \to 0\) and so eventually \(\eta_\alpha - \langle \eta_\alpha, \xi_\phi \rangle \xi_\phi\) can be normalised to a unit vector, \(\zeta_\alpha\) say. It follows that

\[ \langle \phi, \omega_{\eta_\alpha} \circ \pi_\phi \rangle = |\langle \xi_\phi, \zeta_\alpha \rangle|^2 = 0 \]

and \((\phi, \omega_{\eta_\alpha} \circ \pi_\phi) \to (\phi, \psi)\) in \(P(A) \times P(A)\).

Next we give a localized version of the global continuity result for the mapping \(T_0\) on \(R(A)\) [6; Theorem 2.3]. The methods are partly similar to those in [6]. However, some extra arguments are required and we also replace the method of [6; p. 7] by using, and then developing further, a technique of Glimm [16].

**Theorem 4.2.** Let \(A\) be a C*-algebra and let \(T_0 : R(A) \to [0, 1]\) be defined by

\[ T_0(\phi, \psi) = \langle \phi, \psi \rangle \quad ((\phi, \psi) \in R(A)). \]

Let \((\phi, \psi) \in R(A)\). The following conditions are equivalent:
(i) $T_0$ is continuous at $(\phi, \psi)$ for the product topology on $R(A)$,
(ii) either $\langle \phi, \psi \rangle = 0$ or $\pi_\phi$ is a Fell point in $\hat{A}$.

PROOF. (ii) $\Rightarrow$ (i) If $\pi_\phi$ is a Fell point in $\hat{A}$ then the argument in the proof of [6; Theorem 2.3] shows that $(\phi, \psi)$ is a point of continuity for $T_0$. If $\langle \phi, \psi \rangle = 0$ then $(\phi, \psi)$ is a point of continuity for the map $T$ of Theorem 4.1, [6; p. 8]. Hence, by restriction, $(\phi, \psi)$ is a point of continuity for $T_0$.

(i) $\Rightarrow$ (ii) Suppose that $(\phi, \psi)$ is a point of continuity for $T_0$ and that $\langle \phi, \psi \rangle = \delta \neq 0$. We shall show that $\pi_\phi$ is a Fell point in $\hat{A}$.

Write $\pi = \pi_\phi$. We begin by showing that $\{\pi\}$ is open in $S(\pi)$. Suppose, on the contrary, that $\{\pi\}$ is not open in $S(\pi)$. As before let $I$ be a base of open neighbourhoods of 0 in $A^*$. Let $N \in I$ and let $N_1 = 1/2 N$. The canonical image of $(\phi + N_1) \cap P(A)$ in $\hat{A}$ is an open neighbourhood of $\pi$ and hence contains an element $\sigma$ of $S(\pi) \setminus \{\pi\}$. Thus there exists a pure state $\phi'$ associated with $\sigma$ such that $\phi' \in \phi + N_1$. The argument in the proof of [4; Theorem 1 ((ii) $\Rightarrow$ (iii))] shows that there exists $\rho \in \hat{A}$ and unit vectors $\xi$ and $\eta$ in $H_\rho$ such that $\phi_N = \omega_\xi \circ \rho \in \phi' + N_1$, $\psi_N = \omega_\eta \circ \rho \in \psi + N_1$ and $|\langle \xi, \eta \rangle|^2 \leq \delta/2$. Hence $\phi_N \in \phi + N$ and $\psi_N \in \psi + N$. Letting $N$ run through $I$ we have $(\phi_N, \psi_N) \to (\phi, \psi)$ in the product topology on $R(A)$ but lim sup $\langle \phi_N, \psi_N \rangle \leq 1/2 \langle \phi, \psi \rangle$. This contradicts the continuity of $T_0$ at $(\phi, \psi)$ and so $\{\pi\}$ is open in $S(\pi)$.

We show next that $\pi(A) \supseteq LC(H_\alpha)$. Suppose, on the contrary, that $\pi(A) \cap LC(H_\alpha) = \{0\}$. Let $B = \pi(A) + LC(H_\alpha)$. Let $\phi'$ and $\psi'$ be the (unique) states of $B$ which annihilate $LC(H_\alpha)$ and satisfy $\phi' \circ \pi = \phi$ and $\psi' \circ \pi = \psi$. Let $\Delta$ be a base of $w^*$-open neighbourhoods of 0 in $B^*$ and let $M \in \Delta$. By Glimm's vector state space theorem [15], [16] there exists a unit vector $\xi_M$ in $H_\alpha$ such that $\omega_{\xi_M} \in B$ and $M = \pi(\alpha) \subseteq LC(H_\alpha)$. Let $E_M$ be the projection onto the linear span of $\xi_M$. Since $\psi(E_M) = 0$, $\langle \xi_M, \eta_M \rangle = 0$. Thus there exists a unit vector $\eta_M$ such that $\omega_{\eta_M} \in B$ and $|\langle \xi_M, \eta_M \rangle|^2 \leq \delta/2$. As $M$ runs through $\Delta$, $\omega_{\xi_M} \circ \pi \to \phi$ and $\omega_{\eta_M} \circ \pi \to \psi$. Since $|\langle \xi_M, \eta_M \rangle|^2 \to \delta$, this contradicts the assumed continuity of $T_0$ at $(\phi, \psi)$. Hence $\pi(A) \supseteq LC(H_\alpha)$.

Since $\{\pi\}$ is open in $S(\pi)$ there exists an open neighbourhood $U$ of $\pi$ in $\hat{A}$ such that $U \cap S(\pi) = \{\pi\}$. Let $J$ be the closed two-sided ideal of $A$ for which $U = \hat{J}$. Since $\pi(A) \supseteq LC(H_\alpha)$ and $\pi(J) \neq \{0\}$, we have $\pi(J) \supseteq LC(H_\alpha)$. Thus there exists $a \in J^+$ such that $\pi(a)$ is a rank one projection which supports the pure state $\phi$ in the representation $\pi$. Since $\pi$ is a separated point of $\hat{J}$, $\|\sigma(a^2 - a)\| \to 0$ as $\sigma \to \pi$ in $\hat{J}$. By a standard functional calculus argument, we may assume without loss of generality that $\sigma(a)$ is a projection for all $\sigma$ in some neighbourhood $V$ of $\pi$ with $V \subseteq \hat{J}$. By shrinking $V$, if necessary, we may also assume that $\sigma(a) = 0$ for all $\sigma \in V$ [12; 3.3.2].

Suppose that $\pi$ is not a Fell point. Then there exists a net $(\sigma_\alpha)$ in $V$ such that $\sigma_\alpha \to \pi$ and, for each $\alpha$, rank $(\sigma_\alpha(a)) \geq 2$. For each $\alpha$ there is an orthonormal set
\{\xi_\alpha, \zeta_\alpha\} in the Hilbert space for \sigma_\alpha such that \sigma_\alpha(a)\xi_\alpha = \xi_\alpha and \sigma_\alpha(a)\zeta_\alpha = \zeta_\alpha. Let \( x \in A \). Then \( \pi(a)\pi(x)\pi(a) = \lambda\pi(a) \) for some \( \lambda \in \mathbb{C} \). Since \( axa - \lambda a \in J, \|\sigma_\alpha(axa - \lambda a)\| \to 0 \). Hence, arguing as in [16; p. 605]

\[
\lim \langle \sigma_\alpha(x)\xi_\alpha, \xi_\alpha \rangle = \lim \langle \sigma_\alpha(axa)\xi_\alpha, \xi_\alpha \rangle = \lim \langle \sigma_\alpha(\lambda a)\xi_\alpha, \xi_\alpha \rangle = \lambda = \phi(x).
\]

Thus \( \omega_\xi \circ \sigma_\alpha \to \phi \) and similarly \( \omega_\zeta \circ \sigma_\alpha \to \phi \).

Recall that \( \bar{A} \) is \( A \) itself (if \( A \) is unital) or \( A + \mathbb{C}1 \) (if \( A \) is non-unital). There exists a unitary element \( u \in \bar{A} \) such that \( \psi = \phi(u^*u) \). For each \( \alpha \), let \( \eta_\alpha = \tilde{\sigma}_\alpha(u)\zeta_\alpha \), where \( \tilde{\sigma}_\alpha \) is the canonical extension of \( \sigma_\alpha \) to \( \bar{A} \). Then \( \omega_{\eta_\alpha} \circ \sigma_\alpha \to \psi \). Since \( T_0 \) is continuous at \( (\phi, \psi) \), \( |\langle \xi_\alpha, \tilde{\sigma}_\alpha(u)\zeta_\alpha \rangle|^2 \to |\langle \phi, \psi \rangle|^2 \). Thus \( |\langle \xi_\alpha, \tilde{\sigma}_\alpha(u)\zeta_\alpha \rangle|^2 \to \delta \neq 0 \).

On the other hand, \( \pi(a)\tilde{\pi}(u)\pi(a) = \mu \pi(a) \) for some \( \mu \in \mathbb{C} \). Since \( aua - \mu a \in J, \|\sigma_\alpha(aua - \mu a)\| \to 0 \). Hence

\[
\lim \langle \xi_\alpha, \tilde{\sigma}_\alpha(u)\zeta_\alpha \rangle = \lim \langle \xi_\alpha, \sigma_\alpha(aua)\zeta_\alpha \rangle = \lim \langle \xi_\alpha, \sigma_\alpha(\mu a)\zeta_\alpha \rangle = \mu \lim \langle \xi_\alpha, \zeta_\alpha \rangle = 0.
\]

This contradiction shows that \( \pi \) is a Fell point.

**Proposition 4.3.** Let \( A \) be a \( C^* \)-algebra and let \( u \in \bar{A} \) be a unitary element.

Define \( \Phi_u : R(A) \to R(A) \) by

\[
\Phi_u(\phi, \psi) = (\phi, \psi(u \cdot u^*)) \quad (\phi, \psi) \in R(A).
\]

Then \( \Phi_u \) is a bijection and

(i) \( \Phi_u \) is a homeomorphism for \( \tau_p \),

(ii) \( \Phi_u \) is a homeomorphism for \( \tau_q \).

**Proof.** \( \Phi_u \) is a two-sided inverse for \( \Phi_u \), so \( \Phi_u \) is a bijection. For (i) and (ii) it suffices to prove either that \( \Phi_u \) is open or that \( \Phi_u \) is continuous, since this will then also hold for the inverse map \( \Phi_u^{-1} \).

(i) Suppose that \( (\phi_x, \psi_x) \to (\phi, \psi) \) in \( R(A) \). Then \( \phi_x \to \phi, \psi_x \to \psi, \psi_x(u \cdot u^*) \to \psi(u \cdot u^*) \), all in the \( w^* \)-topology, and so \( (\phi_x, \psi_x(u \cdot u^*) \to (\phi, \psi(u \cdot u^*)) \). This shows that \( \Phi_u \) is \( \tau_p \)-continuous.

(ii) Consider the map \( q : (G(A), w^*) \to (R(A), \tau_q) \) given by \( q(\phi) = (|\phi|, |\phi|^*) \) and recall that if \( \phi = \langle \pi(.) \xi, \eta \rangle \) (where \( \pi \) is an irreducible representation and \( \xi \) and \( \eta \) are unit vectors in \( H_\pi \)) then \( |\phi| = \omega_\xi \circ \pi \) and \( |\phi|^* = \omega_\eta \circ \pi \).

Let \( U \) be a non-empty open subset of \( (R(A), \tau_q) \). Then \( q^{-1}(U) \) is an open, \( q \)-saturated subset of \( G(A) \). Let \( V = \{ \psi \in G(A) : \psi = \phi(u) \} \) for some \( \phi \in q^{-1}(U) \). Then \( V \) is an open, \( q \)-saturated subset of \( G(A) \) (note that if \( \phi, \psi \in G(A) \) then \( q(\phi) = q(\psi) \) if and only if \( \phi = \lambda \psi \) for some \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \)). Using the vector functional representation of the previous paragraph, one may check that \( q(V) = \Phi_u(U) \). This shows that \( \Phi_u \) is \( \tau_q \)-open.
Another way of viewing the proof of (ii) is to observe that $\Phi_u$ lifts to a $w^*$-homeomorphism of $G(A)$ given by $\phi \to \phi(u)(\phi \in G(A))$.

**Corollary 4.4.** Let $A$ be a $C^*$-algebra and let 
\[ i : (R(A), \tau_p) \to (R(A), \tau_q) \]
be the identity map. Let $(\phi, \psi) \in R(A)$. Then $i$ is continuous at $(\phi, \psi)$ if and only if $\pi_\phi$ is a Fell point in $\mathcal{A}$.

**Proof.** Suppose that $i$ is continuous at $(\phi, \psi)$. There exists a unitary $u \in \mathcal{A}$ such that $\psi = \phi(u^*u)$. Since $\Phi_{u^*u}$ is $\tau_p$-continuous on $R(A)$ and $\Phi_u$ is $\tau_q$-continuous on $R(A)$, Proposition 4.3, it follows that $i$ is continuous at $(\phi, \phi)$. Since the transition probability map is $\tau_q$-continuous on $R(A)$ [6; Proposition 3.2], $(\phi, \phi)$ is a point of $\tau_p$-continuity for $T_0$. Hence $\pi_\phi$ is a Fell point, by Theorem 4.2.

Conversely, suppose that $\pi_\phi$ is a Fell point. Then, by Theorem 4.2, $(\phi, \phi)$ is a point of $\tau_p$-continuity for $T_0$. By [6; 3.3], $(\phi, \phi)$ is a point of continuity for $i$. Since $\Phi_u$ is $\tau_p$-continuous on $R(A)$ and $\Phi_{u^*u}$ is $\tau_q$-continuous on $R(A)$, $i$ is continuous at $(\phi, \psi)$.

5. **Fell points in $C^*$-algebras of separated topological equivalence relations.**

The work in this section was done jointly with Mark Priest, and we would like to thank him for allowing it to appear here.

The main result of this section is that the Fell points in the spectrum of the reduced $C^*$-algebra of a separated topological equivalence relation can be characterized by the points of continuity of a certain map (Theorem 5.7), exactly as in Corollary 4.4.

Let $R$ be an equivalence relation on a set $X$, with diagonal $R^0$. Let $r, s$ be the projection maps from $R$ to $R^0$ defined by $r((x, y)) = (x, y)$ and $s((x, y)) = (y, y)((x, y) \in R)$. Then $R$ is said to be a separated topological equivalence relation if there is a second countable locally compact, Hausdorff topology $\tau_0$ on $R$ such that $r$ (or, equivalently, $s$) is a local homeomorphism. The separated topological equivalence relations are precisely the second countable $r$-discrete principal groupoids which admit Haar systems [27; I.2.7, I.2.8]. For $(x, y) \in R$ an $R^0$-strip neighbourhood of $(x, y)$ is an open neighbourhood of $(x, y)$ on which both the projection maps, $r$ and $s$, are homeomorphisms onto open subsets of $R^0$. When $R$ is a separated topological equivalence relation the $R^0$-strip neighbourhoods form a base for the topology. For $x \in X$ the equivalence class of $x$ in $X$ will be denoted $[x]$. The set \( \{(y, y) : y \in [x]\} \) is called the orbit of $(x, x)$ in $R^0$. It will be convenient to refer to the set \( \{(y, z) : y, z \in [x]\} \) as an equivalence class in $R$. A subset of $R^0$ is said to be invariant if it contains the orbit of each of its points.

Now let $R$ be a separated topological equivalence relation and let $C_c(R)$ be the
\(\ast\)-algebra of continuous complex functions on \(R\) of compact support, with involution given by

\[ \hat{f}(u, v) = \overline{f(v, u)}, \]

with pointwise addition, and with multiplication given by

\[ f \ast g(u, v) = \sum_{w \sim u} f(u, w)g(w, v) \quad f, g \in C_c(R). \]

The topology ensures that the sum has only finitely many non-zero terms. Let \(\| \cdot \|_{\text{red}}\) be the \(\ast\)-norm on \(C_c(R)\) in [27; II.2.8], [21; 6.3]. The completion of \(C_c(R)\) with respect to this norm is denoted \(C^\ast_{\text{red}}(R)\). We shall give an alternative description of \(\| \cdot \|_{\text{red}}\) after Proposition 5.2. The norm \(\| \cdot \|_{\text{red}}\) dominates the uniform norm on \(R\) so we can regard the elements of \(C^\ast_{\text{red}}(R)\) as continuous functions on \(R\) vanishing at infinity [27; II.4.2].

When \(R\) is a separated topological equivalence relation \(R^0\) is a clopen subset of \(R\) [27; I.2.8], so the \(\ast\)-algebra \(C_0(R^0)\) is an abelian subalgebra of \(A = C^\ast_{\text{red}}(R)\). In fact it is a Cartan (or diagonal) subalgebra of \(A\) [28], [18], [21]. (The definition of “Cartan subalgebra” given in [27; II.4.13] is slightly different, and is no longer used.) For each \(x \in X\), \((x, x)\) can be viewed as a pure state of \(C_0(R^0)\) (given by evaluation at \((x, x)\)), and \((x, x)\) has a unique extension to a pure state \(\phi_x\) on \(A\) (which is also given by evaluation at \((x, x)\)).

Recall that if \(\phi\) is a state on a \(\ast\)-algebra \(A\) then the left kernel of \(\phi\) is the closed left ideal

\[ L_\phi = \{ a \in A : \phi(a^\ast a) = 0 \}. \]

For \(a \in A\) let \(\xi_a\) denote the image of \(a\) in the quotient space \(A/L_\phi\). Then \(<\xi_a, \xi_b> = \phi(b^\ast a)\) is a sesquilinear form on \(A/L_\phi\) defining a pre-Hilbert space structure. Let \(H_\phi\) denote the completed Hilbert space. For \(a, b \in A\) define \(\pi_\phi(a)\xi_b = \xi_{ab}\). Then \((\pi_\phi, H_\phi)\) is the GNS representation corresponding to \(\phi\).

**Lemma 5.1.** Let \(R\) be a separated topological equivalence on a set \(X\). Let \(x \in X\) and for each point \(y \in [x]\) let \(f_y\) be a function in \(C_c(R)\) whose support is contained in an \(R^0\)-strip neighbourhood of \((y, x)\) and such that \(f_y((y, x)) = 1\). Then the set

\[ B = \{ \xi_{f_y} : y \in [x] \} \]

is an orthonormal basis for \(H_{\phi_x}\).

**Proof.** Let \(f_y \in B\). Then

\[ <\xi_{f_y}, \xi_{f_y}> = \phi_x((f_y)^\ast f_y) = ((f_y)^\ast f_y)(x, x) \]

\[ = \sum_{w \sim x} |f_y(w, x)|^2 = |f_y(y, x)|^2 = 1. \]
If \( f_z \) is a different element of \( B \) then

\[
\langle \xi_{f_y}, \xi_{f_x} \rangle = \phi_x((f_z)^* f_y) = ((f_z)^* f_y)(x, x) = \sum_{w \sim x} f_z(w, x) f_y(w, x) = 0
\]

since \( \text{supp} f_y \cap \text{supp} f_z \cap s^{-1}((x, x)) = \emptyset \). Hence the set \( B \) is orthonormal.

To show that the linear span of \( B \) is dense in \( H_{\phi_x} \), suppose that \( a \) is a function in \( C_c(R) \). Then \( a \in L_{\phi_x} \) if and only if \( a|_{s^{-1}((x, x))} = 0 \). Indeed

\[
a^* a(x, x) = \sum_{w \sim x} |a(w, x)|^2
\]

and so \( a^* a(x, x) = 0 \) if and only if \( a|_{s^{-1}((x, x))} = 0 \). The support of \( a \) is compact so there exist only finitely many distinct elements \( y(1), \ldots, y(n) \) in \( X \) for which \( a(y(i), x) \neq 0 \) \((1 \leq i \leq n)\). Hence

\[
\left( a - \sum_{i=1}^{n} a(y(i), x) f_{y(i)} \right)|_{s^{-1}((x, x))} = 0
\]

and we have that

\[
\xi_a = \sum_{i=1}^{n} a(y(i), x) \xi_{f_{y(i)}},
\]

by the preceding observation. Hence \( B \) spans \( C_c(R)/L_{\phi_x} \) which is dense in \( H_{\phi_x} \).

For each \( x \in X \) let \( H_{[x]} \) be a Hilbert space of dimension equal to the cardinality of \( [x] \). Let \( \{e_y \mid y \in [x]\} \) be an orthonormal basis for \( H_{[x]} \). Then Lemma 5.1 shows that the map

\[
e_y \rightarrow \xi_{f_y} \quad (y \in [x])
\]

extends to an unitary operator \( U \) from \( H_{[x]} \) to \( H_{\phi_x} \). Let \( \pi_{[x]} \) be the irreducible representation of \( A \) on \( H_{[x]} \) defined by

\[
\pi_{[x]}(a) = U^* \pi_{\phi_x}(a) U.
\]

**Proposition 5.2.** Let \( R \) be a separated topological equivalence relation on a set \( X \) and let \( A = C^*_\text{red}(R) \). Let \( x \in X \) and let \( \pi_{[x]} \) be the irreducible representation of \( A \) defined above. Then, for all \( a \in A \), \( \langle \pi_{[x]}(a)e_y, e_z \rangle = a(z, y) \) \((y, z \in [x])\).

**Proof.** Since \( \| \cdot \|_{\text{red}} \) dominates the uniform norm on \( R \) [27; II.4.2] it is enough to prove the result for \( a \in C_c(R) \). So let \( a \in C_c(R) \), \( y, z \in [x] \). Then
\[ \langle \pi_{[x]}(a) \xi_{y}, \xi_{z} \rangle = \langle \pi_{[x]}(a) \xi_{f_{y}}, \xi_{f_{z}} \rangle = \phi((f_{z})^* a f_{y}) = (f_{z})^* a f_{y} (x, x) \]
\[ = \sum_{w_1, w_2 \sim x} (f_{z})^* (x, w_1) a (w_1, w_2) f_{y} (w_2, x) \]
\[ = \sum_{w_1, w_2 \sim x} (f_{z}) (x, w_1) a (w_1, w_2) f_{y} (w_2, x) = a(z, y). \]

**Remark.** It follows at once from Lemma 5.2 that \( \cap_{x \in X} \ker \pi_{[x]} = \{0\} \). Hence for all \( a \in C_{\text{red}}^*(R) \)

\[ \|a\|_{\text{red}} = \sup \{ \|\pi_{[x]}(a)\| : x \in X\}. \]

The next lemma follows immediately from the fact that \( \{e_y \} (y \in [x]) \) is an orthonormal basis for \( H_{[x]} \).

**Lemma 5.3.** Let \( R \) be a separated topological equivalence relation on a set \( X \) and let \( A = C_{\text{red}}^*(R) \). If \( x \in X \) and \( a \in A^+ \) then

\[ \text{Tr} \pi_{[x]}(a) = \sum_{y \sim x} a(y, y). \]

**Lemma 5.4.** Let \( R \) be a separated topological equivalence relation on a set \( X \) and let \( A = C_{\text{red}}^*(R) \).

(i) If \( x \in X \) then \( \pi_{[x]}(A) \supseteq \text{LC}(H_{[x]}) \) if and only if the orbit of \( (x, x) \) in \( R^0 \) is discrete.

(ii) If \( \pi \in \tilde{A} \) and if \( \pi(A) \supseteq \text{LC}(H_{\pi}) \) then there exists \( x \in X \) such that \( \pi \) is equivalent to \( \pi_{[x]} \).

**Proof.** (i) The local homeomorphism property of the projection maps \( r \) and \( s \) implies that either every point of the orbit of \( (x, x) \) is isolated, or the orbit of \( (x, x) \) has no isolated points. In the first case there exist elements \( a \) of \( A^+ \) such that \( 0 < \text{Tr} \pi_{[x]}(a) < \infty \); in the second case there do not. Hence in the first case \( \pi_{[x]}(A) \supseteq \text{LC}(H_{[x]}) \), in the second case \( \pi_{[x]}(A) \not\supseteq \text{LC}(H_{[x]}) \).

(ii) Let \( S \) be the closed invariant subset of \( R^0 \) such that \( (\ker \pi)^+ = \{a \in A^+ : a(x, x) = 0 \text{ for all } (x, x) \in S\} [27; II.4.6] \). Let \( a \in A^+ \) such that \( \pi(a) \) is a non-zero compact operator. Since \( a \notin (\ker \pi)^+ \) there exists \( (x, x) \in S \) such that \( a(x, x) \neq 0 \). Hence \( \pi_{[x]}(a) \) is non-zero. Since \( \ker \pi \subseteq \ker \pi_{[x]} \) it follows from [8; 1.3.4] that \( \pi \) is equivalent to \( \pi_{[x]} \).

**Remark.** It follows from 5.4 (ii) that if \( A = C_{\text{red}}^*(R) \) is a postliminal C*-algebra then \( \tilde{A} = \{\pi_{[x]} : x \in X\} \). If \( A \) is a UHF algebra, on the other hand, then \( A \) is isomorphic to \( C_{\text{red}}^*(R) \) for some separated topological equivalence relation \( R \), but it is known that, for each \( x \in X \), \( \phi_x \) is a product state. Since \( A \) has pure states...
which are not unitarily equivalent to product states it follows that the set \( \{ \pi_x : x \in X \} \) is not the whole of \( \hat{A} \). Maybe \( \{ \pi_x : x \in X \} \) is equal to \( \hat{A} \) if and only if \( A \) is postliminal.

We now describe a second topology on a separated topological equivalence relation \( R \). Under the original topology, \( \tau_o \), the diagonal \( R^0 \) is a topological space. We transfer the topology to \( X \), using the natural bijection from \( R^0 \) to \( X \), and then give \( R \) a new topology, \( \tau_p \), by restricting the product topology on \( X \times X \). It is straightforward to check, using the base of \( R^0 \)-strip neighbourhoods for \( \tau_o \), that \( \tau_o \) is finer than \( \tau_p \).

If \( A = C^*_red(R) \) then the mapping \((x, y) \rightarrow (\phi_x, \phi_y)\) embeds \( R \) as a subset of \( R(A) \) in such a way that the \( \tau_q \)-topology on \( R(A) \) restricts to the \( \tau_o \)-topology on \( R \), and the \( \tau_p \)-topology on \( R(A) \) restricts to the \( \tau_p \)-topology on \( R \) [28]. In the light of Section 4 it is natural to investigate the points of continuity for the inclusion map from \((R, \tau_p)\) to \((R, \tau_o)\).

**Lemma 5.5.** Let \( R \) be a separated topological equivalence relation and let \( j : (R, \tau_p) \rightarrow (R, \tau_o) \) be the inclusion map. Let \((x, y) \in R \). The following are equivalent:

(i) \((x, y) \) is a point of continuity for \( j \),

(ii) \((x, y) \) has a \( \tau_o \)-open neighbourhood which contains at most one point from each equivalence class in \( R \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that each neighbourhood of \((x, y) \) contains two points from the same equivalence class in \( R \). Let \( U \) be any fixed \( R^0 \)-strip neighbourhood in \( \tau_o \) of \((x, y) \). By hypothesis there are nets \((a_x, b_x)\) and \((c_x, d_x)\) both converging \( (\tau_o) \) to \((x, y) \) with \((a_x, b_x)\) and \((c_x, d_x)\) in the same equivalence class in \( R \) but not equal, for each \( x \). Since each net is eventually inside \( U \) it follows that eventually \( a_x \neq c_x \) for each \( x \). The continuity of the projection maps \( r \) and \( s \) implies that

\[
(a_x, a_x) \rightarrow (x, x) \quad (\tau_o)
\]

and

\[
(d_x, d_x) \rightarrow (y, y) \quad (\tau_o).
\]

Therefore, by definition

\[
(a_x, d_x) \rightarrow (x, y) \quad (\tau_p).
\]

But \( s \) is injective on \( U \) and \((c_x, d_x)\) is eventually in \( U \), with \( a_x \neq c_x \), so eventually \((a_x, d_x)\) is not in \( U \). Hence \((a_x, d_x) \nrightarrow (x, y) (\tau_o)\), as required.

(ii) \( \Rightarrow \) (i) Conversely, suppose that \((x, y) \) has a \( \tau_o \)-neighbourhood \( U \) containing at most one point from each equivalence class in \( R \). Let \((x_x, y_x)\) be a net converging \( (\tau_p) \) to \((x, y) \). This means, by definition, that \((x_x, x_x) \rightarrow (x, x) \) and
\((y, y) \rightarrow (y, y) (\tau_o)\). Hence eventually \((x, x)\) is in \(r(U)\), so there is a point \((x, z) \in U\) for all \(x\) sufficiently large. Similarly there is a point \((w, y) \in U\) for all \(x\) sufficiently large. But \(U\) contains at most one point from each equivalence class, which implies that \(x = w\) and \(y = z\), that is that \((x, y)\) is eventually in \(U\). The same argument shows that \((x, y)\) is eventually inside any \(\tau_o\)-neighbourhood of \((x, y)\) contained in \(U\), and hence that \((x, y) \rightarrow (x, y) (\tau_o)\).

**Remark.** The local homeomorphism property of \(r\) and \(s\) implies that either every point in an equivalence class in \(R\) satisfies (ii) above, or no point does. Hence either every point in an equivalence class is a point of continuity, or no point is (c.f. Corollary 4.4).

**Lemma 5.6.** Let \(A\) be a \(C^*\)-algebra and let \(b \in A^+\). Then the function from \(\hat{A}\) to the extended interval \([0, \infty]\) defined by \(\pi \rightarrow \text{rank } \pi(b)\) is lower semi-continuous on \(\hat{A}\).

**Proof.** Let \(B = (bAb)^-\) be the hereditary subalgebra of \(A\) generated by \(b\). By [24; 4.1.9] the map \((\pi, H) \rightarrow (\pi|B, \pi(B)H)\) induces a homeomorphism from \(\hat{A} \setminus \text{hull}(B)\) onto \(\hat{B}\). If \(\pi \in \hat{A}\) then \(\dim \pi|B = \text{rank } \pi(b)\) so for each \(n \in \mathbb{N}\) the set \(\{\pi \in \hat{A} : \text{rank } \pi(b) > n\}\) is equal to the set \(\{\pi \in \hat{A} : \dim \pi|B > n\}\), which is open by [12; 3.6.3].

**Theorem 5.7.** Let \(R\) be a separated topological equivalence relation and let \(A = C^*_\text{red}(R)\). Let \(j\) be the map defined in Lemma 5.5.

(i) Let \((x, y) \in R\). If \((x, y)\) is a point of continuity for \(j\) then \(\pi_{[x]}\) is a Fell point in \(\hat{A}\).

(ii) If \(\pi\) is a Fell point in \(\hat{A}\) then there exists \(x \in X\) such that \(\pi \simeq \pi_{[x]}\), and \((y, z)\) is a point of continuity for \(j\) for all \(y, z \in [x]\).

**Proof.** (i) Suppose that \((x, y)\) is a point of continuity for \(j\). By Lemma 5.5 \((x, y)\) has a \(\tau_o\)-open neighbourhood \(U\) which contains at most one point from each equivalence class of \(R\). Let \(f \in C_c(R)\) be supported in \(U\) with \(f(x, y) = 0\). Then for each \(z \in X\), \(\pi_{[z]}(f^*f)\) is an operator of rank less than or equal to one. Since the set \(\{\pi_{[z]} : z \in X\}\) is dense in \(\hat{A}\) it follows from Lemma 5.6 that \(f^*f\) is an abelian element. Since \(\pi_{[x]}(f^*f) = 0\), \(\pi_{[x]}\) is a Fell point in \(\hat{A}\).

(ii) Let \(\pi\) be a Fell point in \(\hat{A}\). By Lemma 5.4 (ii) there exists \(x \in X\) such that \(\pi\) is equivalent to \(\pi_{[x]}\). Let \(a\) be an abelian element in \(A^+\) such that \(\|a\| = \|\pi(a)\| = 1\). Let \(e\) be the unit vector in \(H_{[x]}\) which spans the range of \(\pi(a)\). By Kadison’s Transitivity Theorem [24; 3.13.2] there exists \(b \in A\) with \(\|b\| = 1\) such that \(\pi(b)e_x = e\). Set \(c = b^*ab\). Then \(c\) is an abelian element in \(A^+\) and \(\|c\| = \|\pi(c)\| = 1\). In fact \(\pi(c)\) is the orthogonal projection onto the span of \(e_x\). Hence \(c(x, x) = 1\) and \(c\) vanishes on all other points in the equivalence class of \((x, x)\) in \(R\). Let \(N\) be the \(\tau_o\)-open neighbourhood of \((x, x)\) in \(R\) defined by

\[\{ (y, y) \in R^0 : c(y, y) > 1/2 \}\].
The $N$ contains at most one point from each equivalences class in $R$, for if $(y, y), (z, z) \in N$ are in the same orbit, but not equal, then
\[
\text{Tr } \pi_{xy}(c) \geq c(y, y) + c(z, z) > 1,
\]
contradicting the fact that $c$ is abelian with $\|c\| = 1$. It follows from Lemma 5.5 that $(x, x)$ is a point of continuity for $j$. Hence $(y, z)$ is a point of continuity for $j$ for all $y, z \in [x]$.

An alternative proof of Theorem 5.7 (ii) can be given, based on Corollary 4.4 and the remarks before Lemma 5.5.

**Lemma 5.8.** Let $R$ be a separated topological relation and let $A = C^*_\text{red}(R)$. If $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate identity for $C_0(R^0)$ then it is an approximate identity for $A$.

**Proof.** Let $a \in A$ with $\|a\| = 1$, and let $\varepsilon > 0$ be given. Let $b \in C_\varepsilon(R)$ with $\|a - b\| < \varepsilon/3$ and $\|b\| \leq 1$. Let $S$ denote the support of $b$ and set $T = r(S)$. Then $T$ is a compact subset of $R^0$. Let $f \in C_0(R^0)$ with $f(t) = 1$ for all $t \in T$. Then $fb = b$. Choose $\lambda \in \Lambda$ such that $\|u_\lambda f - f\| < \varepsilon/3$. Then
\[
\|u_\lambda a - a\| = \|u_\lambda a - u_\lambda b + u_\lambda b - fb + fb - a\|
\leq \|u_\lambda\|\|a - b\| + \|u_\lambda f - f\|\|b\| + \|b - a\| < \varepsilon.
\]

**Corollary 5.9.** Let $R$ be a separated topological equivalence relation and let $A = C^*_\text{red}(R)$. Then $A$ is a Fell C*-algebra if and only if the topologies $\tau_p$ and $\tau_o$ coincide on $R$.

**Proof.** If $A$ is a Fell C*-algebra then for every point $(x, y)$ of $R$, $\pi_{xy}$ is a Fell point of $\hat{A}$, so $(x, y)$ is a point of continuity for $j$, by Theorem 5.7. Hence the topologies $\tau_p$ and $\tau_o$ coincide on $R$.

Conversely, suppose that $\tau_p$ and $\tau_o$ coincide on $R$. Let $x$ and $y$ be distinct points of $X$. By Lemma 5.5 there is a $\tau_o$-open neighbourhood $U$ of $(x, x)$ containing at most one point from each equivalence class in $R$. We may assume that $U \subseteq R^0$ and that $(y, y) \notin U$. Let $f \in C_\varepsilon(R^0)^+$ such that $f(x, x) > 0$ and such that the support of $f$ is contained in $U$. Then the argument in the proof of Theorem 5.7 (i) shows that $f$ is an abelian element of $A$.

By the Stone-Weierstrass theorem the set of elements in $C_0(R^0)$ which are abelian in $A$ generates $C_0(R^0)$ as a C*-algebra. Since $C_0(R^0)$ contains an approximate identity for $A$, Lemma 5.8, it follows that the smallest closed ideal of $A$ containing the abelian elements is $A$ itself. This implies, by [24; 6.1.7], that $A$ is a Fell C*-algebra.
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