STRONG BARRELLEDNESS PROPERTIES IN $L_\infty(\mu, X)$

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Throughout this paper $(\Omega, \Sigma, \mu)$ will stand for a finite measure space, $\Sigma$ being a $\sigma$-algebra of subsets of a set $\Omega$, and $X$ is a normed space. $L_\infty(\mu, X)$ will denote the space of all (equivalence classes of) $X$-valued $\mu$-measurable functions defined on $\Omega$ that are essentially bounded. On the other hand, $S(\mu, X)$ will denote the subspace of $L_\infty(\mu, X)$ of all $X$-valued $\mu$-simple functions on $\Omega$ and $S_c(\mu, X)$ will stand for the subspace of $L_\infty(\mu, X)$ formed by the functions that take at most a countable number of different values $\mu$-almost everywhere, all these endowed with the norm

$$\|f\|_\infty = \text{ess sup}\{\|f(\omega)\|: \omega \in \Omega\}.$$  

The subspace $S_c(\mu, X)$ happens to be dense in $L_\infty(\mu, X)$ as a consequence of the Pettis measurability theorem. Finally, $B(\mu, X)$ will denote the closure of $S(\mu, X)$ in $L_\infty(\mu, X)$; it is clear that $S(\mu, X) \subset S_c(\mu, X) \subset L_\infty(\mu, X)$ and $S_c(\mu, X) \subset B(\mu, X)$ if and only if $X$ is finite-dimensional.

When no measure is considered, in [7] it has been shown that the space $S(\Sigma, X)$ of $\Sigma$-simple $X$-valued functions on $\Omega$ is barrelled iff $X$ is finite-dimensional while it is proven in [8] that the space $B(\Sigma, X)$ of all $X$-valued functions that are the uniform limit of $X$-valued $\Sigma$-simple functions is barrelled iff $X$ is barrelled. On the other hand, in [2] it has been shown that if $\mu$ is atomless, $L_\infty(\mu, X)$ is barrelled, and if $\mu$ is atomic and $\sigma$-finite, $L_\infty(\mu, X)$ is barrelled iff $X$ is barrelled. In this paper we will show that if $X$ is barrelled of class $s$, then $S_c(\mu, X)$ and $B(\mu, X)$ are barrelled of class $s$ and, since $S_c(\mu, X)$ is dense in $L_\infty(\mu, X)$, this is also true in $L_\infty(\mu, X)$.

Let us start by recalling that a (real or complex Hausdorff locally convex) space $E$ is Baire-like [9] if, given any increasing sequence of closed absolutely convex subsets of $E$ covering $E$, there is one that is a neighbourhood of the origin. $E$ is said to be db or suprabarrelled [10, 11] if, given any increasing sequence of subspaces of $E$ covering $E$, there is one that is dense and barrelled. Given $s \in \mathbb{N}$, and considering as $\mathcal{C}_s$ the class of Baire-like spaces, a space $E$ is said to be barrelled of class $s$ [5], or briefly $E \in \mathcal{C}_s^-$, if given any increasing sequence of subspaces of $E$ covering $E$, there is one that belongs to $\mathcal{C}_{s-1}$, and $E$ is said to be

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barrelled of class $\mathcal{K}_0$ if $E \in \mathcal{S}_s$ for every $s \in \mathbb{N}$. So $\mathcal{S}_1$ coincides with the class of suprabarrelled spaces and for every $s \in \mathbb{N}$ we have,

$$\text{Baire-like} \Rightarrow \mathcal{S}_{s-1} \Rightarrow \mathcal{S}_s \Rightarrow \text{barrelled of class } \mathcal{K}_0.$$ 

The following definition, [4], will help us to obtain other useful characterization of barrelled spaces of class $s$.

**Definition.** Given a positive integer $s$, a countable family of subspaces $W = \{L_{m_1, \ldots, m_p}: m_r \in \mathbb{N}, 1 \leq r \leq p \leq s\}$ of a linear space $L$ is an $s$-net in $L$ if the sequence $\{L_{m_*}: m \in \mathbb{N}\}$ is increasing, covers $L$ and, for each $p \in \{2, \ldots, s\}$, $\{L_{m_1, \ldots, m_{p-1}, m_*}: m \in \mathbb{N}\}$ is increasing and covers $L_{m_1, \ldots, m_{p-1}}$. The family $\{L_{m_1, \ldots, m_s}: m_* \in \mathbb{N}, 1 \leq i \leq s\}$ will be denoted by $W_s$.

**Proposition 1.** Given $s \in \mathbb{N}$, a space $E$ is barrelled of class $s$ if and only if, given any $s$-net $W$ in $E$, there is some $F \in W_s$ that is Baire-like (or barrelled and dense in $E$).

**Proof.** For $s = 1$ the result is immediate since any dense barrelled subspace of a Baire-like space is Baire-like (see Prop. 1 of [3]).

Let us assume the proposition is true for some $s \in \mathbb{N}$ and suppose $E \in \mathcal{S}_{s+1}$. Let $W = \{E_{m_1, \ldots, m_p}: m_r \in \mathbb{N}, 1 \leq r \leq p \leq s + 1\}$ be an $(s + 1)$-net in $E$, then there is some $m_1 \in \mathbb{N}$ such that $E_{m_1} \in \mathcal{S}_s$ and is dense in $E$. Fixing this $m_1$, $\{E_{m_1, \ldots, m_p}: m_r \in \mathbb{N}, 2 \leq r \leq p \leq s + 1\}$ is an $s$-net in $E_{m_1}$ and, by the induction hypothesis, some $E_{m_1, \ldots, m_{s+1}}$ is barrelled and dense in $E_{m_1}$ and therefore in $E$. On the other hand, assume that given any $(s + 1)$-net $W$ in $E$ there is some $F \in W_{s+1}$ that is barrelled and dense. Suppose that $E \notin \mathcal{S}_{s+1}$, then there is an increasing sequence $\{E_n: n \in \mathbb{N}\}$ of subspaces of $E$ covering $E$ such that no $E_n \in \mathcal{S}_s$. As $E \in \mathcal{S}_s \subset \mathcal{S}_0$, every $E_n$ may be assumed to be dense in $E$. So, by the induction hypothesis, for each $n \in \mathbb{N}$ there will be an $s$-net $W^n = \{F^n_{m_1, \ldots, m_p}: m_r \in \mathbb{N}, 1 \leq r \leq p \leq s\}$ in $E_n$ such that no $F \in (W^n)_s$ is barrelled and dense in $E_n$. Setting $E_{nm_1, \ldots, m_p} := F^n_{m_1, \ldots, m_p}$ for each $n, m_r \in \mathbb{N}, 1 \leq r \leq p \leq s$, then $W := \{E_{m_1, \ldots, m_s}: m_r \in \mathbb{N}, 1 \leq r \leq p \leq s + 1\}$ is an $(s + 1)$-net in $E$ and no $F \in W_{s+1}$ is barrelled and dense in $E$, a contradiction.

In what follows, given $A \in \Sigma$, $e(A)$ will denote the indicator function on $A$, and by a $\mu$-measurable function we shall mean a function from $\Omega$ into $X$ that is the $\mu$-almost everywhere limit of a sequence of $\mu$-simple $X$-valued functions.

**Lemma 1.** If $f \in S_c(\mu, X)$, then there is a countable partition $\{A_n: n \in I\}$ of $\Omega$ formed by nonempty elements of $\Sigma$ such that $f$ is essentially constant on each $A_n$ and takes a different value.

**Proof.** If $f \in S_c(\mu, X)$, then $I$ is finite and the result is obvious. If $f \in S_c(\mu, X) \setminus S(\mu, X)$ let $g$ be a canonical representation of $f$ with countable range $\{x_n: n \in \mathbb{N}\}$. Since $g^{-1}(x_n) \in \Sigma$ for each $n \in \mathbb{N}$ (see for example [1, p. 167]), setting $A_n := g^{-1}(x_n), n \in \mathbb{N}$, the sequence $\{A_n: n \in \mathbb{N}\}$ satisfies the lemma.
Hereafter, given $A \in \Sigma$, $S(\mu, A, X)$ and $S_c(\mu, A, X)$ will stand for the spaces $S(\mu/\Sigma \cap A, X)$, and $S_c(\mu/\Sigma \cap A, X)$, respectively. We identify these spaces with their natural embeddings into $L_\infty(\mu, X)$. Although the two following results can be found enunciated in [2], we give an independent proof of the first of them in order to get in touch with the methods of proof that we use afterwards. On the other hand, a proof of Theorem 2 with similar methods to the ones used in our Theorem 1 can be found in [6].

**Theorem 1.** If $X$ is barrelled, then $B(\mu, X)$ is barrelled.

**Proof.** Suppose that $X$ is barrelled but there is a barrel $T$ in $B(\mu, X)$ which is not a neighbourhood of the origin in $B(\mu, X)$. Then $T$ cannot absorb the unit sphere $S_1$ of $S(\mu, X)$ since if it did so it would also absorb the closed unit ball of $B(\mu, X)$. Hence there must be some $f_1 \in S_1$ such that $f_1 \notin 2T$.

According to Lemma 1, let $\{Q_1^1, Q_2^1, \ldots, Q_k^1\}$ be a partition of $\Omega$ formed by nonempty elements of $\Sigma$ such that $f_1$ is essentially constant on each $Q_i^1$ and takes a different value.

Now given that $S(\mu, X)$ is the topological direct sum of the subspaces $\{S(\mu, Q_i^1, X) : 1 \leq i \leq k_1\}$, $T$ cannot absorb the unit spheres of all of them, and there must be some $m_1 \in \{1, \ldots, k_1\}$ and $f_2 \in S_2$, the closed unit sphere of $S(\mu, Q_{m_1}^1, X)$, such that $f_2 \notin 4T$. Let $\{Q_1^1, Q_2^1, \ldots, Q_{k_1}^1\}$ be a partition of $Q_{m_1}^1$ formed by nonempty elements of $\Sigma$ such that $f_2$ is essentially constant on each $Q_i^1$ and takes a different value.

Going on by recurrence, we obtain a normalized sequence $\{f_n : n \in \mathbb{N}\}$ of $\mu$-simple functions, a sequence $\{m_n : n \in \mathbb{N}\}$ of positive integers and a countable family $\{Q_{m_n}^n : n \in \mathbb{N}\}$ formed by nonempty elements of $\Sigma$ such that for each $n \in \mathbb{N}$, $f_n$ is essentially constant on $Q_{m_n}^n$ in such a way that, for each $n \in \mathbb{N}$,

(i) $\text{supp } f_{n+1} \subset Q_{m_n}^n$.

(ii) $f_n$ is essentially constant in $\text{supp } f_m$ for every $m > n$.

(iii) $Q_{m_n+1}^n \subset Q_{m_n}^n$.

(iv) $f_n \notin 2nT$.

Set $Q := \cap \{Q_{m_n}^n : n \in \mathbb{N}\}$. If $\mu(Q) = 0$ then $e(Q)$ is not the identically null mapping and the mapping $x \mapsto e(Q)x$ is an isometry of $X$ onto its image. Therefore if $x_n$ denotes the value taken by $f_n$ on $Q_{m_n}^n$, then $\|x_n\| \leq 1$ $\forall n \in \mathbb{N}$, since $\{f_n : n \in \mathbb{N}\}$ is normalized, and there must be some $n_0 \in \mathbb{N}$ such that $x_n e(Q) \in n_0 T \forall n \in \mathbb{N}$. Hence $x_n e(Q) \in n T \forall n \geq n_0$.

If for each $n \in \mathbb{N}$ we define $g_n := f_n - x_n e(Q) \notin nT$, then

$$\cap \{\text{supp } g_n : n \geq n_0\} \subset \cap \{Q_{m_n}^n \setminus Q : n \geq n_0\} = \emptyset.$$ 

If $\mu(Q) = 0$, for each $n \in \mathbb{N}$ we define $g_n(\omega) = f_n(\omega)$ if $\omega \notin Q$ and $g_n(\omega) = 0$ if $\omega \in Q$. Taking $n_0 = 1$, then $g_n = f_n \mu$-a.e. $\forall n \geq n_0$ and $\cap \{\text{supp } g_n : n \geq n_0\} = \emptyset$.

In any of these two cases, the sequence $\{g_n : n \geq n_0\}$ is bounded in $S(\mu, X)$. 

Therefore if \( \xi \in l_1 \), \( \sum_{n=n_0}^{\infty} \xi_n g_n \) converges in the completion of \( B(\mu, X) \) and, essentially, takes at most a countable number of values in \( X \). Indeed if \( \omega \in Q \), 
\[ \sum_{n=n_0}^{\infty} \xi_n g_n(\omega) = 0, \]
and if \( \omega \notin Q \), there exists some positive integer \( m_0 \geq n_0 \) such that 
\[ \omega \notin Q_m \] for all \( n > m_0 \) and so 
\[ \sum_{n=n_0}^{\infty} \xi_n g_n(\omega) = \sum_{n=n_0}^{\infty} \xi_n f_n(\omega) = \sum_{n=n_0}^{m_0} \xi_n f_n(\omega) \in X. \]
Therefore, 
\[ \sum_{n=n_0}^{\infty} \xi_n g_n \in B(\mu, X). \]

Hence, denoting by \( B_{l_1} \) the closed unit ball of \( l_1 \), the Banach disk 
\[ D := \left\{ \sum_{n=n_0}^{\infty} \xi_n g_n : \xi \in B_{l_1} \right\} \]
in the completion of \( B(\mu, X) \) is contained in \( B(\mu, X) \).
Thus, by the Baire category theorem, there exists some integer \( q \geq n_0 \) with 
\( D \subset qT \) and hence \( g_q \in qT \), a contradiction.

**Theorem 2.** If \( X \) is barrelled, then \( S_c(\mu, X) \) is barrelled.

In the following results we suppose that \( s \) is any positive integer, 
\( W = \{ E_{m_1, \ldots, m_p} : m_r \in \mathbb{N}, 1 \leq r \leq p \leq s \} \) is an \( s \)-net in \( E \) formed by dense subspaces of \( S_c(\mu, X) \) covering \( S_c(\mu, X) \). For each \( m_1, \ldots, m_s \in \mathbb{N} \), suppose \( T_{m_1, \ldots, m_s} \) is a barrel of \( E_{m_1, \ldots, m_s} \), \( B_{m_1, \ldots, m_s} \) is its closure in \( S_c(\mu, X) \) and \( L_{m_1, \ldots, m_s} := \langle B_{m_1, \ldots, m_s} \rangle \). By decreasing recurrence, for \( p = s - 1, \ldots, 1 \), define the subspaces \( F_{m_1, \ldots, m_{p+1}} := \cap \{ L_{m_1, \ldots, m_p} : m \geq m_{p+1} \} \), \( L_{m_1, \ldots, m_p} := \cup \{ F_{m_1, \ldots, m_p, m} : m \in \mathbb{N} \} \), and \( F_{m_1} := \cap \{ L_{m} : m \geq m_1 \} \). Notice that \( \{ F_m : m \in \mathbb{N} \} \) and \( \{ F_{m_1, m_2, \ldots, m_p} : m \in \mathbb{N} \} \) are 1-nets in \( S_c(\mu, X) \) and \( L_{m_1, \ldots, m_p} \), \( \forall m_r \in \mathbb{N} \), \( 1 \leq r \leq p \leq s - 1 \), and \( E_{m_1, \ldots, m_p} \subset F_{m_1, \ldots, m_p} \), \( \forall m_r \in \mathbb{N} \), \( 1 \leq r \leq p \leq s \).

**Lemma 2.** If \( \{ A_n : n \in \mathbb{N} \} \) is a sequence of nonempty pairwise disjoint elements of \( \Sigma \), then there exists some \( n_0 \in \mathbb{N} \) such that \( S_c(\mu, \cup \{ A_n : n \geq n_0 \}, X) \subset F_{n_0}. \)

**Proof.** Assume the lemma is false and that for each \( p \in \mathbb{N} \) there is some \( f_p \in S_c(\mu, \cup \{ A_n : n \geq p \}, X) \setminus F_p \) so that \( \| f_p \| = 1 \). Then \( \{ f_n : n \in \mathbb{N} \} \) is bounded in 
\( S_c(\mu, X) \) and if \( \xi \in l_1 \), 
\[ \sum_{n=1}^{\infty} \xi_n f_n \] converges in the completion \( L_{\infty}(\mu, X) \) of \( S_c(\mu, X) \).

Now \( \sum_{n=1}^{\infty} \xi_n f_n \) is essentially countably valued in \( X \) since if \( \omega \in \Omega \setminus \cup \{ A_n : n \in \mathbb{N} \} \), 
then \( \sum_{n=1}^{\infty} \xi_n f_n(\omega) = 0 \) and if \( \omega \in \cup \{ A_n : n \in \mathbb{N} \} \) there is some \( r \in \mathbb{N} \) such that \( \omega \in A_r \), i.e. \( \omega \notin \cup \{ A_n : n > r \} \) and, since \( \text{supp } f_n \subset \cup \{ A_i : i \geq n \} \), 
\[ \sum_{n=1}^{\infty} \xi_n f_n(\omega) = \sum_{n=1}^{r} \xi_n f_n(\omega). \]
Moreover, the sequence \( \left\{ \frac{m}{n} \xi_n f_n, \ m \in \mathbb{N} \right\} \) of \( S_c(\mu, X) \) converges to \( \sum_{n=1}^{\infty} \xi_n f_n \) in the completion \( L_\infty(\mu, \hat{X}) \) of \( S_c(\mu, X) \). Hence, \( \sum_{n=1}^{\infty} \xi_n f_n \in S_c(\mu, X) \).

Therefore \( D := \left\{ \sum_{n=1}^{\infty} \xi_n f_n : \xi \in B_1 \right\} \) is a Banach disk and, denoting by \( E_D \) the normed space \( \langle D \rangle \) whose norm is the gauge of \( D \), there is some \( m_1 \in \mathbb{N} \) such that \( F_{m_1} \cap E_D \) is a dense Baire subspace of \( E_D \) \( \forall m_1 \geq m_1 \). By finite induction, suppose that we have found \( m_1 \) and the functions \( m_i(m_1, \ldots, m_{i-1}), 2 \leq i \leq s - 1 \), such that for any positive integer \( m_1 \geq m_1, \ m_i \geq m_i(m_1, \ldots, m_{i-1}), 2 \leq i \leq p \), \( F_{m_1, \ldots, m_i} \cap E_D \) is a dense Baire subspace of \( E_D \). Then, for any \( m_1 \geq m_1, \ldots, m_p \geq m_p(m_1, \ldots, m_{p-1}) \) given that \( \{ F_{m_1, \ldots, m_{p+1}} : m \in \mathbb{N} \} \) covers \( F_{m_1, \ldots, m_p} \), there is some \( m_{p+1}(m_1, \ldots, m_p) \in \mathbb{N} \) such that \( F_{m_1, \ldots, m_{p+1}} \cap E_D \) is a dense Baire subspace of \( E_D \) \( \forall m_{p+1} \geq m_{p+1}(m_1, \ldots, m_p) \). Hence \( D \subseteq L_{m_1, \ldots, m_p} \) if \( m_1 \geq m_1, \ldots, m_s \geq m_s(m_1, \ldots, m_{s-1}) \), since \( B_{m_1, \ldots, m_s} \cap L_{m_1, \ldots, m_s} \cap E_D \) is a barrel and consequently a nonempty subset of the origin in the Baire space \( L_{m_1, \ldots, m_s} \cap E_D \) for \( m_1 \geq m_1, \ldots, m_s \geq m_s(m_1, \ldots, m_{s-1}) \). It follows from this that \( D \subseteq F_{m_1, \ldots, m_s} \) for \( m_1 \geq m_1, \ldots, m_s \geq m_s(m_1, \ldots, m_{s-1}) \) and therefore \( D \subseteq L_{m_1, \ldots, m_s} \) if \( m_1 \geq m_1, \ldots, m_s \geq m_s(m_1, \ldots, m_{s-2}) \). This implies that \( D \subseteq F_{m_1, \ldots, m_{s-1}} \) for \( m_1 \geq m_1, \ldots, m_{s-1} \geq m_{s-1}(m_1, \ldots, m_{s-2}) \). Going on in this way, we obtain that \( D \subseteq F_{m_1} \) for \( m_1 \geq m_1 \), and, consequently, \( f_{m_1} \in F_{m_1} \), a contradiction.

**Lemma 3.** If \( X \) is barrelled of class \( s \), then there exists some \( q \in \mathbb{N} \) such that \( S(\mu, X) \subseteq F_q \).

**Proof.** Suppose the lemma is false and there is some \( f_1 \in S(\mu, X) \), \( f_1 \notin F_1 \) so that \( \| f_1 \| = 1 \). Let \( \{ Q_1^1, Q_2^1, \ldots, Q_r^1 \} \) be a partition of \( \Omega \) formed by nonempty elements of \( \Sigma \) such that \( f_1 \) is essentially constant on each \( Q_i^1 \) and takes a different value.

Now given that \( S(\mu, X) \) is the topological direct sum of the subspaces \( \{ S(\mu, Q_i^1, X) : 1 \leq i \leq k_1 \} \), there must be some \( m_1 \in \{ 1, \ldots, k_1 \} \) such that \( S(\mu, Q_{m_1}^1, X) \) is not contained in \( F_n \) for each \( n \in \mathbb{N} \) and, consequently, there is some \( f_2 \in S(\mu, Q_{m_1}^1, X) \), \( f_2 \notin F_2 \) so that \( \| f_2 \| = 1 \). Let \( \{ Q_1^2, Q_2^2, \ldots, Q_r^2 \} \) be a partition of \( Q_{m_1}^1 \) formed by nonempty elements of \( \Sigma \) such that \( f_2 \) is essentially constant on each \( Q_i^2 \) and takes a different value. Now there is some \( m_2 \in \{ 1, \ldots, k_1 \} \) such that \( S(\mu, Q_{m_2}^1, X) \) is not contained in \( F_n \) for each \( n \).

Assume that we have obtained by induction a sequence \( \{ f_n : n \in \mathbb{N} \} \) of \( \mu \)-simple functions, a sequence of positive integers \( \{ k_n : n \in \mathbb{N} \} \), and a countable family \( \{ Q_i^m : n \in \mathbb{N}, 1 \leq i \leq k_n \} \) formed by nonempty elements of \( \Sigma \) such that:

a) for each \( n \in \mathbb{N} \), \( f_n \) is essentially constant on each \( Q_i^m \) and takes a different value, and
b) for each \( n \in \mathbb{N} \), the following properties are satisfied

(i) \( \| f_n \| = 1 \).

(ii) \( \operatorname{supp} f_{n+1} \subset Q_{m_n}^n \) for some \( m_n \in \{1, \ldots, k_n\} \).

(iii) \( Q_{m_n+1}^{n+1} \subset Q_{m_n}^n \).

(iv) \( f_n \notin F_n \).

Set \( Q := \cap \{ Q_{m_n}^n : n \in \mathbb{N} \} \). In case \( \mu(Q) \neq 0 \) we define \( g_n := f_n - x_n e(Q) \) for each \( n \in \mathbb{N} \) where \( x_n \) denotes the value taken by \( f_n \) on \( Q_{m_n}^n \). Then, since the mapping of \( X \) into \( S_c(\mu, X) \) such that \( x \mapsto e(Q)x \) is an isometry and \( X \in C_s \), using Proposition 1 it is easy to find some \( m_1 \in \mathbb{N} \) such that \( e(Q)x_j \in F_n \forall j \in \mathbb{N} \) and for all \( n \geq m_1 \). Thus, \( g_n \notin F_n \) for each \( n \geq m_1 \) and

\[ \cap \{ \operatorname{supp} g_n : n \in \mathbb{N} \} = \emptyset. \]

In case \( \mu(Q) = 0 \), we define \( g_n(\omega) := f_n(\omega) \) for \( \omega \notin Q \) and \( g_n(\omega) := 0 \) for \( \omega \in Q \) for each \( n \in \mathbb{N} \). Then \( g_n = f_n \mu \text{-a.e.} \) and \( \cap \{ \operatorname{supp} g_n : n \in \mathbb{N} \} = \emptyset \) as well.

As in Theorem 1, \( D := \left\{ \sum_{n=1}^{\infty} \xi_n g_n : \xi \in B_{l_1} \right\} \) is a Banach disk in \( L_\infty(\mu, \hat{X}) \) which is contained in \( S_c(\mu, X) \) and there must be some \( p \geq m_1 \) such that \( D \subset F_p \).

Hence \( g_p \in F_p \), a contradiction.

**Theorem 3.** Given \( s \in \mathbb{N} \), if \( X \) is barrelled of class \( s \) then \( S_c(\mu, X) \) is barrelled of class \( s \).

**Proof.** By Theorem 2, \( S_c(\mu, X) \in C_0 \) since it is a metrizable barrelled space. Proceeding by recurrence, let \( p \in \{1, \ldots, s\} \) and assume \( S_c(\mu, X) \in C_{p-1} \setminus C_p \). Then, by Proposition 1, there is a \( p \)-net \( W := \{ E_{m_1, m_2} : m_2 \in \mathbb{N}, 1 \leq r \leq i \leq p \} \) in \( S_c(\mu, X) \) formed by dense subspaces such that no \( E_{m_1, m_2} \in W_1 \) is barrelled of class \( i - 1, 1 \leq i \leq p \). And as \( S_c(\mu, X) \) is metrizable, no \( E_{m_1, m_2} \) is barrelled. For each \( m_1, m_2 \in \mathbb{N} \), suppose \( T_{m_1, m_2} \) is a barrel of \( E_{m_1, m_2} \) which is not a neighbourhood of the origin in \( E_{m_1, m_2} \), let \( B_{m_1, m_2} \) be the closure of \( T_{m_1, m_2} \) in \( S_c(\mu, X) \) and let \( L_{m_1, m_2} := \{ B_{m_1, m_2} \} \). By decreasing recurrence, for \( i = p - 1, \ldots, 1 \), define the subspaces \( F_{m_1, \ldots, m_i+1} := \cap \{ L_{m_1, \ldots, m_i} : m \geq m_{i+1} \} \), \( L_{m_1, \ldots, m_i} := \cup \{ F_{m_1, \ldots, m_i} : m \in \mathbb{N} \} \), and \( F_{m_1} := \cap \{ L_m : m \geq m_1 \} \). Then \( \{ F_m : m \in \mathbb{N} \} \) and \( \{ F_{m_1, \ldots, m_i} : m \in \mathbb{N} \} \) are respectively 1-nets in \( S_c(\mu, X) \) and in \( L_{m_1, \ldots, m_i}, \forall m_i \in \mathbb{N}, 1 \leq r \leq i \leq p - 1 \). Besides \( E_{m_1, \ldots, m_i} \subset F_{m_1, \ldots, m_i}, \forall m_i \in \mathbb{N} \) with \( 1 \leq r \leq i \leq p \). Now if there is a \( m_1 \in \mathbb{N} \) such that \( S_c(\mu, X) \) coincides with \( F_{m_1} \), then \( L_{m_1} \in C_{p-1} \) and there must be some \( m_2 \in \mathbb{N} \) such that \( F_{m_1, m_2} \in C_{p-2} \) and is dense in \( S_c(\mu, X) \). Thus, \( L_{m_1, m_2} \in C_{p-2} \) and is dense in \( S_c(\mu, X) \). Continuing in this way we would find some \( F_{m_1, \ldots, m_p} \in C_0 \). So \( B_{m_1, \ldots, m_p} \cap F_{m_1, \ldots, m_p} \) would be a neighbourhood of zero in \( F_{m_1, \ldots, m_p} \) and \( E_{m_1, \ldots, m_p} \) would be barrelled, a contradiction.

Hence, no \( F_n \) may coincide with \( S_c(\mu, X) \). Now by the previous Lemma we may assume that \( S(\mu, X) \subset F_n \forall n \in \mathbb{N} \). Let \( f_1 \in S_c(\mu, X) \setminus F_1 \) be such that \( \| f_1 \| = 1 \) and
let \( \{Q^1_i : i \in \mathbb{N}\} \) be a partition of \( \Omega \) formed by nonempty elements of \( \Sigma \) determined by Lemma 1 and defined by the \( \mu \)-measurable function \( f_1 \) so that \( f_1 \) is essentially constant on each \( Q^1_i \) and takes a different value.

By Lemma 2 there is some positive integer \( n_2 > n_1 = 1 \) so that \( S_c(\mu, \cup \{Q^1_i : i \geq n_2\}, X) \subseteq F_{n_2} \). Thus, setting \( \Omega_1 := \cup \{Q^1_i : 1 \leq i \leq n_2\} \), \( S_c(\mu, \Omega_1, X) \) cannot be contained in any \( F_n \), \( n \geq n_2 \), and there must be some \( f_2 \in S_c(\mu, \Omega_1, X) \cap F_{n_2} \) so that \( \|f_2\| = 1 \). Let \( \{Q^n_i : i \in \mathbb{N}\} \) be a partition of \( \Omega_1 \) formed by nonempty elements of \( \Sigma \) determined by the \( \mu \)-measurable function \( f_2 \) so that \( f_2 \) is constant on each \( Q^n_i \) and takes a different value.

Continuing in this way, we obtain a sequence of positive integers \( \{n_i : i \in \mathbb{N}\} \) and a sequence \( \{f_n : n \in \mathbb{N}\} \) of \( \mu \)-measurable functions of \( S_c(\mu, X) \) which determine the sequence \( \{Q^n_i : n \in \mathbb{N}, i \in \mathbb{N}\} \) formed by nonempty elements of \( \Sigma \) such that, for each \( n \in \mathbb{N} \), \( f_n \) is essentially constant on each \( Q^n_i \) and takes a different value, in such a way that setting \( \Omega_n := \cup \{Q^n_i : 1 \leq i \leq n_{i+1}\} \) \( \forall n \in \mathbb{N} \), for each \( i \in \mathbb{N} \) we have that,

(i) \( \text{supp } f_{i+1} \subseteq \Omega_i \).
(ii) \( e(\Omega_i)f_i \in S(\mu, \Omega_i, X) \subseteq S(\mu, X) \).
(iii) \( \Omega_{i+1} \subseteq \Omega_i \).
(iv) \( f_i \notin F_{n_i} \).

Now let \( g_i := f_i - e(\Omega_i)f_i \) for each \( i \in \mathbb{N} \). Then \( g_i \notin F_{n_i} \) for each \( i \in \mathbb{N} \) and \( \text{supp } g_i \cap \text{supp } g_j = \emptyset \) for \( i \neq j \). Hence \( \langle \{g_n : n \in \mathbb{N}\} \rangle \), where the closure is in \( L_\infty(\mu, \hat{X}) \), is a copy of \( c_0 \) since \( \{g_n/\|g_n\| : n \in \mathbb{N}\} \) is equivalent to the unit vector basis of \( c_0 \). As it is easy to see that \( \langle \{g_n : n \in \mathbb{N}\} \rangle \subseteq S_c(\mu, X) \), using the Baire category theorem as above, there must be some \( q \in \mathbb{N} \) such that \( \{g_n : n \in \mathbb{N}\} \subseteq F_k \) for each \( k \geq n_q \). Hence \( g_q \in F_{n_q} \), a contradiction. Thus \( S_c(\mu, X) \subseteq c_p \).

Hence \( S_c(\mu, X) \subseteq c_p \) and the proof is over.

**THEOREM 4.** If \( X \) is a barrelled space of class \( s \) (barrelled of class \( \kappa_0 \)), then both \( L_\infty(\mu, X) \) and \( B(\mu, X) \) are barrelled of class \( s \) (barrelled of class \( \kappa_0 \)).

**PROOF.** The first affirmation is an obvious consequence of the previous theorem, since \( S_c(\mu, X) \) is dense in \( L_\infty(\mu, X) \). The argument to prove the second affirmation is analogous to the one given in theorems above, but working with \( B(\mu, X) \) instead of \( S_c(\mu, X) \), and using Theorem 1 instead of Theorem 2.

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