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ESSENTIAL SINGULARITIES OF QUASIMEROMORPHIC MAPPINGS

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In their paper "Lindelöf's theorem for normal quasimeromorphic mappings" [1] Heinonen and Rossi raised a question concerning removable singularities for these mappings (Remark 2.4). In this note we give an affirmative answer to their question. Let B^n denote the open unit ball in euclidean n-space and S^n the unit sphere in euclidean $(n+1)$-space with chordal metric $q(x, y)$ normalised so that $q(\mathsf{S}^n) = 1$. S^n is conformally equivalent to $\overline{\mathsf{R}^n} = \mathsf{R}^n \cup \{\infty\}$ via the stereographic projection. We refer to [4] and [7] for basic definitions and properties of quasimeromorphic mappings. Our main result is the following theorem.

THEOREM. *Let f: $\mathsf{B}^n - \{0\} \to \mathsf{S}^n$ be quasimeromorphic and suppose*

$$\limsup_{|x| = |y| = r \to 0} q(f(x), f(y)) < 1.$$

Then 0 is a removable singularity for f.

Heinonen and Rossi proved this theorem with the bound 1/4 on the right hand side. Of course the bound 1 is sharp. Actually we will show a slightly stronger result to be true, namely that in every deleted neighbourhood of an essential singularity there is an essential round sphere on which f assumes antipodal values. With this formulation our result is even stronger than the classical result of Lehto [3] for meromorphic functions (cf. the comments by Minda on page 71 of [5]). Our proof is based on the following two topological lemmas. In these lemmas we denote by S and N respectively the south and north poles of S^n.

LEMMA 1. *Let f: $\mathsf{S}^{n-1} \times [-1, 1] \to \mathsf{S}^n$ be a continuous function with the following properties:*

(1) $S \in f(\mathsf{S}^{n-1} \times \{-1\})$;
(2) $N \in f(\mathsf{S}^{n-1} \times \{1\})$;
(3) *$f(\mathsf{S}^{n-1} \times (-1, 1))$ is contained in $\mathsf{S}^n - \{S, N\}$;*

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(4) f: $\mathsf{S}^{n-1} \times \{0\} \to \mathsf{S}^n - \{S, N\}$ *is not homotopic to a constant.*
Then there is an $r \in [-1, 1]$ *such that* $q(f(\mathsf{S}^{n-1} \times \{r\})) = 1$.

PROOF. Suppose to the contrary that $q(f(\mathsf{S}^{n-1} \times \{r\})) < 1$, for each r. Let

$$A = \{(x_0, \ldots, x_n) \in \mathsf{S}^n \colon x_n > \tfrac{1}{2}\}, \quad B = \{x \in \mathsf{S}^n \colon -x \in A\}$$

and let

$$\theta \colon \mathsf{S}^n - (A \cup B) \to \mathsf{S}^{n-1} \times [-1, 1]$$

be the homeomorphism defined by

$$\theta(x_0, \ldots, x_n) = \left(\frac{(x_0, \ldots, x_{n-1})}{|(x_0, \ldots, x_{n-1})|}, 2x_n \right).$$

Extend $f\theta$ to a continuous function g: $\mathsf{S}^n \to \mathsf{S}^n$ by coning over $\mathrm{cl}(A)$ and $\mathrm{cl}(B)$ along great circles. More precisely, define $g|A$ as follows. Set $g(N) = N$. If $x \in A - \{N\}$ then there is a unique $x' \in \partial A$ with x', x, N lying in that order on a segment of a great circle of S^n from N to ∂A. If $f\theta(x') = N$ then set $g(x) = N$. Otherwise let $g(x)$ be the unique point on the arc of the great circle from $f\theta(x')$ to N so that

$$\frac{q(g(x), N)}{q(f\theta(x'), N)} = \frac{q(x, N)}{q(x', N)}.$$

Similar comments apply to $g|B$ using S as the cone point.

Next define h: $\mathsf{S}^{n-1} \to \mathsf{S}^{n-1}$ by

$$h(x_0, \ldots, x_{n-1}) = \rho f((x_0, \ldots, x_{n-1}), 0),$$

where ρ: $\mathsf{S}^n - \{S, N\} \to \mathsf{S}^{n-1}$ is the retraction of the n-sphere minus its poles along great circles to its equator which we identify with S^{n-1} in the usual way. It is a straightforward exercise to show that the map g is homotopic to the suspension of h. Thus by (4) and the fact that the suspension of such a map is not homotopic to a constant (cf., for example, Theorem 2.10 of [2]) it follows that g is not homotopic to a constant.

On the other hand, suppose that p_n: $\mathsf{S}^n \to [-1, 1]$ is projection on the last coordinate and let α: $[-1, 1] \to \mathsf{S}^{n-1}$ be a path so that $f(\alpha(t), -1) = S$ when $t \leqq -\frac{1}{2}$ and $f(\alpha(t), 1) = N$ when $t \geqq \frac{1}{2}$. Extend the function f to $\mathsf{S}^{n-1} \times [-2, 2]$ by setting $f(x, t) = f(x, 1)$ for $t \geqq 1$ and $f(x, t) = f(x, -1)$ *for* $t \leqq -1$.

Define the homotopy H_t: $\mathsf{S}^n \to \mathsf{S}^n$ by

$$H_t(x) = \frac{(1 - 2t)f\theta(x) + 2tf(\alpha p_n(x), 2p_n(x))}{|(1 - 2t)f\theta(x) + 2tf(\alpha p_n(x), 2p_n(x))|}$$

when $x \in \mathsf{S}^n - (A \cup B)$ and $0 \leqq t \leqq \frac{1}{2}$, extending this homotopy, with $0 \leqq t \leqq \frac{1}{2}$, over $A \cup B$ by coning in the same way as g extends $f\theta$ and defining $H_t(x) = f(\alpha((2-2t)p_n(x)), 2(2-2t)p_n(x))$ when $\frac{1}{2} \leqq t \leqq 1$. Then H_t is a homotopy from g to a constant map. From this contradiction it follows that $q(f(\mathsf{S}^{n-1} \times \{r\})) = 1$ for some r.

In the sequel we identify $\pi_{n-1}(\mathsf{S}^n - \{S, N\})$ with the infinite cyclic group $\mathbf{Z}$ by the isomorphism which sends the homotopy class of the inclusion of S^{n-1} in $\mathsf{S}^n - \{S, N\}$ to the integer 1. $\mathsf{S}^{n-1}(x, r)$ denotes the sphere and $\mathsf{B}^n(x, r)$ the open ball in euclidean n-space of radius r centred at x; when x is the origin it will be omitted from this notation.

LEMMA 2. *Suppose f: $\mathsf{B}^n - \{0\} \to \mathsf{S}^n$ is a discrete, open, orientation preserving and continuous function, $r \in (0, 1)$, and $S \in f(\mathsf{S}^{n-1}(r))$ but $N \notin f(\mathsf{S}^{n-1}(r))$. Then there is a number ε with $0 < \varepsilon < r$ and $\varepsilon < 1 - r$ satisfying:*

(i) *for each t, if $0 < |t - r| < \varepsilon$, then $f(\mathsf{S}^{n-1}(t))$ lies in $\mathsf{S}^n - \{S, N\}$;*

(ii) *if $\alpha, \beta \in \pi_{n-1}(\mathsf{S}^n - \{S, N\})$ are such that α (respectively β) is the homotopy class of the composition of the radial contraction $\mathsf{S}^{n-1} \to \mathsf{S}^{n-1}(t)$ followed by f for some $t \in (r - \varepsilon, r)$ (respectively $(r, r + \varepsilon)$), (i.e., α and β are the respective degrees of these maps) then $\alpha < \beta$.*

PROOF. Use the discreteness of f to choose ε satisfying (i). Up to homotopy the composition in (ii) depends only on whether $t < r$ or $t > r$, for $0 < |t - r| < \varepsilon$. Thus α and β are well defined. Suppose $x \in \mathsf{S}^{n-1}(r)$ with $f(x) = S$. Choose δ so that $0 < 2\delta < \varepsilon$ and $f^{-1}(S) \cap \mathsf{B}^n(x, 2\delta) = \{x\}$. If x is the only point of $f^{-1}(S) \cap \mathsf{S}^{n-1}(r)$, then $\beta - \alpha$ is represented by $f \,|\, \mathsf{S}^{n-1}(x, \delta)$ and this is positive as f is orientation preserving. Otherwise, $f^{-1}(S) \cap \mathsf{S}^{n-1}(r)$ contains more than one point (but still finitely many) and $\beta - \alpha$ is a sum of positive integers, one for each point of $f^{-1}(S) \cap \mathsf{S}^{n-1}(r)$.

REMARK 1. Interchanging the roles of S and N in Lemma 2 leads to $\alpha > \beta$ since in this case the map $f \,|\, \mathsf{S}^{n-1}(x, \delta)$ represents a negative integer in $\pi_{n-1}(\mathsf{S}^n - \{S, N\})$.

PROOF OF THEOREM. Suppose that 0 is an essential singularity. Let U be a neighbourhood of 0 in B^n. We assume $U = \mathsf{B}^n(a)$ and will find r such that $0 < r < a$ and $q(f(\mathsf{S}^{n-1}(r))) = 1$.

If there is r with $0 < r < a$ and $\{S, N\} \subset f(\mathsf{S}^{n-1}(r))$ then the proof is complete, so suppose that this is not the case. As f is discrete, the set

$$D = \{r \in (0, a) \,/\, \{S, N\} \cap f(\mathsf{S}^{n-1}(r)) \neq \emptyset\}$$

is discrete. By the Big Picard Theorem for quasimeromorphic mappings [6] the set D contains arbitrarily small s [but by our supposition there is no value of $s < a$ for which $\{S, N\} \subset f(\mathsf{S}^{n-1}(s))$].

We will select positive numbers b, c, d with $b < c < d < a$ such that

(i) $\{S, N\} \cap f(\mathsf{B}^n(d) - \mathrm{cl}\,\mathsf{B}^n(b)) = \emptyset$,

(ii) $f \mid \mathsf{S}^{n-1}(c)$: $\mathsf{S}^{n-1}(c) \to \mathsf{S}^n - \{S, N\}$ has non-zero degree,

(iii) either $S \in f(\mathsf{S}^{n-1}(b))$ and $N \in f(\mathsf{S}^{n-1}(d))$ or $S \in f(\mathsf{S}^{n-1}(d))$ and $N \in f(\mathsf{S}^{n-1}(b))$.

Consider the degree of $f \mid \mathsf{S}^{n-1}(r)$, i.e. the integer which is the homotopy class in $\pi_{n-1}(\mathsf{S}^n - \{S, N\})$ determined by $f \mid \mathsf{S}^{n-1}(r)$: this integer is undefined when $r \in D$ and is constant on any interval disjoint from D. By Lemma 2 and Remark 1, when r, decreasing from a, crosses a point $s \in D$, the degree decreases if $S \in f(\mathsf{S}^{n-1}(s))$ and increases if $N \in f(\mathsf{S}^{n-1}(s))$. Thus the degree determined by $f \mid \mathsf{S}^{n-1}(r)$ increases, decreases, increases, decreases, etc. as r decreases to 0. In particular for some values of r this degree will be non-zero; more precisely there will be $b, d \in D$ with $b < d$, $(b, d) \cap D = \emptyset$, $f \mid \mathsf{S}^{n-1}(r)$ has non-zero degree for $r \in (b, d)$ and either $S \in f(\mathsf{S}^{n-1}(b))$ and $N \in f(\mathsf{S}^{n-1}(d))$ or $S \in f(\mathsf{S}^{n-1}(d))$ and $N \in f(\mathsf{S}^{n-1}(b))$. Choose any $c \in (b, d)$.

Now define the homeomorphism

$$k\colon \mathrm{cl}\,\mathsf{B}^n(d) - \mathsf{B}^n(b) \to \mathsf{S}^{n-1} \times [-1, 1]$$

by

$$k(w) = \begin{cases} \left(\dfrac{w}{|w|}, \dfrac{|w| - c}{d - c}\right) & \text{if} \quad c \leqq |w| \leqq d \\ \left(\dfrac{w}{|w|}, \dfrac{|w| - c}{c - b}\right) & \text{if} \quad b \leqq |w| \leqq c \end{cases}$$

Then either fk^{-1} or $-fk^{-1}$ satisfies the conditions demanded of f in Lemma 1, so there is $t \in [-1, 1]$ with $q(fk^{-1}(\mathsf{S}^{n-1} \times \{t\})) = 1$. Finally, since $k^{-1}(\mathsf{S}^{n-1} \times \{t\}) = \mathsf{S}^{n-1}(r)$ for some r, we have $q(f(\mathsf{S}^{n-1}(r))) = 1$ as required.

Remark 2. For simplicity of exposition we have used the Big Picard Theorem for quasiregular mappings in our proof. This is perhaps the deepest known result concerning quasimeromorphic mappings and is due to Rickman [6]. However closer examination will reveal that the Casorati-Weierstrass property, together with the discrete open and orientation preserving properties of quasiregular mappings, suffice for our proof. One needs only slight modifications of our argument. The details are easy and it is interesting to note how entirely topological methods result in this analytic result.

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