L-THEORY AND DIHEDRAL HOMOLOGY

GUILLERMO CORTIÑAS*

Abstract.

Let \( k \) be a commutative ring with \( 1/2 \in k \), \( A \) an involutive \( k \)-algebra, \( \varepsilon = \pm 1 \). We construct a Chern class map, \( \text{ch}' \) from \( L^*_{\varepsilon}(A) \) to the dihedral homology \( \text{HD}_{\varepsilon}(A) \), in such a way that, if \( \text{ch} \) is Karoubi's Chern class, \( y \) is Loday's involution and \( H := \) the hyperbolic functor, the following is a commutative diagram

\[
\begin{array}{ccc}
\text{ch}'_{q} : L^*_{\varepsilon}(A) & \longrightarrow & \text{HD}^*_{\varepsilon+2q}(A) \\
\uparrow H & & \uparrow 1+y \\
\text{ch}_{q} : K^*_{\varepsilon}(A) & \longrightarrow & \text{HC}^*_{\varepsilon+2q}(A) \\
\uparrow & & \uparrow \\
\text{ch}'_{q} : L^*_{\varepsilon}(A) & \longrightarrow & \text{HD}^*_{\varepsilon+2q}(A)
\end{array}
\]

§0. Introduction.

Let \( k \) be a commutative ring, \( 1/2 \in k \), \( A \) an involutive (or hermitian) \( k \)-algebra with an identity. M. Karoubi has defined Chern classes ([Ka-I])

\[
\text{ch}'_{q} : K_{n}(A) \longrightarrow \text{HC}^*_{n+2q}(A) \quad (n, q \geq 0)
\]

where HC stands for cyclic homology. Since \( A \) is involutive, for \( \varepsilon = \pm 1 \) we can as well consider its \( L \)-theory (in the sense of Karoubi [Ka-II]), that is related to \( K \)-theory by means of the "forget" functor. In particular, we can compose \( \text{ch}'_{q} \) with the forgetful map \( \varepsilon L^*_{\varepsilon}(A) \to K^*_{\varepsilon}(A) \) and study its image. When \( 1/2 \in k \), there is a splitting due to Loday [Lo] \( \text{HC}^*_{\varepsilon}(A) = +1 \text{HC}^*_{\varepsilon}(A) \oplus -1 \text{HC}^*_{\varepsilon}(A); \quad +1 \text{HC}^*_{\varepsilon}(A) \) is the dihedral homology of \( A \), and it is often denoted by \( \text{HD}^*_{\varepsilon}(A) \). In this paper we prove:

* Supported by CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas de La Argentina. The author is indebted to Rutgers University for its hospitality during his CONICET sponsored stay.

Received February 11, 1992.
Theorem (§3.3). The image of $\text{ch}_q^*: L_n^q(A) \to \text{HC}_{n+2q}(A)$ lies in the subgroup $\text{HC}_{n+2q}(A)$.

In order to explain the organization of this paper, we give a sketch of the proof of this theorem. First recall that Karoubi's $\text{ch}_q^n$ has a factorization

$$L_n^q(A) = \pi_n(B(eO(A))^+) \longrightarrow K_n(A) = \pi_n(BGl(A)^+)$$

$$\downarrow$$

$$H_n(eO(A), k) \longrightarrow H_n(Gl(A))$$

$$\downarrow$$

$$\text{HN}_n(k[eO(A)]) \longrightarrow \text{HN}_n(k[Gl(A)])$$

$$\text{colim}_m \text{HN}_n(M_m(A))$$

$$\text{Tr}$$

$$\text{HN}_n(A)$$

$$\downarrow$$

$$\text{HP}_n(A)$$

$$\downarrow$$

$$\text{HC}_{n+2q}(A)$$

Where $\alpha$ is Goodwillie's map ([G, II.3.1]), $\text{Tr}$ is the trace map, and the composite of the right column is the Chern class $\text{ch}_q^n$. In §1 we give appropriate definitions for the $e\text{HP}_*, e\text{HN}_*$ of an involutive algebra, and we provide $M_{2m}(A)$ with an involution $e$ (ex.1.2) which coincides with the standard involution on $eO(A)$. Thus we only need to prove

i) The image of $\alpha: H_n(eO(A), k) \to \text{HN}_n(k[eO(A)])$ lies in $-1\text{HN}_n(k[eO(A)])$. This is proved for any group $G$ (instead of $eO(A)$) in §2

ii) $\text{Tr}: \text{HN}_*(M_{2m}(A)) \to \text{HN}_*(A)$ sends $+1\text{HN}_*(M_{2m}(A))$ to $+1\text{HN}_*(A)$. This proved in §3

§4 is devoted to the study of the compatibility between our Chern classes and the hyperbolic functor $H$. We prove the
THEOREM (4.6). If \( p: \text{HC}_*(A) \to +_1 \text{HC}_*(A) \) denotes the projection, then \( \text{ch}_q^r \circ H = 2p \circ \text{ch}_q^r \).

I am indebted to C. Weibel who had the patience to read the originals and made useful comments.

§1. Definitions and Notations.

1.0. Let \( k \) be a commutative with \( 1/2 \in k \), \( G \) any fixed group; \( R := k[G] \), the group algebra, \( A \) any unital associative \( k \)-algebra. We consider the following chain complexes (\( \otimes : = \otimes_k, \bar{A} : = A/k, \bar{A}^n : = A^\otimes n \))

- The bar resolution of \( k \) as a left \( R \)-module ([C-E])
  \[
  (X_*, \partial) \quad X_n := R \otimes \bar{R}^n \quad (n \geq 0)
  \]
  \[
  \partial(g[g_1, \ldots, g_n]) := g(g_1[g_2, \ldots, g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1, \ldots, g_i] g_{i+1}, \ldots, g_n] +
  \]
  \[
  + (-1)^n [g_1, \ldots, g_{n-1}] g_n)
  \]

- The bar resolution of \( k \) as a right \( R^{op} \)-module
  \[
  (X_*, \partial^{op}) \quad X_n^{op} := \bar{R}^n \otimes R^{op} \quad (n \geq 0)
  \]
  \[
  \partial^{op}([g_1, \ldots, g_n] g) := ([g_2, \ldots, g_n] + b \sum_{i=1}^{n-1} (-1)^i [g_1, \ldots, g_i] g_{i+1}, \ldots, g_n] +
  \]
  \[
  + (-1)^n [g_1, \ldots, g_{n-1}] g_n) g
  \]

- The bar complex of \( G([C-E]) \)
  \[
  (\bar{X}_*, \partial) = k \otimes_R (X_*, \partial) \equiv (X_*, \partial^{op}) \otimes_R k.
  \]

- The Hochschild bar resolution of \( A([C-E]) \) as a left \( A^{op} := A \otimes A^{op} \)-module
  \[
  (U_*(A), b^1) \quad U_n(A) = A \otimes \bar{A}^n \otimes A^{op} \quad (n \geq 0)
  \]
  \[
  b^1(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}
  \]

- The Hochschild bar complex
  \[
  (\bar{U}_*(A), b) = A \otimes_{A^*} (U_*(A), b^1)
  \]

We use the following notations for homology

- The \((k-)\) group homology of \( G \), \( H_n(G) = H_n(\bar{X}_*, \partial) \)
- The Hochschild homology of \( A \), \( \text{HH}_n(A) = H_n(\bar{U}_*(A), b) \)

The following classical complex maps will be considered (see [Ka-I])
\( \tau: \hat{X}_\bullet \to \hat{U}_\bullet(R): \tau([g_i, \ldots, g_n]) = (g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \ldots \otimes g_n \)

\( \pi: \hat{U}_\bullet(R) \to \hat{X}_\bullet: \pi(g_0 \otimes \ldots \otimes g_n) = \begin{cases} [g_1, \ldots, g_n] & \text{if } \prod_{i=1}^n g_i = 1 \\ 0 & \text{otherwise} \end{cases} \)

Following [G], we provide \((\hat{U}_\bullet(A), b)\) with a cyclic module structure and consider its cyclic periodic and negative cyclic homologies, that we denote respectively by

\[ \text{HC}_\bullet(A), \text{HP}_\bullet(A), \text{HN}_\bullet(A). \]

Namely, let \( B: \hat{U}_\bullet(A) \to \hat{U}_{\bullet+1}(A) \) as in [L-Q] and put

\[ V_{p,q}(A) := \hat{U}_{q-p} := \hat{U}_{q-p}(A) = A \otimes \hat{A}^{q-p} q, p \in \mathbb{Z}, q \geq p. \]

Pictorially

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow \\
\ldots & \leftarrow B \hat{U}_5 & \leftarrow B \hat{U}_4 & \leftarrow B \hat{U}_3 & \leftarrow B \hat{U}_2 & \leftarrow B \hat{U}_1 & \leftarrow B \hat{U}_0 \\
B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow \\
\ldots & \leftarrow B \hat{U}_4 & \leftarrow B \hat{U}_3 & \leftarrow B \hat{U}_2 & \leftarrow B \hat{U}_1 & \leftarrow B \hat{U}_0 \\
B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow \\
\ldots & \leftarrow B \hat{U}_3 & \leftarrow B \hat{U}_2 & \leftarrow B \hat{U}_1 & \leftarrow B \hat{U}_0 \\
B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow \\
\ldots & \leftarrow B \hat{U}_2 & \leftarrow B \hat{U}_1 & \leftarrow B \hat{U}_0 \\
B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow \\
\ldots & \leftarrow B \hat{U}_1 & \leftarrow B \hat{U}_0 \\
B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow & B \downarrow \\
\ldots & \leftarrow B \hat{U}_0 \\
p = -2 & p = -1 & p = 0 & p = 1 & p = 2 & p = 3
\end{array}
\]

figure 1. The double complex \( V_{\bullet \bullet} \)

then the periodic homology of \( A \) is the homology of \( W_\bullet(A) = \text{Tot}(V_{\bullet \bullet}(A), b, B) \). Similarly, if \( V_{\bullet \bullet}^+(A) \) (resp. \( V_{\bullet \bullet}^-(A) \)) denotes the second (resp. first) quadrant truncation of \( V_{\bullet \bullet}(A) \), then the negative cyclic (resp. cyclic) homology of \( A \) is the homology of \( W^-(A) = \text{tot}_*(V^-(A)) \) (resp. of \( W^+(A) = \text{tot}_*(V^+(A)) \)).

1.1. Let \( A \) be an involutive \( k \)-algebra with involution "\(-"”, i.e. \( \lambda = \lambda, a\bar{b} = \bar{b}a, \lambda \in k, b \in A \) and \( e = \pm 1 \). Following Loday [Lo] we consider the natural map of chain complexes

\[ y: \hat{U}_\bullet(A) \to \hat{U}_{\bullet}(A): y(a_0 \otimes \ldots \otimes a_n) = (-1)^{n(n+1)} \frac{n}{2} (a_{\bar{0}} \otimes \bar{a}_n \otimes \bar{a}_{n-1} \otimes \ldots \otimes \bar{a}_1) \]
Let $\bar{\mathcal{U}}_n = \{ x \in \mathcal{U}_n : yx = \varepsilon x \}$, $\varepsilon HH_n(A) = H_n(\varepsilon \bar{\mathcal{U}}_n(A))$, the $\varepsilon$-Hochschild homology; then $HH_*(A) = +1 HH_*(A) \oplus -1 HH_*(A)$ ([Lo]). Also from [Lo] we learn that $B(\varepsilon \bar{\mathcal{U}}_*) \subseteq (-\varepsilon \bar{\mathcal{U}}_*)$, so that we can consider the complex

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B & \varepsilon \bar{U}_4 & B & -\varepsilon \bar{U}_3 & B & \varepsilon \bar{U}_2 & B & -\varepsilon \bar{U}_1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B & \varepsilon \bar{U}_3 & B & -\varepsilon \bar{U}_2 & B & \varepsilon \bar{U}_1 & B & -\varepsilon \bar{U}_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B & \varepsilon \bar{U}_2 & B & -\varepsilon \bar{U}_1 & B & \varepsilon \bar{U}_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B & \varepsilon \bar{U}_1 & B & -\varepsilon \bar{U}_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B & \varepsilon \bar{U}_0 \\
\end{array}
\]

Figure 2. The double complex $\varepsilon V_{**}$

Then we put $\varepsilon W_* = \text{tot}_*(\varepsilon V_{**})$, $\varepsilon HP_*(A) = H_*(\varepsilon W(A))$. We can as well define $\varepsilon W^+, \varepsilon W^-, \varepsilon \text{HN}(A), \varepsilon \text{HC}(A)$ in the obvious way; $+1 \text{HC}(A)$ is called the dihedral homology ([Lo]). In the rest of the paper, we will be interested in the following

1.2. EXAMPLES. i) Let $R = k[G]$, as above; we view $R$ as a hermitian algebra with the involution $g \rightarrow g^{-1}$. For reasons that will be uncovered in the next §, the corresponding involution on $\bar{\mathcal{U}}_*(R)$ will be denoted by $y_2$.

ii) Let $n \geq 1$, $M_{2n}(A)$ the matrix ring, $I \in M_n(A)$ the identity matrix, $\varepsilon = \pm 1$, put

$$h = \varepsilon h := \begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix}$$

We consider two involutions on $M_{2n}(A)$;

$$a = (a_{ij}) \rightarrow \bar{a} := (\bar{a}_{ij})$$

$$a \rightarrow \tilde{\varepsilon} a := h^{-1}(\bar{a})h$$

The corresponding involutions on $\bar{\mathcal{U}}_*(M_{2n}(A))$ will be named $y$, $y_\varepsilon$ respectively.

1.3. REMARK. With the notations in the example ii) above, let $G = \varepsilon 0_{n,n}(A) = \{ g \in M_{2n}(A) : \check{g}(\varepsilon h)g = \varepsilon h \}$; then the natural morphism $R \rightarrow (M_{2n}(A), \tilde{\varepsilon})$ is a morphism of hermitian algebras, i.e. $\check{g} = g^{-1} (g \in G)$. 

§2. Relation between $H_*(G)$ and $\text{^1HH}(k[G])$.

2.0. Following the notations in §1, we define $k$-linear maps

$$y'_1: X_n \to X_n^{\text{op}}: y'_1(g[g_1, \ldots, g_n]) = (-1)\frac{n(n+1)}{2} [g_1^{-1}, \ldots, g_n^{-1}] g^{-1}$$

$$y'_2: U_n(R) \to U_n(R): y'_2(g \otimes g_1 \otimes \ldots \otimes g_n \otimes h) = h^{-1} \otimes g_1^{-1} \otimes \ldots \otimes g_n^{-1} \otimes g^{-1}$$

$$\tau': X_n \to U_n(R): g[g_1, \ldots, g_n] \to g \otimes g_1 \otimes \ldots \otimes g_n \otimes (g_1 \ldots g_n)^{-1} g^{-1}$$

$$\eta': X_n^{\text{op}} \to U_n(R): [g_1, \ldots, g_n] \to h^{-1} (g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \ldots \otimes g_n \otimes h$$

$$\Theta: X_n \to X_n^{\text{op}}: \Theta(g[g_1, \ldots, g_n]) = [g_1, \ldots, g_n] (gg_1 \ldots g_n)^{-1}.$$

2.1 Proposition. i) All the above are chain complex maps and induce maps

$$\bar{y}'_1 = y'_1: (\bar{X}, \bar{\partial}) \to (\bar{X}^{\text{op}}, \bar{\partial}^{\text{op}})$$

$$\bar{y}'_2 = y'_2: (\bar{U}, b) \to (\bar{U}, b)$$

$$\bar{\tau}' = \bar{\eta}' = (\bar{X}, \bar{\partial}) \to (\bar{U}, b)$$

$$\bar{\Theta} = \text{identity: } (\bar{X}, \bar{\partial}) \to (\bar{X}, \bar{\partial}).$$

where $\tau, y_2$ are the same as those defined in §1.0.

ii) The following diagram commutes

$$
\begin{array}{ccc}
X_* & \xrightarrow{\tau'} & U_*(R) \\
\downarrow{y'_1} & & \downarrow{y'_2} \\
X_*^{\text{op}} & \xrightarrow{\eta'} & U_*(R)
\end{array}
$$

iii) There is a chain homotopy $h$ such that

$$\theta - y'_1 = h \partial + \partial^{\text{op}} h$$

$$h(gx) = h(x)g^{-1} \text{ for all } g \in G, x \in X_*$$

Proof. All the commutativity issues involved are straightforward. As to the induced maps, it is enough to observe that all $\bar{X}, \bar{U}$ are quotients of $X, U$ respectively and then one checks easily that the above maps come down to those quotients. As to iii), we first need some notations; let $M$ be an $R$-left module, $N$ an $R^{\text{op}}$-right module, $(\dagger)$ a group morphism is called an $R$-skew morphism if $f(gm) = f(m)g^{-1}$ for all $m \in M, g \in G$. For example $\Theta, y'_1$ are $R$-skew morphisms; moreover, if $\varepsilon: X_* \to k, \varepsilon': X_*^{\text{op}} \to k$ denote the standard augmentations, then the square

$(\dagger)$ Or equivalently, an $R$-left module.
is commutative. Then we are done if we prove the following statement: Given a chain complex of free \( R \)-left modules \((C_*, d)\) and an acyclic complex of \( R^{\text{op}} \)-right modules \((D_*, d')\) and a skew morphism \( f: C_0 \to D_0 \) then there exists a chain skew morphism \( \alpha_*: C_* \to D_* \) such that \( \alpha_0 = f \) and \( \alpha_* \) is unique up to skew-homotopy.

The above becomes a well known fact ([C-E]) if we replace "skew morphism" by morphism. We can as well mimic the proof in [C-E]; the only crucial point to prove is that if \( L \) is a left \( R \)-free module and \( M \xrightarrow{\pi} N \to 0 \) is an exact sequence of \( R^{\text{op}} \)-right modules then for any skew morphism \( \alpha: L \to N \) there is a lifting \( \bar{\alpha} \) (i.e. \( \pi \bar{\alpha} = \alpha \)). Now let \( \{e_i; i \in \bar{\alpha}\} \) be a basis for \( L \), \( n_i = \alpha(e_i) \) and pick \( m_i \in M \) such that \( \pi(m_i) = n_i \); then \( \bar{\alpha}(\sum_i g_i e_i) = \sum_i m_i g_i^{-1} \) does the job.

2.2. Corollary. The maps

\[ y_2 \tau \text{ and } \tau: (\bar{X}_*, \partial) \to (\bar{U}_*(R), b) \]

are chain homotopic.

**Proof.** From 2.1. i), iii) we get that \( y_1 \) is homotopic to the identity map \( \hat{\Theta} \). Then i) and ii) yield the desired result.

As a straightforward consequence of 2.2, we get the

2.3. Theorem. Let \( \tau: H_*(G) \to \text{HH}_*(k[G]) \) as in 1.0; then the image of \( \tau \) lies in the subgroup \( _+\text{HH}_*(k[G]) \)

**Proof.** In view of 2.2., both \( \tau \) and \( \frac{\tau + y_2 \tau}{2} \) are the same thing in homology.

2.4. In [G; II-3.1] it is shown that the chain map \( \tau: (\bar{X}_*, \partial) \to (\bar{U}, b) \) has a natural lifting (unique up to natural homotopy)

\[
\begin{array}{ccc}
\bar{X}_* & \xrightarrow{\tau} & \bar{U}_* \\
\alpha \downarrow & & \pi \downarrow \\
W^- & \xrightarrow{\alpha} & \bar{U}_* \\
\end{array}
\]

where \( W^- \) is as in §1 and \( \pi \) is the canonical projection. This result, together with 2.3 give the following
2.5. COROLLARY. Let $\alpha: H_n(G) \to \text{HN}^{-}_{n}(k[G])$ be Goodwillie's map (see 2.4); then the image of $\alpha$ lands in $+_{1}\text{HN}_{n}(k[G])$.

**Proof.** We will go through some steps. First, we prove the existence of a complex map $\tilde{\alpha}: \tilde{X} \to +_{1}W$ (using the notations of §1.1) that makes the following into a commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\alpha}} & +_{1}W \\
pr & \downarrow & \\
+_{1}\tilde{U}(R) & \xrightarrow{\tilde{\alpha}} & +_{1}U(R)
\end{array}
$$

For that sake, we mimic Goodwillie's proof of [G, II.3.1]: the only crucial point is that, if $F_p$ is the free group in $p$ letters, then $H_{n}(+_{1}W^{-}_{*}(F_p))$ must be zero for $n \geq p - 1$. But $H_{n}(+_{1}W^{-}_{*}(F_p))$ is a direct summand of $H_{n}(W^{-}_{*}(F_p))$, which is zero for $n \geq p - 1$, done.

Next, we know from 2.2. that $\tau, y_2 \tau$ are naturally chain homotopic through say, $h$; as a second step, we choose a natural lifting $\tilde{h}$, such that $\pi \tilde{h} = h$. The existence of such a lifting is clear from Goodwillie's arguments, i.e., we can choose a lifting for the free groups and then extend by naturally. Last, let $\Delta: W^{-}_{*} \to W^{-}_{*} - 1$ be the boundary map, and put $\beta = \frac{1}{2}(\tilde{h} \delta + \Delta \tilde{h}), \alpha' = \tilde{\alpha} + \beta$. Then

$$
\pi \alpha' = p\tau + \frac{\pi \tilde{h} \delta + \pi \Delta \tilde{h}}{2} = p\tau + \frac{h \delta + bh}{2} = \frac{\tau + y_2 \tau}{2} + \frac{\tau - y_2 \tau}{2} = \tau
$$

In view of the uniqueness of $\alpha$ (2.4), $\alpha'$ is chain homotopic to $\alpha$, so that $H_{*}(\alpha) = H_{*}(\alpha') = H_{*}(\tilde{\alpha})$, done.

§3. Orthogonal Chern Characters.

Let $A$ be a hermitian $k$-algebra (§1.1), $\varepsilon = \pm 1$ the purpose of this § to define Chern characters

$$
\text{ch}_{q}^{(n)}: L_{n}^{\varepsilon}(A) \to +_{1}\text{HP}_{n}(A) \to +_{1}\text{HC}_{n+2q}(A) \quad (n, q \geq 0)
$$

We first need to have a corresponding Dennis trace map; this is provided by the following

3.0. **Theorem.** Let $n \geq 0$, $A$ a hermitian $k$-algebra, $D$ the Dennis trace map. There is a commutative diagram

$$
\begin{array}{ccc}
L_{n}^{\varepsilon}(A) & \xrightarrow{D'} & +_{1}\text{HH}_{n}(A) \\
\downarrow & & \downarrow i \\
K_{n}(A) & \xrightarrow{D} & \text{HH}_{n}(A)
\end{array}
$$
where the left column is the "forget" morphism, i is the natural split inclusion. In other words, the image of D lies in \( \delta_{1} \text{HH}_{*}(A) \).

In view of 2.0., it seems natural to state the following

3.1. **Definition.** The map \( D' \) of 2.0. will be called the *orthogonal Dennis trace map*.

In order to prove 3.0., we need the following (refer to 1.1 for notations).

3.2. **Lemma**

i) Let \( n \geq 1 \), \( \text{Tr} := \) the trace map; the diagram

\[
\begin{array}{ccc}
\tilde{U}_{*}(M_{n}(A)) & \xrightarrow{\text{Tr}} & \tilde{U}_{*}(A) \\
y & & y \\
\tilde{U}_{*}(M_{n}(A)) & \xrightarrow{\text{Tr}} & \tilde{U}_{*}(A)
\end{array}
\]

is commutative.

ii) Let \( P \in \text{Gl}_{n}(A) \), and consider the map

\[
f_{P} : U_{*}(M_{n}(A)) \rightarrow U_{*}(M_{n}(A)) ; \\
f_{P}(a_{0} \otimes \cdots \otimes a_{n}) := a_{0} P^{-1} \otimes \cdots \otimes P_{n-1} P^{-1} \otimes P_{n}
\]

then \( f_{P} \) is an \( A^{s} \)-chain complex map, \( k \)-linearly chain homotopic to the identity.

iii) Let \( y_{t} : \tilde{U}_{*}(M_{n}(A)) \rightarrow \tilde{U}_{*}(M_{n}(A)) \) be the chain complex involution corresponding to \( \xi \) (cf. 1.1); the following diagram is commutative

\[
\begin{array}{ccc}
\text{HH}_{*}(M_{2n}(A)) & \xrightarrow{\text{Tr}} & \text{HH}_{*}(A) \\
y_{t} & & y \\
\text{HH}_{*}(M_{2n}(A)) & \xrightarrow{\text{Tr}} & \text{HH}_{*}(A)
\end{array}
\]

iv) Let \( m \geq 1 \), \( e \in M_{2m}(A) \), \( \xi h \in M_{2m}(A) \) as in 1.1. and suppose \( e^{2} = e \) and \( (\xi_{e} h(1 - e) = 0 \)

(that is, \( e \) is an \( \varepsilon \)-orthogonal projector) then

\[
\xi e = e.
\]

**Proof.** Let \( r \geq 0 \), \( a_{0} \otimes \cdots \otimes a_{r} \in \tilde{U}_{r}(M_{n}(A)) \), \( a_{l}^{i,j} 0 \leq l \leq r \), \( 1 \leq i,j \leq n \) will denote the \((i,j)\)-th entry of \( a_{l} \). Then

\[
\text{Tr}(y(a_{0} \otimes \cdots \otimes a_{n})) = (-1)^{\frac{n(n+1)}{2}} \sum_{i_{0},\ldots,i_{n}} a_{l}^{i_{0},i_{1}} \otimes a_{n}^{i_{2},i_{1}} \otimes \cdots \otimes a_{1}^{i_{0},i_{n}} =
\]

\[
(-1)^{\frac{n(n+1)}{2}} \sum_{i_{0},\ldots,i_{n}} a_{l}^{i_{0},i_{1}} \otimes a_{n}^{i_{2},i_{1}} \otimes \cdots \otimes a_{1}^{i_{0},i_{n}} = y(\text{Tr}(a_{0} \otimes \cdots \otimes a_{n}))
\]
This proves i).

ii) Since proving that \( f_p \) is a chain map is trivial, we only prove our second assertion. Let \( \mu : M_n(A^e) \to M_n(A) \) be the canonical augmentation of \( U_* \); then
\[
\begin{array}{ccc}
U_*(M_n(A)) & \xrightarrow{\mu} & M_n(A) \\
\downarrow f_p & & \\
U_*(M_n(A)) & \xrightarrow{\mu} & M_n(A)
\end{array}
\]
is commutative. The assertion is now clear since \( \mu \) is a relative projective resolution.

iii) is a trivial consequence of i) and ii), since, for \( P = h = e h, \) if \( f_h^* : \bar{U}_* \to \bar{U}_* \) is the induced map, then \( y_e = f_h^* \circ y. \)

iv) Let \( h = e h; \) then
\[
e - \bar{e} = e - h^{-1} t e h = e - h^{-1} t(e) h e = (1 - h^{-1} t(e)) h e =
\]
\[
= h^{-1} (1 - t(e)) h e = h^{-1} e (t(e) h (1 - e)) = 0
\]

**Proof of 3.1.** First consider the case \( n = 0. \) An element in \( L_n^0(A) \) can be thought of as the equivalence class of a projector \( e \) as in 3.2iv). Then \( D_0(e) \) is the homology class of \( \text{Tr}(e); \) but in view of 3.2 iii), iv)
\[
y \text{Tr}(e) \cong \text{Tr}(y_e(e)) = \text{Tr}(e) = \text{Tr}(e).
\]
As to the case \( n \geq 1, \) 3.2 iii) and 2.4 yield the commutative diagram (recall 1.3.)
\[
L_n^0(A) = \pi_n(B(e O(A))^+) \xrightarrow{\text{forget}} \pi_n(B \text{Gl}(A)^+) = K_n(A)
\]
\[
\xymatrix{H_n(B(e O(A))^+, k) \ar[d] \ar[r] & H_n(B \text{Gl}(A)^+, k) \\
H_n(e O(A), k) \ar[r] \ar[d] & H_n(Gl(A), k) \\
(+) \HH_n(k[e O(A)]) \ar[d] \ar[r] & \HH_n(k(Gl(A))) \\
colim(+) \HH_n(M_m(A)) \ar[r] \ar[d]_{\text{Tr}} & \colim \HH_n(M_m(A)) \ar[d]_{\text{Tr}} \\
+ \HH_n(A) & \HH_n(A)
}
\]
The ordinary Dennis map is the composite of the right side column in the above diagram; the theorem is done by calling \( D' := \) the composite of the left column.

Just as we have done in 2.4., we can extend 3.0. to cyclic homology.

3.3. Theorem. Let

\[
\text{ch}^0_q : K_n(A) \longrightarrow \text{HN}_n(A) \longrightarrow \text{HP}_n(A) \longrightarrow \text{HC}_{n+2q}(A)
\]

be the Karoubi Chern character (see [Ka-I], [We]). Then there is a commutative diagram

\[
\begin{array}{c}
\text{ch}^0_q : K_n(A) \\
\uparrow \\
\text{HN}_n(A) \\
\uparrow \\
\text{HP}_n(A) \\
\uparrow \\
\text{HC}_{n+2q}(A)
\end{array}
\]

\[
\begin{array}{c}
\text{ch}'_q : L^n(A) \\
\uparrow \\
+1\text{HN}_n(A) \\
\uparrow \\
+1\text{HP}_n(A) \\
\uparrow \\
+1\text{HC}_{n+2q}(A)
\end{array}
\]

Proof. Taking into account Karoubi's explicit computation for \( \text{ch}^0_q \), the \( n = 0 \) case is trivial after 3.1. Let \( n \geq 1 \); in view of 1.1., we only need to prove the commutativity of the left side square. Now that square can be decomposed as follows

\[
\begin{array}{c}
L^n(A) \\
\text{Hurewicz} \\
H_n(B_+(O(A))^+, k) \\
\downarrow \\
H_n(\text{Gl}(A), k)
\end{array}
\]

where \( \alpha, \bar{\alpha} \) are as in 2.4. Moreover, 2.4., 3.1 prove that the above is a commutative diagram

\[
\begin{array}{c}
colim_{+1} \text{HC}_{n}^{-}(M_n(A)) \\
\downarrow \text{Tr} \\
+1\text{HC}_{n}^{-}(A) \\
\downarrow \\
\text{HC}_{n}^{-}(A)
\end{array}
\]

\[
\begin{array}{c}
colim_{+1} \text{HC}_{n}^{-}(M_n(A)) \\
\downarrow \text{Tr} \\
+1\text{HC}_{n}^{-}(A) \\
\downarrow \\
\text{HC}_{n}^{-}(A)
\end{array}
\]

\[
\begin{array}{c}
colim_{+1} \text{HC}_{n}^{-}(M_n(A)) \\
\downarrow \text{Tr} \\
+1\text{HC}_{n}^{-}(A) \\
\downarrow \\
\text{HC}_{n}^{-}(A)
\end{array}
\]

where \( \alpha, \bar{\alpha} \) are as in 2.4.
3.4. Definition. The map $\text{ch}_q^{(n)}: L^e_n(A) \to \pm_1 \text{HC}_{n+2q}(A)$ above is the orthogonal Chern class.

§4. Compatibility with the hyperbolic map.

4.0. I am grateful to C. Weibel, who insisted that something like 4.1. should hold. Let $A$ be a hermitian $k$-algebra, $1/2 \in k$; put

$\mathcal{P}(A) :=$ the category of all finitely-generated-projective-right-$A$-modules

and for $\varepsilon = \pm 1$ ([Ka-II])

$\varepsilon \mathcal{Q}(A) :=$ the category of all right-$A$-$\varepsilon$-quadratic modules in the sense of Karoubi.

Recall that the hyperbolic functor is (see for example [Ka-II]).

$H = \varepsilon H : \mathcal{P}(A) \to \varepsilon \mathcal{Q}(A)$

$H(P) = \left( P \oplus \bar{P}, \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix} \right) \quad (P \in \mathcal{P}(A))$

$H(\alpha) = \begin{bmatrix} \alpha & 0 \\ 0 & (\bar{\alpha})^{-1} \end{bmatrix} \quad (\alpha \text{ an isomorphism in } \mathcal{P}(A)).$

where

$\bar{P} = \{ f: P \to A, k \text{ linear } / f(p \lambda) = \bar{\lambda} f(p), p \in P, \lambda \in A \}$

then $H$ induces maps from $K_*(A)$ to $L_*(A)$ and from $H_*(\text{Gl}(A))$ to $H_*(\varepsilon \mathcal{O}(A))$, all of which will be denoted by $H$. We have a commutative diagram

$\begin{array}{ccc}
K_*(A) & \xrightarrow{H} & H_*(\text{Gl}(A)) \\
\downarrow H & & \downarrow H \\
L^e_n(A) & \xrightarrow{H} & H_*(\varepsilon \mathcal{O}(A))
\end{array}$

In what follows, we study the composite map $\text{ch}_q^{(n)} \circ H$.

4.1. Theorem. With the notations of 3.0, 3.1, the following is a commutative diagram

$\begin{array}{ccc}
K_n(A) & \xrightarrow{D} & \text{HH}_n(A) \\
\downarrow H & & \downarrow 1+y \\
L^e_n(A) & \xrightarrow{D'} & \text{HH}_n(A)
\end{array}$

(n $\geq 0$)
PROOF. We first do the case \( n = 0 \). Let \( \pi, Q \in \mathcal{D}(A) \) such that \( P \oplus Q = A^n \) \( (n \geq 1) \), and consider the idempotent matrix \( e \in M_n(A) \) of the projection to \( P \) with respect to the canonical basis. Then \( e \oplus \, ^t \bar{e} \) is the matrix corresponding to the orthogonal projection from \( H(A^n) \) to \( H(P) \). Now 3.2.i) yields \( D'(H(P)) = \text{Tr}(e \oplus \, ^t \bar{e}) = \text{Tr}(e \oplus ye) = \text{Tr}(e) + y\text{Tr}(e) = (1 + y)D(P) \). We now proceed with the case \( n \geq 1 \). Consider the involution on the complex \( \bar{X} \) of 1.0. given by

\[
(4.2) \quad y_3 : \bar{X}(\text{Gl}(A)) \to \bar{X}(\text{Gl}(A)) : y_3 [g_1, \ldots, g_n] = (-1)^{n(n+1)/2} [\, ^t \bar{g}_n, \ldots, \, ^t \bar{g}_1]
\]

It is easy to see that \( y_3 \) is a chain map; moreover

\[
y_1y_3 [g_1, \ldots, g_n] = y_1 \left( (-1)^{n(n+1)/2} [\, ^t \bar{g}_n, \ldots, \, ^t \bar{g}_1] \right) = [(\, ^t \bar{g}_1)^{-1}, \ldots, (\, ^t \bar{g}_n)^{-1}].
\]

Consequently we get that \( H = 1 + y_1y_3 \) as chain morphisms from \( \bar{X}(\text{Gl}(A)) \) to \( \bar{X}(eO(A)) \). We know from 2.1 iii) that \( y_1 \) is homotopic to the identity, so that, at the level of group homology, \( H \) is the same as \( 1 + y_3 \). But if \( g \in eO(A) \), then \( ^t \bar{g} = h(h^{-1}(\, ^t \bar{g})h)h^{-1} = hg^{-1}h^{-1} \), which in view of 3.2. ii) implies that the restriction of \( y_3 \) to \( \bar{X}(O(A)) \) is homotopic to the identity. Summing up, we have a commutative diagram

\[
\begin{array}{ccc}
H_\ast(\text{Gl}(A)) & \xrightarrow{H} & H_\ast(eO(A)) \\
\downarrow & & \downarrow 2 \downarrow \\
H_\ast(O(A)) & \xrightarrow{1 + y_3} & H_\ast(eO(A))
\end{array}
\]

Next, if we let \( y_4 : \bar{U}(k[\text{Gl}(A)]) \to \bar{U}(k[\text{Gl}(A)]) \) be the "\( y \)-map corresponding to the involution \( g \to ^t \bar{g} \) (i.e. \( y_4(g_0 \otimes \ldots \otimes g_n) = (-1)^{n(n+1)/2} (\, ^t \bar{g}_n \otimes \ldots \otimes \, ^t \bar{g}_1) \)). Then it is easy to see that \( \tau y_4 = y_4 \tau \) (\( \tau \) is defined in §1.0), and in view of 3.2.i), we can complete (4.3) to yield

\[
\begin{array}{cccccc}
D : K_\ast(A) & \longrightarrow & H_\ast(\text{Gl}(A)) & \xrightarrow{\tau} & HH_\ast(k[\text{Gl}(A)]) & \xrightarrow{\text{Tr}} & HH_\ast(A) \\
\downarrow H & & \downarrow H & & \downarrow H_y & & \downarrow H_y \\
D' : L_\ast(A) & \longrightarrow & H_\ast(eO(A)) & \xrightarrow{\tau} & \ell HH_\ast(k[e HH_\ast(eO(A))]) & \xrightarrow{\text{Tr}} & HH_\ast(A)
\end{array}
\]

where \( \ell HH_\ast(k[e O(A)]) \) is the "\( + \)"-summand in the decomposition corresponding to the involution \( \alpha \to ^t \bar{\alpha} \) (see 1.1.).

4.4. Consider the involution \( y_3 \) defined in (4.2), and let \( \varepsilon = \pm 1 \). In the spirit of 1.1., it is natural to define \( (t \text{ stands for transpose) \)
and, since we are assuming that $1/2 \in k$,

$$H_{\bullet}(\text{Gl}(A)) = \iota H_{\bullet}(\text{Gl}(A)) \oplus -\iota H_{\bullet}(\text{Gl}(A))$$

Now by choosing the diagram (4.3) we get the

4.5. Corollary. If $1/2 \in k$, then $H_{\bullet}(\epsilon O(A), k)$ is a direct summand of $H_{\bullet}(\text{Gl}(A), k)$. Moreover, with the notations of 3.8,

$$H_{\bullet}(\epsilon O(A), k) = \iota H_{\bullet}(\text{Gl}(A), k).$$

Proof. See 4.4. above.

4.6. Corollary. With the notations of 3.3, 3.4., the following diagram is commutative

$$\begin{array}{ccc}
\text{ch}_q : K_{\bullet}(A) & \longrightarrow & H_{\bullet + 2}(A) \\
\uparrow H & & \uparrow 1+y \\
\text{ch}_q' : L^2_{\bullet}(A) & \longrightarrow & +_1 HD_{\bullet + 2}(A)
\end{array}$$

Proof. The case $n = 0$ is derived from the arguments in the proof of 3.3, taking into account Karoubi's explicit computation of $\text{ch}_{q}^0$ ([Ka-I, 2.17]). Now we go to the case $n \geq 1$. Consider the automorphism $\beta : \text{Gl}(A) \rightarrow \text{Gl}(A)$ $\beta(g) = (\bar{g})^{-1}$; then $\beta$ induces chain automorphisms on both $\bar{X}(\text{Gl}(A))$, $\bar{U}(k[\text{Gl}(A)])$ that we denote also by $\beta$. Next, let $\alpha : H_{\bullet}(\text{Gl}(A)) \rightarrow \text{HN}_{\bullet}(k[\text{Gl}(A)])$ be Goodwillie's map (as in 2.4.); in the proof of 2.5, we showed that $y_2 \alpha = \alpha y_1$; keeping in mind the naturality of $\alpha$, and the fact that $y_2, y_4$ can both be extended to $\text{HN}_{\bullet}$ (see 1.1.), we get

$$\alpha y_3 = (\alpha y_3 y_1) y_1 = (\alpha \beta) y_1 = \beta \alpha y_1 = (y_4 y_2) \alpha y_1 = y_4 \alpha y_1^2 = y_4 \alpha.$$

Now this computation, together with 4.3 and 3.2 i) yield the desired result.

References


MATHEMATICS DEPARTMENT
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903
U.S.A.

CURRENT ADDRESS:
MATH. DEPARTMENT
WASHINGTON UNIVERSITY
ONE BROOKINGS DRIVE
ST. LOUIS, MO 63130
U.S.A.