MATH. SCAND. 73 (1993), 15–20

ROOT MULTIPLICITIES AND IDEALS IN QUASISIMPLE LIE ALGEBRAS

HELMER ASLAKSEN, TERJE WAHL and TORE WENTZEL-LARSEN

1. Introduction.

The purpose of this note is to elaborate on some of the results of a paper by Høegh-Krohn and Torresani [1]. They determined the possible root systems for quasisimple Lie algebras, but they did not discuss the multiplicities of the roots. The nonisotropic roots can only occur with multiplicity one, and the multiplicities of isotropic roots in the affine Lie algebras are known. But for general quasisimple Lie algebras, the multiplicities of the isotropic roots are more complicated. We call the dimension of the span of the isotropic roots the *type* and denote it by v. We will give examples of quasisimple Lie algebras of type two with the same sets of roots but with different multiplicities. This shows that although Høegh-Krohn and Torresani have determined the possible root systems (not counting multiplicities) and given explicit realizations for the root systems of type $v \leqq 2$ it does not follow that we know all quasisimple Lie algebras with $v \leqq 2$.

There is also the question of whether the root system determines the Lie algebra. Høegh-Krohn and Torresani mentioned that V. G. Kac had suggested there might be many Lie algebras corresponding to a given root system when $v > 2$. Our examples will show that the root system (not counting multiplicities) fails to determine the Lie algebra even for $v = 2$. The question of whether the root system *with* multiplicities suffices to determine the isomorphism class of the Lie algebra will not be addressed in this paper.

Høegh-Krohn and Torresani obtained many of their result through extensive use of the Weyl group. But since the isotropic roots are fixed by the Weyl group, it is harder to get infomation about the multiplicities of the isotropic roots. We believe that the multiplicities of the isotropic roots are hard to determine and that it will be difficult to classify all the quasisimple Lie algebras. We will, however, give examples of families of quasisimple Lie algebras with the same root system.

If we say that a Lie algebra $\mathfrak{g}$ is X-simple (e.g., simple, semisimple or quasisimple) we should be able to say something about the ideals (or lack of such) in $\mathfrak{g}$.

Received July 6, 1992.

Let $\mathfrak{g}(A)$ be the Kac-Moody algebra corresponding to an indecomposable generalized Cartan matrix A. It is well-known that any ideal is either contained in the center of $\mathfrak{g}(A)$ or contains the derived algebra $\mathfrak{g}'(A)$. We will show that this holds for some types of quasisimple Lie algebras, but not all.

The first author would like to thank the NDRE for letting him work on Lie algebras while doing part of his military service there. The other authors would like to express their gratitude to the late R. Høegh-Krohn for introducing them to the fascinating realm of quasisimple Lie algebras.

2. Extensions of Quasisimple Lie Algebras.

Let us briefly recall the definition of a quasisimple Lie algebra from [1].

DEFINITION 1. A *quasisimple Lie algebra* is a complex Lie algebra $\mathfrak{g}$ satisfying the following conditions.

1. $\mathfrak{g}$ has a Killing form $\langle , \rangle$, i.e., a nondegenerate invariant symmetric bilinear form.
2. $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$, i.e., a maximal finite-dimensional Abelian subalgebra such that the adjoint of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalizable and the root system is discrete.
3. $\mathrm{Ad}(X)$ is nilpotent for any X in the root space of a nonisotropic root, i.e., a root α such that $\langle \alpha, \alpha \rangle \neq 0$.

Let $\mathfrak{g}$ be a quasisimple Lie algebra with Cartan subalgebra $\mathfrak{h}_0$ and set $h = \dim \mathfrak{h}_0$. Notice that $h = \operatorname{rank} \mathfrak{g} + v$ where v is the type of $\mathfrak{g}$, i.e., the dimension of the span of the isotropic roots. Generalizing a construction from [1], we set

$$\mathfrak{g}_n = \bigoplus_{N \in \mathrm{R}} \mathfrak{g} \otimes Z^N \oplus C^n \oplus D^n$$

where for $N = (N_1, \ldots, N_n)$ we write $Z^N = z_1^{N_1} \ldots z_n^{N_n}$ and C^n and D^n denote the spans of $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$, respectively. If we set $A \cdot C = \sum_{i=1}^n a_i c_i$ and $B \cdot D = \sum_{i=1}^n b_i d_i$ we have

$$[X_N Z^N + A \cdot C + B \cdot D, X'_M Z^M + A' \cdot C + B' \cdot D]$$
$$= [X_N, X'_M] Z^{N+M} + (B \cdot M) X'_M Z^M - (B' \cdot N) X_N Z^N + \delta(N, -M) \langle X_N, X'_M \rangle N \cdot C$$

and

$$\langle X_N Z^N + A \cdot C + B \cdot D, X'_M Z^M + A' \cdot C + B' \cdot D \rangle = \delta(N, -M) \langle X_N, X'_M \rangle + A \cdot B' + B \cdot A'.$$

It is easy to see that $\mathfrak{g}_n$ also is a quasisimple Lie algebra with Cartan subalgebra $\mathfrak{h}_n = \mathfrak{h}_0 \oplus C^n \oplus D^n$. If $\alpha \in \mathfrak{h}_0^*$ is a root of $\mathfrak{g}_0$ we can extend it to a root of $\mathfrak{g}_n$ (that we for simplicity of notation will also denote by α) by defining it to be equal to 0 on $C^n \oplus D^n$. If $N \in Z^n$ we can define an element of $\mathfrak{h}_n^*$ (that we for simplicity will also denote by N) by $N(H, A, B) = B \cdot N$. The multiplicities of the roots of $\mathfrak{g}_n$ are given in Table 1.

root	root space	multiplicity
0	$\mathfrak{h}_n$	$h + 2n$
$N \neq 0$	$\mathfrak{h}_0 Z^N$	h
$\alpha + N$	$\mathfrak{g}_\alpha Z^N$	1

Table 1. Multiplicities of the roots of $\mathfrak{g}_n$

We can also construct

$$\mathfrak{g}_{nm} = (\mathfrak{g}_n)_m = \left\{ \sum_{M \in Z^m} \left(\sum_{N \in Z^n} X_N Z^N \oplus C^n \otimes D^n \right) Z^M \oplus C^m \oplus D^m \right\}$$

where $Z^M = z_{n+1}^{M_1} \ldots z_{n+m}^{M_m}$, C^n is the span of $c_1, \ldots, c_n$, C^m is the span of $c_{n+1}, \ldots, c_{n+m}$ and similarly for D^n and D^m. It is easy to see that

$$\mathfrak{h}_{nm} = (\mathfrak{h}_0 \oplus C^n \oplus D^n) \oplus C^m \oplus D^m$$

is a Cartan subalgebra and we have

$$\begin{aligned}
&[(H + A_1' \cdot C_1 + B_1' \cdot D_1) + A_2' \cdot C_2 + B_2' \cdot D_2, \\
&\quad \textstyle\sum(\sum X_N Z^N + A_1 \cdot C_1 + B_1 \cdot D_1) Z^M + A_2 \cdot C_2 + B_2 \cdot D_2] \\
&= \textstyle\sum[H + A_1' \cdot C_1 + B_1' \cdot D_1, \sum X_N Z^N + A_1 \cdot C_1 + B_1 \cdot D_1] Z^M \\
&\quad + \textstyle(B_2' \cdot M) \sum(\sum X_N Z^N + A_1 \cdot C_1 + B_1 \cdot D_1) Z^M \\
&= \textstyle\sum([H, X_N] + (B_1' \cdot N) X_N + (B_2' \cdot M) X_N) Z^N Z^M \\
&\quad + \textstyle\sum (B_2' \cdot M)(A_1 \cdot C_1) Z^M + \sum (B_2' \cdot M)(\mathrm{B}_1 \cdot \mathrm{D}_1) Z^M.
\end{aligned}$$

We extend the isotropic roots as above, and for $N \in Z^n$ and $M \in Z^m$ we define $(N, M) \in \mathfrak{h}_{n,m}^*$ by $(N, M)(H, A_1, B_1, A_2, B_2) = B_1 \cdot N + B_2 \cdot M$. The multiplicities of the roots of $\mathfrak{g}_{nm}$ are given in Table 2.

Let us now compare $\mathfrak{g}_2$ and $\mathfrak{g}_{11}$. We have

$$\mathfrak{g}_2 \subset \mathfrak{g}_{11}$$

and the elements of the form $(ac_1 + bd_1)z_2^r$ are in $\mathfrak{g}_{11}$ but not in $\mathfrak{g}_2$. The two tables show that the multiplicity of $(0, r)$ is h in $\mathfrak{g}_2$ but $h + 2$ in $\mathfrak{g}_{11}$. Taking $\mathfrak{g}$ to be

root	root space	multiplicity
0	$\mathfrak{h}_{nm}$	$h + 2n + 2m$
$(N, M) \neq (0, M)$	$\mathfrak{h}_0 Z^N Z^M$	h
$(0, M) \neq (0, 0)$	$(\mathfrak{h}_0 + C^n + D^n) Z^M$	$h + 2n$
$\alpha + (N, M)$	$\mathfrak{g}_\alpha Z^N$	1

Table 2. Multiplicities of the roots of $\mathfrak{g}_{nm}$

a finite-dimensional, simple Lie algebra, the table shows that the same root system can occur with different multiplicities even for quasisimple Lie algebras of type two. Hence a classification of the root systems (not counting multiplicities) does not give us all the possible quasisimple Lie algebras. It also shows that V. G. Kac's suggestion that there might be several quasisimple Lie algebras corresponding to a given root system (not counting multiplicities) is true, and in fact it can happen already for quasisimple Lie algebras of type two. The question of whether the root system *with* multiplicities suffices to determine the isomorphism class of the Lie algebra will not be addressed in this paper.

This is in fact just a special example of a more general construction. We can construct the following two hierarchies of quasisimple Lie algebras.

$$\mathfrak{g}_n \subset \mathfrak{g}_{n-1,1} \subset \cdots \subset \mathfrak{g}_{1\ldots1}$$

and

$$\mathfrak{g}_n \subset \mathfrak{g}_{1,n-1} \subset \cdots \subset \mathfrak{g}_{1\ldots1}.$$

All these quasisimple Lie algebras have the same root system.

In general, to each set of positive integers $p_1, \ldots, p_r$ with $\sum p_i = n$, we can form the extension $\mathfrak{g}_{p_1,\ldots,p_r}$. There are 2^{n-1} such extensions and they all have the same root system, but with different multiplicities for the isotropic roots.

If $\mathfrak{g}$ has a root system of the form (R, ν, τ) or $(BC_n, \nu, \tau, \mu_1, \mu_2)$, then $\mathfrak{g}_{p_1,\ldots,p_r}$ will have a root system of the form $(R, \nu + n, \tau)$ or $(BC_n, \nu + n, \tau, \mu_1, \mu_2)$.

3. Ideals in Quasisimple Lie Algebras.

If $\mathfrak{g}(A)$ is a Kac-Moody algebra corresponding to an indecomposable generalized Cartan matrix A, then any ideal is either contained in the center of $\mathfrak{g}(A)$ or contains the derived algebra $\mathfrak{g}'(A)$ ([2, Proposition 1.7 and exercise 1.1] or [4, 1.10]).

Let $\mathfrak{g}$ be a finite-dimensional, simple Lie algebra and let σ be an automorphism of $\mathfrak{g}$ of order k where $k = 1, 2$ or 3. We first decompose $\mathfrak{g}$ in the standard way

$$\mathfrak{g} = \bigoplus_{i=0}^{k-1} \mathfrak{g}_i.$$

We then choose a grading mod k of Z^N for $N \in \mathbf{Z}^n$ by choosing a homomorphism from $\mathbf{Z}^n$ to $\mathbf{Z}$ and considering the grading induced by the standard grading on $\mathbf{Z}$. We then define

$$\mathfrak{g}_n^{(k)} = \bigoplus_{N \in \mathbf{Z}^n} \mathfrak{g}_{N \bmod k} \otimes Z^N \oplus C^n \oplus D^n.$$

For $k = 1$ we get $\mathfrak{g}_n^{(1)} = \mathfrak{g}_n$. We will call $\mathfrak{g}_n$. We will call $\mathfrak{g}_n$ the *untwisted extension* of $\mathfrak{g}$. For $k = 2$ or 3 we will call $\mathfrak{g}_n^{(k)}$ the *twisted extension* of $\mathfrak{g}$. These xtensions are the quasisimple Lie algebras described in Theorems 20 and 21 in [1]. For $n = 1$ they coincide with the affine Lie algebras $\mathfrak{g}(X_m^{(k)})$ ([2]).

It is easy to see that the center of $\mathfrak{g}_n^{(k)}$ is $\mathscr{Z}(\mathfrak{g}_n^{(k)}) = C^n$ and the derived algebra is $\bigoplus_{N \in \mathbf{Z}^n} \mathfrak{g}_{N \bmod k} \oplus Z^N \oplus C^n$. The following was proved in [3, Theorem 1].

PROPOSITION 1. *Let $\mathfrak{g}_n^{(k)}$ be a untwisted or twisted extension of a finite-dimensional, simple Lie algebra $\mathfrak{g}$. Any ideal in $\mathfrak{g}_n^{(k)}$ is either contained in the center or contains the derived algebra. In particular, any ideal in $\mathfrak{g}_n^{(k)}$ has either finite dimension or finite codimension.*

Unfortunately, this does not generalize to arbitrary quasisimple Lie algebras. If $\mathfrak{g}$ is an arbitrary quasisimple Lie algebra, then it is easy to see that the center of $\mathfrak{g}_n$ is $\mathscr{Z}(\mathfrak{g}_n) = \mathscr{Z}(\mathfrak{g}) \oplus C^n$ and the derived algebra of $\mathfrak{g}_n$ is $\mathfrak{g}_n' = \bigoplus_{N \in \mathbf{Z}^n} \mathfrak{g} \otimes Z^N \oplus C^n$. But then

$$I = \sum_{M \in \mathbf{Z}^m} \mathscr{Z}(\mathfrak{g}_n) \otimes Z^M \oplus C^m \subset \mathfrak{g}_{nm}$$

is an ideal of both infinite dimension and infinite codimension in $\mathfrak{g}_{nm}$ with

$$\mathscr{Z}(\mathfrak{g}_{nm}) \subset I \subset \mathfrak{g}_{nm}'.$$

This shows that in general quasisimple Lie algebras are very far from being simple, and in this respect the name "quasisimple" may not be that appropriate.

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DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE 0511
SINGAPORE

NORWEGIAN DEFENCE RESEARCH ESTABLISHMENT
P.O. BOX 25
N-2007 KJELLER
NORWAY

BODØ GRADUATE SCHOOL OF BUSINESS
P.O. BOX 6003
N-80016 MØRKVED
NORWAY