TRIVIAL FIXED POINT SUBALGEBRAS
OF THE ROTATION ALGEBRA

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Abstract.

Here we prove that the fixed point subalgebras of the rotation algebra \( \mathcal{A}_\theta \) under parabolic and hyperbolic automorphisms induced by \( \text{SL}(2, \mathbb{Z}) \) in the standard representation are trivial.

In this note we study the fixed point subalgebras of the rotation algebra \( \mathcal{A}_\theta \), the universal \( C^* \)-algebra generated by two unitaries \( U \) and \( V \) satisfying \( VU = \rho UV \) with \( \rho = e^{2\pi i \theta} \) and \( 0 \leq \theta < 1 \), induced by \( \text{SL}(2, \mathbb{Z}) \), where any \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \) induces the automorphism \( \tau_A \) of \( \mathcal{A}_\theta \),

\[ \tau_A(U) = e^{ac\pi i \theta} U^a V^c, \quad \tau_A(V) = e^{bd\pi i \theta} U^b V^d, \]

see [11]. We will prove the results we announced in [5], concerning the fixed point subalgebras of the infinite order automorphisms of \( \mathcal{A}_\theta \), namely the parabolic and the hyperbolic ones (See Theorem 2.0.8 of [5]). These results are valid for any \( \theta \in [0, 1) \). If \( A \in \text{SL}(2, \mathbb{Z}) \), we shall denote by \( \mathcal{A}_{\theta}^A \) the fixed point subalgebra of the automorphism \( \tau_A \) of \( \mathcal{A}_\theta \).

Many new results concerning the fixed point subalgebras associated to elements of \( \text{SL}(2, \mathbb{Z}) \) have been found recently. In [1] and [2] Bratteli, Elliott, Evans and Kishimoto started studying \( \mathcal{A}_\theta^{-t_2} \), \( \theta \in [0, 1) \). Subsequently Kumjian [10] computed the \( K \)-theory of \( \mathcal{A}_\theta^{-t_2} \) for \( \theta \) irrational. In [6], [7], [8] and [9] we give a characterization of the fixed point subalgebra associated to any finite order, i.e. elliptic, element of \( \text{SL}(2, \mathbb{Z}) \), respectively in the rational and irrational case. Very recently in [3] Bratteli and Kishimoto have shown that \( A_\theta^{-t_2} \) is an AF algebra, while in [4] Elliott and Evans have shown that \( \mathcal{A}_\theta \) is an inductive limit of direct sums of two circle algebras (both results require \( \theta \) to be irrational). There are still many open problems. For example an interesting question to ask is if the fixed point subalgebras of the elliptic elements are AF for \( \theta \) irrational. Unfortunately

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the techniques in [3] and [4] are not directly applicable in these more general examples.

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We now describe the situation for the (parabolic and hyperbolic) infinite order elements of SL(2, Z). Recall the following proposition from [5].

**Proposition.** If $A$ is an infinite order element of $SL(2, \mathbb{Z})$, then $|\text{Trace}(A)| \geq 2$. Moreover if $A \in SL(2, \mathbb{Z})$, $|\text{Trace}(A)| = 2$, and $A^n = I_2$ for any $n \in \mathbb{Z} \setminus \{0\}$, then $A$ is conjugate in $SL(2, \mathbb{Z})$ to $\pm W^n$ for some $n \in \mathbb{Z} \setminus \{0\}$, where $W = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

**Theorem.** 1. If $A \in SL(2, \mathbb{Z})$, $A \neq I_2$, and $\text{Trace}(A) = 2$, then $\mathcal{A}_A^A \cong C(S^1)$.

2. If $A \in SL(2, \mathbb{Z})$, $A \neq -I_2$, and $\text{Trace}(A) = -2$, then $\mathcal{A}_A^A \cong C([-2, +2])$.

3. If $A \in SL(2, \mathbb{Z})$, and $|\text{Trace}(A)| > 2$, then $\mathcal{A}_A^A \cong \mathbb{C}$.

For the proof of 3. of this theorem in the irrational case see also [11].

**Proof.** 1. Since $A$ is conjugate to $W^k$ for some $k \in \mathbb{Z} \setminus \{0\}$, we consider the case in which

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad \tau_A(U) = U, \quad \tau_A(V) = e^{\pi i\theta k} U^k V, \quad k \in \mathbb{Z}, \ k \neq 0.$$ 

Let $x$ be a fixed point of $\tau_A$ and take its Fourier expansion,

$$x = \sum_{n,m \in \mathbb{Z}} c_{n,m} U^n V^m.$$ 

So,

$$\tau_A(x) = \sum_{n,m} c_{n,m} \tau_A(U^n V^m) = \sum_{n,m} c_{n,m} e^{\pi i\theta km^2} U^n V^m.$$ 

Therefore, by recursion,

$$\tau_A^K(x) = \sum_{n,m} c_{n,m} e^{\pi i\theta km^2} U^{n+Kmk} V^m, \quad \forall K \in \mathbb{N}.$$ 

But if $x \in \mathcal{A}_A$ is a fixed point of $\tau_A$, it follows by equating coefficients that,

$$c_{n,m} = c_{n+Kmk,m} e^{-\pi i\theta km^2}, \quad \forall K \in \mathbb{N}.$$ 

Thus,

$$\|c_{n,m}\| = \|c_{n+Kmk,m}\|.$$ 

But by Riemann-Lebesgue's lemma, we have $c_{n,m} \to 0$ as the indices tend to
infinity, so \( c_{n,m} = 0 \) for all \( m \neq 0 \). Therefore \( x = \sum c_n U^n \), i.e. the fixed point subalgebra of \( \tau_A \) is \( C(S^1) \). This proves 1.

2. Since \( A \) is conjugate to \(-W^k\) for some \( k \in \mathbb{Z} \setminus \{0\} \), we now consider the case in which \( A = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}, k \neq 0 \). Then \( A \) induces the automorphism \( \tau_A \) of \( \mathcal{A}_\theta \) defined by \( \tau_A(U) = U^{-1}, \tau_A(V) = e^{-\pi i \theta} U^k V^{-1} \). Let \( x \) be a fixed point of \( \tau_A \) and let,

\[
x = \sum_{n,m \in \mathbb{Z}} c_{n,m} U^n V^m,
\]

be the Fourier decomposition of \( x \). Then one computes by recursion,

\[
\tau_A^{2K}(x) = \sum c_{n,m} \rho^{-m^2 \theta K} U^{n-2K m} V^m, \forall K \in \mathbb{N},
\]

\[
\tau_A^{2K+1}(x) = \sum c_{n,m} e^{-(2K+1)\pi i \theta km^2} U^{(2K+1)m} V^m, \forall K \in \mathbb{N}.
\]

Since \( x \in \mathcal{A}_\theta \) is a fixed point of \( \tau_A \), one obtains by equating coefficients,

\[
c_{n,m} = c_{n-2K m, m} \rho^{m^2 \theta K}, \text{ and}
\]

\[
c_{n,m} = c_{(2K+1) m, m} e^{(2K+1)\pi i \theta km^2}, \forall K \in \mathbb{N}.
\]

Therefore \( c_{n,m} = 0 \) for all \( m \neq 0 \) and \( c_{n,0} = c_{-n,0} \) for all \( n \). Thus, \( x = \sum_{n \geq 0} c_{n,0} (U^n + U^{-n}) - c_{0,0} \), so that \( \mathcal{A}_\theta^A \cong C^*(U + U^*) \cong C[-2, +2] \). This proves 2.

3. Now suppose that,

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

where \(|T| > 2, T = \text{Trace}(A)\). Note that \( (\tau_A)^K = \tau_{(A^K)} \), and so we will compute \( \tau_{(A^K)} \). By induction on \( K \geq 1 \),

\[
A^K = \begin{pmatrix} S_K a - S_K b & S_K b \\ S_K c & S_K d - S_K b \end{pmatrix},
\]

where \( S_0 = 0, S_1 = 1 \) and \( S_{K+1} = S_K T - S_K -1 \). Note that \( b \) and \( c \) cannot be zero since the requirement that \(|a + d| > 2 \) and \( ad = 1 \) is impossible in \( \mathbb{Z} \). Similarly, if \( K \geq 1 \),

\[
A^{-K} = \begin{pmatrix} S_K d - S_K b & -S_K b \\ -S_K c & S_K a - S_K b \end{pmatrix}.
\]

So if \( x = \sum_{n,m \in \mathbb{Z}} c_{n,m} U^n V^m \) is a fixed point of \( \tau_A \), then using \( \tau_{A^K} \), one computes, as in cases 1 and 2, that
\[ \|c_{n,m}\| = \|c(S_Kd - S_{K^{-1}})n - S_Kbm, -S_Kcn + (S_Ka - S_{K^{-1}})m\| \]
\[ = \|c_{S_KX - S_{K^{-1}}Y, S_KX' - S_{K^{-1}}Y'}\|, \forall K \geq 1, \]

where \( X = dn - bm, X' = -cn + am, Y = n \) and \( Y' = m \). Note that \( X, X', Y \) and \( Y' \) are integers and if \((n,m) \neq (0,0)\), then \((X, Y), (X', Y') \neq (0,0)\). Now,
\[ S_K = \frac{1}{\sqrt{T^2 - 4}} \left[ \left( \frac{T + \sqrt{T^2 - 4}}{2} \right)^K - \left( \frac{T - \sqrt{T^2 - 4}}{2} \right)^K \right], |T| > 2, \]
so assuming \((X, Y) \neq (0,0)\) we have,
\[ S_KX - S_{K^{-1}}Y = \frac{1}{2\sqrt{T^2 - 4}} \left( \frac{T + \sqrt{T^2 - 4}}{2} \right)^{K-1} [(T + \sqrt{T^2 - 4})X - 2Y] - \]
\[ \frac{1}{2\sqrt{T^2 - 4}} \left( \frac{T - \sqrt{T^2 - 4}}{2} \right)^{K-1} [(T - \sqrt{T^2 - 4})X - 2Y]. \]

But as \(|T| > 2\), at least one of the numbers \( \frac{T \pm \sqrt{T^2 - 4}}{2} \) has absolute value greater than one, and since the product of these numbers is one, the other has absolute value less than one. Since \( T \) is an integer with \(|T| > 2\), \( T \pm \sqrt{T^2 - 4} \) are irrational and hence \((T \pm \sqrt{T^2 - 4})X - 2Y \neq 0\). Thus \( \lim_{K \to \infty} |S_KX - S_{K^{-1}}Y| = \infty. \)

In combination with the above relation for \( \|c_{n,m}\| \), this implies \( \|c_{n,m}\| = 0 \forall (n,m) \neq (0,0) \). So any fixed point is a constant, i.e., \( \mathcal{A}_\theta^4 = C \).

This ends the proof of the Theorem.

REFERENCES


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