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PENROSE'S TENSORS. II.

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This is a direct continuation of calculations from [1]. The notations and reference are those of [1].

Let $V = V(m|0)$ and $U = U(0|n)$ be the standard (identity) $\mathfrak{gl}(m)$- and $\mathfrak{gl}(n)$-modules. (Hereafter $\mathfrak{gl}(m) = \mathfrak{gl}(m|0)$, $\mathfrak{gl}(n) = \mathfrak{gl}(0|n)$, etc.)

In what follows we will consider the standard (compatible) $\mathbf{Z}$-grading of $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m \leqq n$ and let the degrees of all even roots be zero. This yields the $\mathbf{Z}$-grading of the form:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \text{ where } \mathfrak{g}_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbf{C},\ \mathfrak{g}_{-1} = \mathfrak{g}_1^* = U \otimes V^*.$$

Let $\hat{\mathfrak{g}}_0$ be the Levi subalgebra of $\mathfrak{g}_0$, i.e., $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n)$. The weights are given with respect to the bases $\varepsilon_1, \ldots, \varepsilon_m$ and $\delta_1, \ldots, \delta_n$ of the dual spaces to the maximal tori of $\mathfrak{gl}(m|n)$. Let $e_1, \ldots, e_m$ be the weight basis of V and $f_1, \ldots, f_n$ be the weight basis of U. Let $\tilde{e}_1, \ldots, \tilde{e}_m$ and $\tilde{f}_1, \ldots, \tilde{f}_n$ be the bases of the dual spaces to V and U, respectively, normed so that $\tilde{e}_i(e_j) = \tilde{f}_i(f_j) = \delta_{ij}$. If $\oplus_\lambda k_\lambda V_\lambda$ is a direct sum of irreducible $\mathfrak{g}_0$-modules (here k_λ is the multiplicity of V_λ) with highest weight λ, denote by v_λ^i the highest weight vectors of the corresponding components: $i = 1, \ldots, k_\lambda$. We will often represent the elements of $\mathfrak{gl}(m|n)$ by the matrices

$$X = \operatorname{diag}(A, D) + \operatorname{antidiag}(B, C)$$

where the dimensions of the matrices A, B, C, and D are $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively. Denote by $A_{i,j}$ the matrix X whose components B, C, and D are zero and all the entries of A are also zero except for the (i,j)th. The matrices $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$ are defined similarly.

Denote by S^iV and Λ^iV the ith symmetric and exterior powers of a vector (super)space V, respectively. Set $S^*V = \oplus_{i \geqq 0} S^iV$.

Let $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ be the Cartan prolongation of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

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1. Theorems.

1.1. THEOREM.
a) *If* $m = 1, n > 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(0|n)$, $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{svect}(0|n)$;
b) *if* $m, n > 1$ *and* $m \neq n$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$, $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0$;
c) *if* $m = n = 2$, *then* $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{h}(0|4)$, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{h}(0|4) \ltimes S^*(\mathfrak{g}^*_{-1})$;
d) *if* $m = n > 2$, *then* $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{psl}(n|n)$, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{psl}(n|n) \ltimes S^*(\mathfrak{g}^*_{-1})$.

1.2. THEOREM. a) If $m = 1, n > 1$, *then*

$$H^{k,2}_{\mathfrak{g}_0} = 0 \text{ for any } k > 0;$$

$$H^{k,2}_{\hat{\mathfrak{g}}_0} = \Pi^n(\mathbf{C})\delta_{kn}.$$

b) *If* $m, n > 1$, *then* $H^{k,2}_{\mathfrak{g}_0} = 0$ *for* $k > 2$ *and the* $\mathfrak{g}_0$-*modules* $H^{1,2}_{\mathfrak{g}_0}$ *and* $H^{2,2}_{\mathfrak{g}_0}$ *are the direct sums of irreducible submodules whose highest weights are given in Table 1. If* $m = n$, *then* $H^{k,2}_{\hat{\mathfrak{g}}_0} = H^{k,2}_{\mathfrak{g}_0}$ *for any k and if* $m \neq n$, then $H^{1,2}_{\hat{\mathfrak{g}}_0} = H^{1,2}_{\mathfrak{g}_0}$ *whereas*

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \text{ if either } m = 2 \text{ or } n = 2;$$

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n} \text{ if } m, n > 2.$$

2. Proof of Theorem 1.1.

2.1. $m = 1, n \geqq 2$. Then $\mathfrak{g}_0 = \mathfrak{sl}(n) \oplus \mathbf{C} = \mathfrak{gl}(n)$ and $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(n)$, where $\mathfrak{g}_{-1}$ is the standard $\mathfrak{g}_0$ (or $\hat{\mathfrak{g}}_0$) module. Therefore, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(0|n)$, $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{svect}(0|n)$.
Notice that if $m \neq n$, then

$$\mathfrak{sl}(m|n) \subset (\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \tag{2.1.1}$$

and if $m = n$, then

$$\mathfrak{psl}(n|n) \subset (\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*. \tag{2.1.2}$$

Indeed, the Lie superalgebras $\mathfrak{sl}(m|n)$, where $m \neq n$, and $\mathfrak{psl}(n|n)$ are simple and therefore, they are transitive (i.e., if there exists $g \in \mathfrak{g}_i$ $(i \geqq 0)$ such that $[\mathfrak{g}_{-1}, g] = 0$, then $g = 0$. It follows that $\mathfrak{g}_1$ is embedded into $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ (or $\hat{\mathfrak{g}}_0 \otimes \mathfrak{g}^*_{-1}$). The Jacobi identity implies $\mathfrak{g}_1 \subset \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}^*_{-1}$.

2.2. *Calculation of the first term of the Cartan prolongation for* $m, n \geqq 2, m \neq n$. Let $\mathfrak{g}'_1$ be the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. Let us show that $\mathfrak{g}'_1 = \mathfrak{g}_1$. By definition,

$\mathfrak{g}'_1 = (\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}) \cap (\mathfrak{g}_{-1} \otimes S^2\mathfrak{g}^*_{-1})$, where, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} \cong [(V \otimes V^*)/\mathbf{C} \oplus (U \otimes U^*)/\mathbf{C} \oplus \mathbf{C}] \otimes (U^* \otimes V).$$

Note that if $g \in \mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$, then

$g \in \mathfrak{g}_{-1} \otimes S^2 \mathfrak{g}^*_{-1}$ if and only if $g(g_1)(g_2) = -g(g_2)(g_1)$ for any $g_1, g_2 \in \mathfrak{g}_{-1}$, since $\mathfrak{g}_{-1}$ is purely odd.

LEMMA. *The* $\mathrm{gl}(m) \oplus \mathrm{gl}(n)$*-module* $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ *is the direct sum of irreducible submodules whose highest weights and highest vectors are listed in Table 2.*

PROOF. The proof of the Lemma consists of:

a) a verification of the fact that vectors v_λ from Table 2 are indeed highest with respect to $\mathrm{gl}(m) \oplus \mathrm{gl}(n)$, i.e. $A_{i,j} v_\lambda = D_{i,j} v_\lambda = 0$ for $i < j$;

b) a calculation of dimension of $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ and of dimensions of the irreducible submodules of $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ by the formula from the Appendix.

Let us show that if

$$\lambda = 2\varepsilon_1 - \varepsilon_m - \delta_n,\ \varepsilon_1 + \delta_1 - 2\delta_n,\ \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n \text{ (if } m \geqq 3),$$

or

$$\lambda = \varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n \text{ (if } n \geqq 3),$$

then $v_\lambda \notin \mathfrak{g}'_1$. For this it suffices to indicate $g_1, g_2 \in \mathfrak{g}_{-1}$ such that

$$v_\lambda(g_1)(g_2) \neq -v_\lambda(g_2)(g_1) \tag{2.2.1}$$

or, perhaps, there exists just one $g \in \mathfrak{g}_{-1}$ such that

$$v_\lambda(g)(g) \neq 0. \tag{2.2.2}$$

Let $\lambda = 2\varepsilon_1 - \varepsilon_m - \delta_n$. Then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = A_{1,m}(f_n \otimes \tilde{e}_1) = -f_n \otimes \tilde{e}_m \neq 0.$$

If $\lambda = \varepsilon_1 + \delta_1 - 2\delta_n$, then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = D_{1,n}(f_n \otimes \tilde{e}_1) = f_1 \otimes \tilde{e}_1 \neq 0.$$

If $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$ (for $m \geqq 3$), then

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = A_{1,m}(f_{n-1} \otimes \tilde{e}_1) = -f_{n-1} \otimes \tilde{e}_m,$$

but
$$v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = 0.$$

Finally, if $\lambda = \varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$ (for $n \geqq 3$), then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = D_{1,n-1}(f_{n-1} \otimes \tilde{e}_2) = f_1 \otimes \tilde{e}_2,$$

but
$$v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = 0.$$

Now, let us show that if $\lambda = \varepsilon_1 - \delta_n$, then $\mathfrak{g}'_1$ contains precisely one irreducible $\mathrm{gl}(m) \oplus \mathrm{gl}(n)$-module with highest weight λ. Notice that by (2.1.1) $\mathfrak{g}'_1$ contains at least one such module. Let

$$v_\lambda = k_1 v_\lambda^1 + k_2 v_\lambda^2 + k_3 v_\lambda^3, \quad \text{where } k_1, k_2, k_3 \in \mathbf{C},$$

be a linear combination of highest vectors of weight λ. Then the condition

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = -v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2)$$

implies

(2.2.3) $$mk_1 = nk_2,$$

whereas the condition

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = 0$$

implies

$$k_1(m-1) + k_2(1-n) + k_3(m-n) = 0.$$

Hence,

(2.2.4) $$k_2 = mk_1/n \text{ and } k_3 = -k_1/n.$$

Thus, $\mathfrak{g}_1' = V_{\varepsilon_1 - \delta_n}$ and $\mathfrak{g}_1' = \mathfrak{g}_1$.

2.3. *Calculation of the second term of the Cartan prolongation for* $m, n \geqq 2$, $m \neq n$. Let $\mathfrak{g}_2$ be the second term of the Cartan prolongation of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. Let us show that $\mathfrak{g}_2 = 0$. Indeed, by definition, $\mathfrak{g}_2 = (\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_0 \otimes S^2 \mathfrak{g}_{-1}^*)$. Notice that, as $\mathfrak{g}_0$-module,

$$\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^* \cong (U^* \otimes V) \otimes (U^* \otimes V) =$$
$$S^2 U^* \otimes S^2 V \oplus \Lambda^2 U^* \otimes \Lambda^2 V \oplus \Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V.$$

This decomposition and Table 5 of [OV] imply the following

LEMMA. *The* $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$*-module* $(U^* \otimes V) \otimes (U^* \otimes V)$ *is the direct sum of irreducible submodules whose highest weights and the corresponding highest vectors are listed in Table 3.*

Let us show that $v_\lambda \notin \mathfrak{g}_2$, where v_λ is any of the highest vectors listed in Table 3. Let us indicate $g_1, g_2 \in \mathfrak{g}_{-1}$ for which either (2.2.1) or (2.2.2) holds.
Let $\lambda = 2\varepsilon_1 - 2\delta_n$. Then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = B_{1,n}(f_n \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2, \text{ but } v_\lambda(f_n \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = 0.$$

If $\lambda = \varepsilon_1 + \varepsilon_2 - 2\delta_n$, then

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_2) = B_{1,n}(f_n \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2 \neq 0.$$

If $\lambda = 2\varepsilon_1 - \delta_{n-1} - \delta_n$, then

$$v_\lambda(f_{n-1}\otimes\tilde e_1)(f_n\otimes\tilde e_2) = -B_{1,n}(f_n\otimes\tilde e_2) = -e_1\otimes\tilde e_2, \text{ but}$$

$$v_\lambda(f_n\otimes\tilde e_2)(f_{n-1}\otimes\tilde e_1) = 0.$$

Let $\lambda = \varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$. Then if $n>2$, we have

$$v_\lambda(f_{n-1}\otimes\tilde e_2)(f_1\otimes\tilde e_1) = B_{1,n}(f_1\otimes\tilde e_1) = f_1\otimes\tilde f_n,$$

but $v_\lambda(f_1\otimes\tilde e_1)(f_{n-1}\otimes\tilde e_2) = 0$. If $m>2$, then

$$v_\lambda(f_{n-1}\otimes\tilde e_2)(f_n\otimes\tilde e_m) = B_{1,n}(f_n\otimes\tilde e_m) = e_1\otimes\tilde e_m,$$

but $v_\lambda(f_n\otimes\tilde e_m)(f_{n-1}\otimes\tilde e_2)=0$. Therefore, $\mathfrak{g}_2=0$ and $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*=\mathfrak{g}$. Note that by (2.2.4) we have $(\mathfrak{g}_{-1},\hat{\mathfrak{g}}_0)_*=\mathfrak{g}_{-1}\oplus\hat{\mathfrak{g}}_0$. This proves part b) of Theorem 1.1.

2.4. $m=n$. Let $m=n=2$. Since $\hat{\mathfrak{g}}_0=\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)=\mathfrak{o}(4)$ and $\mathfrak{g}_{-1}$ is the standard $\mathfrak{o}(4)$-module (considered as purely odd superspace), then $(\mathfrak{g}_{-1},\hat{\mathfrak{g}}_0)_*=\mathfrak{h}(0|4)$.
Let $m=n>2$ and $\mathfrak{g}_1'$ be the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\hat{\mathfrak{g}}_0)$.
Let us show that $\mathfrak{g}_1'=\mathfrak{g}_1$. Indeed, by (2.1.2) $\mathfrak{g}_1\subset\mathfrak{g}_1'$. By sec.2.2 and Table 2 we see that the only highest weights of $\mathfrak{g}_1'$ are all equal to $\varepsilon_1-\delta_n$. Then formula (2.2.3) implies that the highest vector of such weight in $\mathfrak{g}_1'$ is precisely one and therefore, $\mathfrak{g}_1'=\mathfrak{g}_1$. By sec.2.3 the second term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\hat{\mathfrak{g}}_0)$ is zero. Hence, for $m=n>2$ we have $(\mathfrak{g}_{-1},\hat{\mathfrak{g}}_0)_*=\mathfrak{psl}(n|n)$.
Let $m=n>1$ and let $\mathfrak{g}_k$ be the kth term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\mathfrak{g}_0)$. Recall that $\mathfrak{g}_k=(\mathfrak{g}_0\otimes S^k\mathfrak{g}_{-1}^*)\cap(\mathfrak{g}_{-1}\otimes S^{k+1}\mathfrak{g}_{-1}^*)\,(k\geqq 1)$. Observe that

$\mathfrak{g}_0\otimes S^k\mathfrak{g}_{-1}^*=(\hat{\mathfrak{g}}_0\oplus\langle z\rangle)\otimes S^k\mathfrak{g}_{-1}^*$, where $z=1_{2n}$ is the center of $\mathfrak{sl}(n|n)$.

Note that

$$\langle z\rangle\otimes S^k\mathfrak{g}_{-1}^*\subset\mathfrak{g}_{-1}\otimes S^{k+1}\mathfrak{g}_{-1}^*.$$

Therefore,

$$(\mathfrak{g}_{-1},\mathfrak{g}_0)_*=(\mathfrak{g}_{-1},\hat{\mathfrak{g}}_0)_*\ltimes S^*(\mathfrak{g}_{-1}^*).$$

This proves Theorem 1.1.

3. Proof of Theorem 1.2.

3.1. Lemma ([St]). *Let $\mathfrak{g}_{-1}$ be a faithful module over a Lie superalgebra $\mathfrak{g}_0$, and let*

$$\mathfrak{g}_{k-1}\otimes\mathfrak{g}_{-1}^*\xrightarrow{\partial_{\mathfrak{g}_0}^{k+1,1}}\mathfrak{g}_{k-2}\otimes\Lambda^2\mathfrak{g}_{-1}^*\xrightarrow{\partial_{\mathfrak{g}_0}^{k,2}}\mathfrak{g}_{k-3}\otimes\Lambda^3\mathfrak{g}_{-1}^*\quad(k\geqq 1)$$

be the corresponding Spencer cohomology sequence. Then

$$\operatorname{Im}\partial_{\mathfrak{g}_0}^{k+1,1}\cong(\mathfrak{g}_{k-1}\otimes\mathfrak{g}_{-1}^*)/\mathfrak{g}_k. \tag{3.1.1}$$

The proof easily follows from the definitions.

3.2. *Calculation of $H^{1,2}_{\mathfrak{g}_0}$ and $H^{1,2}_{\hat{\mathfrak{g}}_0}$ for $m, n \geqq 2$, $m \neq n$.* For $k = 1$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0.$$

Observe that

$$\begin{aligned}
&\mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-1} = (U \otimes V^*) \otimes \Lambda^2(U^* \otimes V) \cong \\
&(U \otimes V^*) \otimes (\Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V) \cong \\
&(\Lambda^2 U^* \otimes U) \otimes (S^2 V \otimes V^*) \oplus (S^2 U^* \otimes U) \otimes (\Lambda^2 V \otimes V^*), \\
&\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = (V \otimes V^*/\mathbf{C} \oplus U \otimes U^*/\mathbf{C} \oplus \mathbf{C}) \otimes (U^* \otimes V), \\
&\mathfrak{g}_1 = U^* \otimes V.
\end{aligned}$$

Therefore, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

(3.2.1) $$\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} \cong (V \otimes V^*/\mathbf{C} \oplus U \otimes U^*/\mathbf{C}) \otimes (U^* \otimes V)$$

and

(3.2.2) $$H^{1,2}_{\mathfrak{g}_0} \cong (\Lambda^2 U^* \otimes U/U^*) \otimes (S^2 V \otimes V^*/V) \oplus (S^2 U^* \otimes U/U^*) \otimes (\Lambda^2 V \otimes V^*/V).$$

Note that

$$\begin{aligned}
S^2 V \otimes V^*/V &= V_{2\varepsilon_1 - \varepsilon_m}, \\
\Lambda^2 V \otimes V^*/V &= V_{\varepsilon_1 + \varepsilon_2 - \varepsilon_m} \text{ for } m > 2, \\
\Lambda^2 V \otimes V^*/V &= 0 \text{ for } m = 2.
\end{aligned}$$

Since U is purely odd, we deduce with the help of Table 5 of [OV] that

$$\begin{aligned}
\Lambda^2 U^* \otimes U/U^* &= V_{\delta_1 - 2\delta_n}, \\
S^2 U^* \otimes U/U^* &= V_{\delta_1 - \delta_{n-1} - \delta_n} \text{ for } n > 2, \\
S^2 U^* \otimes U/U^* &= 0, \text{ for } n = 2.
\end{aligned}$$

Therefore, we have

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_n} \text{ if } m = 2, n > 2$$

and

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \varepsilon_m + \delta_1 - \delta_{n-1} - \delta_n} \text{ if } m, n > 2.$$

By part b) of Theorem 1.1 $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0$. Therefore, by (3.1.1), we have

$$\operatorname{Im} \partial^{2,1}_{\hat{\mathfrak{g}}_0} = \hat{\mathfrak{g}}_0 \otimes \mathfrak{g}^*_{-1} = \operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\hat{\mathfrak{g}}_0} = H^{1,2}_{\mathfrak{g}_0}$.

3.3. *Calculation of $H^{1,2}_{\hat{\mathfrak{g}}_0}$ for $m = n > 1$.* Since by parts c) and d) of Theorem 1.1 the first term of the Cartan prolongation $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*$ is $U^* \otimes V$, then by (3.1.1)

$$\operatorname{Im} \partial^{2,1}_{\hat{\mathfrak{g}}_0} = [(V \otimes V^*/\mathrm{C} \oplus U \otimes U^*/\mathrm{C}) \otimes (U^* \otimes V)]/(U^* \otimes V).$$

Therefore, by (3.2.1) and (3.2.2),

$$H^{1,2}_{\hat{\mathfrak{g}}_0} \cong (\Lambda^2 U^* \otimes U/U^*) \otimes (S^2 V \otimes V^*/V) \oplus (S^2 U^* \otimes U/U^*) \otimes (\Lambda^2 V \otimes V^*/V) \oplus (U^* \otimes V).$$

Hence,

$$H^{1,2}_{\hat{\mathfrak{g}}_0} = V_{2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_2} \oplus V_{\varepsilon_1 - \delta_2} \text{ for } n = 2$$

and

$$H^{1,2}_{\hat{\mathfrak{g}}_0} = V_{2\varepsilon_1 - \varepsilon_n + \delta_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \varepsilon_n + \delta_1 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 - \delta_n} \text{ for } n > 2.$$

3.4. *Calculation of $H^{2,2}_{\mathfrak{g}_0}$ for $m, n > 1$, $m \neq n$.* For $k = 2$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{3,1}_{\mathfrak{g}_0}} \mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,2}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^3 \mathfrak{g}^*_{-1}.$$

Observe that

$$\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} = (V \otimes V^*/\mathrm{C} \oplus U \otimes U^*/\mathrm{C} \oplus \mathrm{C}) \otimes (\Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V),$$

$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} = (U^* \otimes V) \otimes (U^* \otimes V),$$

$$\mathfrak{g}_2 = 0.$$

LEMMA. *As $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module, $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ is the direct sum of the irreducible submodules whose highest weights and highest vectors are listed in Table 4 (s and t denote the cyclic permutations of $(1, 2, 3)$ and $(n-2, n-1, n)$, respectively).*

The proof follows from the formula given in the Appendix.

Let us show that if

$$\lambda = 3\varepsilon_1 - \varepsilon_m - 2\delta_n,\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n\ (m > 2),\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n,$$
$$2\varepsilon_1 + \delta_1 - 3\delta_n,\ 2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n\ (n > 2), \text{ or } \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n,$$

then $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$. Recall that if $v \in \mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v(g_1, g_2, g_3) = -v(g_1, g_2) g_3 - v(g_1, g_3) g_2 - v(g_2, g_3) g_1 \tag{3.4.1}$$

for any $g_1, g_2, g_3 \in \mathfrak{g}_{-1}$.
Let $\lambda = 3\varepsilon_1 - \varepsilon_m - 2\delta_n$. Then

$$\partial_{\mathfrak{g}_0}^{2,2} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = -3v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) =$$
$$= 3A_{1,m}(f_n \otimes \tilde{e}_1) = -3f_n \otimes \tilde{e}_m \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n$ $(m > 2)$. Then

$$\partial_{\mathfrak{g}_0}^{2,2} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) =$$
$$= -A_{2,m}(f_{n-1} \otimes \tilde{e}_2) = f_{n-1} \otimes \tilde{e}_m \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n$. Then

$$\partial_{\mathfrak{g}_0}^{2,2} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -2v_\lambda(f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) =$$
$$= -A_{1,m}(f_n \otimes \tilde{e}_1) = f_n \otimes \tilde{e}_m \neq 0.$$

The proof of the fact that $v_\lambda \notin \operatorname{Ker} \partial_{\mathfrak{g}_0}^{2,2}$ for $\lambda = 2\varepsilon_1 + \delta_1 - 3\delta_n, 2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n (n > 2)$, and $\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n$ is similar.
Let $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n$ $(m > 3)$. Let us show that if $n = 2$, then $v_\lambda \in \operatorname{Ker} \partial_{\mathfrak{g}_0}^{2,2}$ and if $n > 2$, then $v_\lambda \notin \operatorname{Ker} \partial_{\mathfrak{g}_0}^{2,2}$. Indeed, if $n = 2$, then for $j = 0, 1, 2$ we have

$$\partial_{\mathfrak{g}_0}^{2,2} v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)}, f_1 \otimes \tilde{e}_{s^j(1)}) =$$
$$- v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)})(f_1 \otimes \tilde{e}_{s^j(1)}) -$$
$$- v_\lambda(f_2 \otimes \tilde{e}_{s^j(3)}, f_1 \otimes \tilde{e}_{s^j(1)})(f_1 \otimes \tilde{e}_{s^j(2)}) =$$
$$A_{s^j(1),m}(f_1 \otimes \tilde{e}_{s^j(1)})/2 - A_{s^j(2),m}(f_1 \otimes \tilde{e}_{s^j(2)})/2 =$$
$$- f_1 \otimes \tilde{e}_m/2 + f_1 \otimes \tilde{e}_m/2 = 0,$$
$$\partial_{\mathfrak{g}_0}^{2,2} v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)}, f_2 \otimes \tilde{e}_{s^j(1)}) =$$
$$- v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)})(f_2 \otimes \tilde{e}_{s^j(1)}) -$$
$$- v_\lambda(f_2 \otimes \tilde{e}_{s^j(1)}, f_1 \otimes \tilde{e}_{s^j(2)})(f_2 \otimes \tilde{e}_{s^j(3)}) =$$
$$A_{s^j(1),m}(f_2 \otimes \tilde{e}_{s^j(1)})/2 - A_{s^j(3),m}(f_2 \otimes \tilde{e}_{s^j(3)})/2 =$$
$$- f_2 \otimes \tilde{e}_m/2 + f_2 \otimes \tilde{e}_m/2 = 0.$$

Therefore, $v_\lambda \in \operatorname{Ker} \partial_{\mathfrak{g}_0}^{2,2}$. If $n > 2$, then

$$\partial_{\mathfrak{g}_0}^{2,2} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3, f_1 \otimes \tilde{e}_1) = -v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3)(f_1 \otimes \tilde{e}_1) =$$
$$= A_{1,m}(f_1 \otimes \tilde{e}_1)/2 = -f_1 \otimes \tilde{e}_m/2 \neq 0.$$

The proof of the fact that if $\lambda = \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geqq 4)$, then $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ for $m = 2$ and $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ for $m > 2$ is similar.
Finally, let us show that if

$$\lambda = 2\varepsilon_1 - 2\delta_n, \varepsilon_1 + \varepsilon_2 - 2\delta_n, 2\varepsilon_1 - \delta_{n-1} - \delta_n, \text{ or } \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$$

and $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, then $v_\lambda \in \operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$. Note that since $\mathfrak{g}_2 = 0$, then, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

$$\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} = (U^* \otimes V) \otimes (U^* \otimes V).$$

Therefore, by Table 3,

$$\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n}.$$

Let $\lambda = \varepsilon_1 + \varepsilon_2 - 2\delta_n$. By Table 4 $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ contains two irreducible components with the indicated highest weight, and one of the corresponding highest vectors is v^1_λ. Observe that

$$\partial^{2,2}_{\mathfrak{g}_0} v^1_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -v^1_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) =$$
$$= -A_{2,1}(f_{n-1} \otimes \tilde{e}_2) = f_{n-1} \otimes \tilde{e}_1 \neq 0.$$

Therefore, $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one irreducible submodule with highest weight $\varepsilon_1 + \varepsilon_2 - 2\delta_n$ and this submodule belongs to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$. Similarly, $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ contains two irreducible submodules with highest weight $2\varepsilon_1 - \delta_{n-1} - \delta_n$, one of which belongs to $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ and, therefore, to $\operatorname{Im}^{3,1}_{\mathfrak{g}_0}$.
Let $\lambda = 2\varepsilon_1 - 2\delta_n$. Then by Table 4 any $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-highest vector of weight λ, which belongs to $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, is

$$v_\lambda = k_1 v^1_\lambda + k_2 v^2_\lambda + k_3 v^3_\lambda, \text{ where } k_1, k_2, k_3 \in \mathbf{C}.$$

If $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, then the condition $\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = 0$ implies

(3.4.2) $$k_1(m-1) - k_2(n-1) + k_3(m-n) = 0,$$

and the condition $\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_1) = 0$ implies that

(3.4.3) $$k_1 m - k_2 n = 0.$$

Thus, for $m \neq n$ we have

(3.4.4) $$k_2 = mk_1/n, k_3 = -k_1/n.$$

Therefore, $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one irreducible submodule with highest weight $2\varepsilon_1 - 2\delta_n$ and this submodule belongs to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$.
Finally, let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$. Then by Table 4 any highest vector with weight λ, which belongs to $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, is

$$v_\lambda = k_1 v^1_\lambda + k_2 v^2_\lambda + k_3 v^3_\lambda, \text{ where } k_1, k_2, k_3 \in \mathbf{C},$$

and if $m = 2$, then $k_1 = 0$. Note that if $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_1, f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1) = 0$$

implies that

$$k_1 + k_2 + k_3(n - m) = 0. \tag{3.4.5}$$

Thus, if $m = 2$, then

$$k_2 = (2 - n)k_3. \tag{3.4.6}$$

If $m, n > 2$, then the condition

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_m, f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1) = 0 \tag{3.4.7}$$

implies that $k_1 + k_2 = 0$. Hence,

$$k_2 = -k_1, k_3 = 0. \tag{3.4.8}$$

Therefore, $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one highest vector of weight $\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$ which belongs to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$. Thus, we have the description of $H^{2,2}_{\mathfrak{g}_0}$ given in Table 1.

3.5. *Calculation of $H^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m, n > 1, m \neq n$.* By part b) of Theorem 1.1

$$(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0.$$

Therefore, the Spencer cohomology in sequence for $k = 2$ takes the form

$$0 \xrightarrow{\partial^{3,1}_{\hat{\mathfrak{g}}_0}} \hat{\mathfrak{g}}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,2}_{\hat{\mathfrak{g}}_0}} \mathfrak{g}_{-1} \otimes \Lambda^3 \mathfrak{g}^*_{-1}.$$

Note that since $\mathfrak{g}_0 = \hat{\mathfrak{g}}_0 \oplus \mathbf{C}$, then

$$\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} = \hat{\mathfrak{g}}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} \oplus V_{2\varepsilon_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n}. \tag{3.5.1}$$

As we have shown in sec.3.4, if λ is one of the weights from Table 4, then an irreducible module with highest weight λ is contained in the decomposition of $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ into irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules if and only if

$$\begin{aligned} \lambda &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n (m > 3), \\ &\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n (n > 3), \\ &\varepsilon_1 + \varepsilon_2 - 2\delta_n, 2\varepsilon_1 - \delta_{n-1} - \delta_n, 2\varepsilon_1 - 2\delta_n \text{ or } \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n \end{aligned} \tag{3.5.2}$$

and its multiplicity is 1. Therefore, by (3.5.1), if

$$\begin{aligned} \lambda &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n (m > 3), \\ &\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n (n > 3), \\ &\varepsilon_1 + \varepsilon_2 - 2\delta_n, \text{ or } 2\varepsilon_1 - \delta_{n-1} - \delta_n, \end{aligned}$$

then the corresponding submodule is contained in $\operatorname{Ker} \delta_{\hat{\mathfrak{g}}_0}^{2,2}$ as well.
Let $\lambda = 2\varepsilon_1 - 2\delta_n$ and $v_\lambda \in \operatorname{Ker} \partial_{\mathfrak{g}_0}^{2,2}$. Then (3.4.4) where $k_3 = 0$, implies that $v_\lambda \notin \operatorname{Ker} \partial_{\hat{\mathfrak{g}}_0}^{2,2}$.
Let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$, $v_\lambda \in \operatorname{Ker} \partial_{\mathfrak{g}_0}^{2,2}$. Then (3.4.6) implies that $v_\lambda \notin \operatorname{Ker} \partial_{\hat{\mathfrak{g}}_0}^{2,2}$ for $m = 2$, and (3.4.8) implies that $v_\lambda \in \operatorname{Ker} \partial_{\hat{\mathfrak{g}}_0}^{2,2}$ for $m, n > 2$. Thus, we have

$$H_{\hat{\mathfrak{g}}_0}^{2,2} = H_{\mathfrak{g}_0}^{2,2} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \text{ if } m = 2$$

and

$$H_{\hat{\mathfrak{g}}_0}^{2,2} = H_{\mathfrak{g}_0}^{2,2} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n} \text{ if } m, n > 2.$$

3.6. *Calculation of $H_{\hat{\mathfrak{g}}_0}^{2,2}$ for $m = n > 1$.* By parts c) and d) of Theorem 1.1 the first term of $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*$ is $U^* \otimes V$ and the second one is C for $n = 2$ and zero for $n > 2$. By formula (3.1.1) we have

$$\operatorname{Im} \partial_{\hat{\mathfrak{g}}_0}^{3,1} = (U^* \otimes V) \otimes (U^* \otimes V) \text{ for } n > 2 \tag{3.6.1}$$

and

$$\operatorname{Im} \partial_{\hat{\mathfrak{g}}_0}^{3,1} = (U^* \otimes V) \otimes (U^* \otimes V)/\mathrm{C} \text{ for } n = 2.$$

Therefore, by Table 3,

$$\operatorname{Im} \partial_{\hat{\mathfrak{g}}_0}^{3,1} = V_{2\varepsilon_1-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \text{ for } n = 2$$

and

$$\operatorname{Im}{}_{\hat{\mathfrak{g}}_0}^{3,1} = V_{2\varepsilon_1-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n} \text{ for } n > 2.$$

Therefore, by (3.4.6) and (3.5.2),

$$H_{\hat{\mathfrak{g}}_0}^{2,2} = 0 \text{ for } n = 2, 3$$

and

$$H_{\hat{\mathfrak{g}}_0}^{2,2} = V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_n-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ for } n > 3.$$

3.7. *Computation of $H_{\mathfrak{g}_0}^{3,2}$ for $m, n > 1$, $m \neq n$.* For $k = 3$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{4,1}} \mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{3,2}} \mathfrak{g}_0 \otimes \Lambda^3 \mathfrak{g}_{-1}^*.$$

Observe that

$$\mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}_{-1}^* = (U^* \otimes V) \otimes \Lambda^2 (U^* \otimes V) \cong$$

$$(\Lambda^2 U^* \otimes U^*) \otimes (S^2 V \otimes V) \oplus (S^2 U^* \otimes U^*) \otimes (\Lambda^2 V \otimes V),$$

$$\mathfrak{g}_2 = 0.$$

By Table 5 from [OV]

$$S^2 V \otimes V = V_{3\varepsilon_1} \oplus V_{2\varepsilon_1+\varepsilon_2},$$

$$\Lambda^2 V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \text{ for } m > 2,$$

$$\Lambda^2 V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \text{ for } m = 2.$$

Since U is purely odd,

$$\Lambda^2 U^* \otimes U^* = V_{-3\delta_n} \oplus V_{-\delta_{n-1}-2\delta_n},$$

$$S^2 U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \oplus V_{-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ for } n > 2,$$

$$S^2 U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \text{ for } n = 2.$$

LEMMA. *The* $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$*-module* $\mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ *is the direct sum of irreducible submodules whose highest weights and highest vectors are listed in Table 5 (s and t denote the cyclic permutations of* $(1, 2, 3)$ *and* $(n-2, n-1, n)$*, respectively).*

Let us show that $\operatorname{Ker} \partial^{3,2}_{\mathfrak{g}_0} = 0$. Let $\lambda = 3\varepsilon_1 - 3\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = 3B_{1,n}(f_n \otimes \tilde{e}_1) = 3(e_1 \otimes \tilde{e}_1 + f_n \otimes \tilde{f}_n) \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - 3\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = -3B_{2,n}(f_n \otimes \tilde{e}_1) = -3e_2 \otimes \tilde{e}_1 \neq 0.$$

Let $\lambda = 3\varepsilon_1 - \partial_{n-1} - 2\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = 3B_{1,n-1}(f_n \otimes \tilde{e}_1) = 3f_n \otimes \tilde{f}_{n-1} \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$. Since by Table 5 $\mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ contains two irreducible submodules with highest weight λ, then any highest vector of weight λ in $\mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ is of the form

$$v_\lambda = k_1 v^1_\lambda + k_2 v^2_\lambda, \text{ where } k_1, k_2 \in \mathbf{C}.$$

Let $v_\lambda \in \operatorname{Ker} \partial^{3,2}_{\mathfrak{g}_0}$. If $m > 2$, then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_m) = -(1/2)k_2 B_{1,n}(f_n \otimes \tilde{e}_m) =$$

$$-(1/2)k_2 e_1 \otimes \tilde{e}_m = 0.$$

Therefore, $k_2 = 0$. Moreover,

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_m) = k_1 B_{2,n-1}(f_{n-1} \otimes \tilde{e}_m) = k_1 e_2 \otimes \tilde{e}_m = 0.$$

Hence, $k_1 = 0$. If $n > 2$, then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_1) = -(1/2)k_2 B_{1,n}(f_1 \otimes \tilde{e}_1) =$$

$$-(1/2)k_2 f_1 \otimes \tilde{f}_n = 0.$$

Therefore, $k_2 = 0$. Moreover,

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2) = k_1 B_{2,n-1}(f_1 \otimes \tilde{e}_2) = k_1 f_1 \otimes \tilde{f}_{n-1} = 0.$$

Hence, $k_1 = 0$.
Let $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-1} - 2\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3, f_n \otimes \tilde{e}_2) = (1/2)B_{1,n}(f_n \otimes \tilde{e}_2) = (1/2)e_1 \otimes \tilde{e}_2 \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \delta_{n-2} - \delta_{n-1} - \delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_1) =$$
$$(1/2)B_{1,n-2}(f_{n-1} \otimes \tilde{e}_1) = (1/2)f_{n-1} \otimes \tilde{f}_{n-2} \neq 0.$$

Finally, let $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-2} - \delta_{n-1} - \delta_n$. Then

$\partial^{3,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3, f_{n-2} \otimes \tilde{e}_1) =$

$(1/2)(B_{1,n-2}(f_{n-2} \otimes \tilde{e}_1) + B_{3,n}(f_n \otimes \tilde{e}_3) + B_{2,n-1}(f_{n-1} \otimes \tilde{e}_2)) =$

$(1/2)(e_1 \otimes \tilde{e}_1 + f_{n-2} \otimes \tilde{f}_{n-2} + e_3 \otimes \tilde{e}_3 + f_n \otimes \tilde{f}_n + e_2 \otimes \tilde{e}_2 + f_{n-1} \otimes \tilde{f}_{n-1}) \neq 0.$

Thus, $H^{3,2}_{\mathfrak{g}_0} = 0$.

3.8. *Calculation of* $H^{3,2}_{\hat{\mathfrak{g}}_0}$ *for* $m = n > 1$. By part d) of Theorem 1.1 for $n > 2$ the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)$ is $U^* \otimes V$ and the second one is zero. Therefore, by arguments similar to those from sec. 3.7 we get $H^{3,2}_{\hat{\mathfrak{g}}_0} = 0$.
If $n = 2$, then by part c) of Theorem 1.1 the first term of $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*$ is $U^* \otimes V$, the second one is the 1-dimensional $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-module with highest weight $\varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2$, and the third one is zero. Thus, by (3.1.1),

$$\operatorname{Im} \partial^{4,1}_{\hat{\mathfrak{g}}_0} = V_{2\varepsilon_1 + \varepsilon_2 - \delta_1 - 2\delta_2}.$$

By Table 5 $(U^* \otimes V) \otimes \Lambda^2(U^* \otimes V)$ contains two irreducible $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-modules with highest weight $\lambda = 2\varepsilon_1 + \varepsilon_2 - \delta_1 - 2\delta_2$ and one of the corresponding highest vectors is v^2_λ. Since

$$\partial^{3,2}_{\hat{\mathfrak{g}}_0} v^2_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2, f_2 \otimes \tilde{e}_2) = B_{1,2}(f_2 \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2 \neq 0,$$

then $\operatorname{Ker} \partial^{3,2}_{\hat{\mathfrak{g}}_0} = \operatorname{Im} \partial^{4,1}_{\hat{\mathfrak{g}}_0}$. Thus, $H^{3,2}_{\hat{\mathfrak{g}}_0} = 0$.

3.9. *Calculation of* $H^{4,2}_{\hat{\mathfrak{g}}_0}$ *for* $m = n = 2$. For $k = 4$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_3 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{5,1}_{\hat{\mathfrak{g}}_0}} \mathfrak{g}_2 \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\partial^{4,2}_{\hat{\mathfrak{g}}_0}} \mathfrak{g}_1 \otimes \Lambda^3 \mathfrak{g}^*_{-1}.$$

By part c) of Theorem 1.1 the second term of $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*$ is $\mathfrak{g}_2 = V_{\varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2} =$

$\langle g \rangle$, the 1-dimensional $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-module, and the third one is zero. Since by Table 3

$$\Lambda^2 \mathfrak{g}_{-1}^* = V_{2\varepsilon_1 - 2\delta_2} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2},$$

then

$$\mathfrak{g}_2 \otimes \Lambda^2 \mathfrak{g}_{-1}^* = V_{3\varepsilon_1 + \varepsilon_2 - \delta_1 - 3\delta_2} \oplus V_{2\varepsilon_1 + 2\varepsilon_2 - 2\delta_1 - 2\delta_2}.$$

Let $\lambda = 3\varepsilon_1 + \varepsilon_2 - \delta_1 - 3\delta_2$. Then by Table 3 $v_\lambda = g \otimes (\tilde{f}_2 \otimes e_1) \wedge (\tilde{f}_2 \otimes e_1)$. Let v be an element from the basis of $\mathfrak{g}_{-1}$ such that $g(v) \neq 0$. If $v = f_2 \otimes \tilde{e}_1$, then

$$\partial_{\mathfrak{g}_0}^{4,2} v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1, v) = -3 v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1)(v) = 3g(v) \neq 0,$$

and if $v \neq f_2 \otimes \tilde{e}_1$, then

$$\partial_{\mathfrak{g}_0}^{4,2} v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1, v) = -v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1)(v) = g(v) \neq 0,$$

Let $\lambda = 2\varepsilon_1 + 2\varepsilon_2 - 2\delta_1 - 2\delta_2$. Then by Table 3

$$v_\lambda = g \otimes ((\tilde{f}_2 \otimes e_1) \wedge (\tilde{f}_1 \otimes e_2) - (\tilde{f}_2 \otimes e_2) \wedge (\tilde{f}_1 \otimes e_1) -$$
$$- (\tilde{f}_1 \otimes e_1) \wedge (\tilde{f}_2 \otimes e_2) + (\tilde{f}_1 \otimes e_2) \wedge (\tilde{f}_2 \otimes e_1)).$$

Let v be an element of the basis of $\mathfrak{g}_{-1}$ such that $g(v) \neq 0$. Then

$$\partial_{\mathfrak{g}_0}^{4,2} v_\lambda(f_2 \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2, v) = -2 v_\lambda(f_2 \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2)(v) = g(v) \neq 0$$

if either $v = f_2 \otimes \tilde{e}_1$ or $v = f_1 \otimes \tilde{e}_2$,

and

$$\partial_{\mathfrak{g}_0}^{4,2} v_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2, v) = -2 v_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2)(v) = -g(v) \neq 0$$

if either $v = f_2 \otimes \tilde{e}_2$ or $v = f_1 \otimes \tilde{e}_1$.

Therefore, $H_{\mathfrak{g}_0}^{4,2} = 0$.

3.10. *Calculation of* $H_{\mathfrak{g}_0}^{k,2}$ *for* $m = n > 1, k > 0$.

LEMMA. $H_{\mathfrak{g}_0}^{k,2} = H_{\hat{\mathfrak{g}}_0}^{k,2}$.

PROOF. Note that if $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{g}_{-1} \oplus (\oplus_{k \geqq 0} \hat{\mathfrak{g}}_k)$ is the Cartan prolongation of the pair $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)$, then, since $\mathfrak{g}_k = \hat{\mathfrak{g}}_k \oplus S^k(\mathfrak{g}_{-1}^*)$ $(k \geqq 0)$, the Spencer cohomology sequence is of the form

$$(\hat{\mathfrak{g}}_0 \oplus \mathbf{C}) \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{2,1}} \mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{1,2}} 0 \qquad \text{for } k = 1,$$

$$(\hat{\mathfrak{g}}_{k-1} \oplus S^{k-1}(\mathfrak{g}_{-1}^*)) \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{k+1,1}} (\hat{\mathfrak{g}}_{k-2} \oplus S^{k-2}(\mathfrak{g}_{-1}^*)) \otimes \Lambda^2 \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{k,2}}$$
$$(\hat{\mathfrak{g}}_{k-3} \oplus S^{k-3}(\mathfrak{g}_{-1}^*)) \otimes \Lambda^3 \mathfrak{g}_{-1}^* \qquad \text{for } k > 1.$$

Note that since $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = (\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* \ltimes S^*(\mathfrak{g}_{-1}^*)$, then the sequence

$$S^{k-1}(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\bar\partial^{k+1,1}_{\mathfrak{g}_0}} S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\bar\partial^{k,2}_{\mathfrak{g}_0}} S^{k-3}(\mathfrak{g}^*_{-1}) \otimes \Lambda^3 \mathfrak{g}^*_{-1}$$

for $k \geqq 1$,

where $\bar\partial^{k+1,1}_{\mathfrak{g}_0}$ and $\bar\partial^{k,2}_{\mathfrak{g}_0}$ are the restrictions of the operators $\partial^{k+1,1}_{\mathfrak{g}_0}$ and $\partial^{k,2}_{\mathfrak{g}_0}$ to $S^{k-1}(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}^*_{-1}$ and $S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, respectively, and $S^k(\mathfrak{g}^*_{-1}) = 0$ for $k < 0$, is well-defined. Hence the corresponding cohomology groups

$$\bar H^{k,2}_{\mathfrak{g}_0} = \operatorname{Ker} \bar\partial^{k,2}_{\mathfrak{g}_0} / \operatorname{Im} \bar\partial^{k+1,1}_{\mathfrak{g}_0}$$

are well-defined and $H^{k,2}_{\mathfrak{g}_0} = H^{k,2}_{\hat{\mathfrak{g}}_0} \oplus \bar H^{k,2}_{\mathfrak{g}_0}$.

Let us show that $\bar H^{k,2}_{\mathfrak{g}_0} = 0$ for $k > 0$. For $k = 1$ this is obvious. Let $k = 2$. Since $S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2 \mathfrak{g}^*_{-1} = \langle z \rangle \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, where z is a generator of the center of $\mathfrak{gl}(n|n)$, then

$$\operatorname{Ker} \bar\partial^{k,2}_{\mathfrak{g}_0} \cong \Lambda^2 \mathfrak{g}^*_{-1}.$$

By formula (3.1.1)

$$\operatorname{Im} \bar\partial^{k+1,1}_{\mathfrak{g}_0} \cong \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} / S^2 \mathfrak{g}^*_{-1} = \Lambda^2 \mathfrak{g}^*_{-1}.$$

Therefore, $\bar H^{2,2}_{\mathfrak{g}_0} = 0$. Let $k = 3$. Observe that

$$S^2(U^* \otimes V) \otimes (U^* \otimes V) = (S^2 U^* \otimes U^*) \otimes (S^2 V \otimes V) \oplus (\Lambda^2 U^* \otimes U^*) \otimes (\Lambda^2 V \otimes V).$$

By Table 5 from [OV] we get:

$$S^2 V \otimes V = V_{3\varepsilon_1} \oplus V_{2\varepsilon_1+\varepsilon_2},\ \Lambda^2 V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \text{ if } n = 2,$$

$$\Lambda^2 V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \text{ if } n > 2.$$

Since U is odd,

$$\Lambda^2 U^* \otimes U^* = V_{-3\delta_n} \oplus V_{-\delta_{n-1}-2\delta_n},$$

$$S^2 U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \text{ if } n = 2,$$

$$S^2 U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \oplus V_{-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ if } n > 2.$$

Therefore,

$$S^2(U^* \otimes V) \otimes (U^* \otimes V) = V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus 2V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n}$$

if $n = 2$

and

$$S^2(U^* \otimes V) \otimes (U^* \otimes V) =$$

$$V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus 2V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \oplus V_{3\varepsilon_1-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus$$

$$V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-3\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n} \qquad \text{if } n > 2.$$

Moreover, we have

$$S^3(U^* \otimes V) = V_{2\varepsilon_1+\varepsilon_2-\delta_1-2\delta_2} \qquad \text{if } n = 2 \text{ and}$$

$$S^3(U^* \otimes V) = V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-3\delta_n} \oplus V_{3\varepsilon_1-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \quad \text{if } n > 2.$$

Thus, by (3.1.1)

$$\operatorname{Im} \bar{\partial}^{4,1}_{\mathfrak{go}} = V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \qquad \text{if } n = 2,$$

$$\operatorname{Im} \bar{\partial}^{4,1}_{\mathfrak{go}} = V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \oplus$$

$$\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n} \qquad \text{if } n > 2.$$

Finally, the decomposition of the $\mathfrak{gl}(n) \oplus \mathfrak{gl}(n)$-module $\mathfrak{g}^*_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ into the direct sum of irreducible components is given in Table 5. Checking the action of $\bar{\partial}^{3,2}_{\mathfrak{go}}$ on the highest vectors we get:

$$\operatorname{Im} \bar{\partial}^{4,1}_{\mathfrak{go}} = \operatorname{Ker} \bar{\partial}^{3,2}_{\mathfrak{go}}.$$

Note that for $k > 3$ the cohomology groups $\bar{H}^{k,2}_{\mathfrak{go}}$ coincide with the Spencer cohomology groups $H^{k-2,2}_{\mathfrak{o}(n^2)}$ corresponding to the Cartan prolongation $(V(0|n^2), \mathfrak{o}(n^2))_* = \mathfrak{h}(0|n^2)$, where $\mathfrak{o}(n^2)$ is the orthogonal Lie algebra and $V(0|n^2)$ is the standard odd $\mathfrak{o}(n^2)$-module. These groups are vanishing for $k > 3$, cf. [P4].

Appendix.

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis of the dual space to the space of diagonal marices in $\mathfrak{gl}(n)$, V_λ be the irreducible $\mathfrak{sl}(n)$-module with highest weight $\lambda = k_1\varepsilon_1 + k_2\varepsilon_2 + \ldots + k_n\varepsilon_n$, where $k_i \in \mathbf{Z}$. Then

$$\dim V_\lambda = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} \left(1 + \frac{k_i - k_{i+j}}{j}\right).$$

Proof. A weight λ is the highest of an irreducible $\mathfrak{sl}(n)$-module if and only if λ is a dominant integer form, i.e., if

$$2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbf{Z}_+.$$

It is known that the inner products of the weights ε_i and of weight ρ, where $\rho = (\sum_{\beta \in \Delta_+} \beta)/2$, with fundamental weights α_j are:

$$(\varepsilon_1, \alpha_1) = 1/(2n), (\varepsilon_1, \alpha_j) = 0 \qquad \text{for } 2 \leqq j \leqq n-1;$$

$$(\varepsilon_i, \alpha_{i-1}) = -1/(2n), (\varepsilon_i, \alpha_i) = 1/(2n), (\varepsilon_i, \alpha_j) = 0 \; (j \neq i-1, i)$$

$$\text{for } 2 \leqq i \leqq n-1;$$

$$(\varepsilon_n, \alpha_{n-1}) = -1/(2n), (\varepsilon_n, \alpha_j) = 0 \qquad \text{for } 1 \leqq j \leqq n-2;$$

$$(\rho, \alpha_i) = 1/(2n) \qquad \text{for } 1 \leqq i \leqq n-1.$$

Thus,

$$(\lambda, \alpha_i) = \frac{k_i - k_{i+1}}{2n} \text{ and } k_i \geqq k_{i+1}.$$

By Weyl's character formula

$$\dim V_\lambda = \prod\nolimits_{\beta \in \Delta_+} \left(1 + \frac{(\lambda, \beta)}{(\rho, \beta)}\right).$$

For $\mathfrak{sl}(n)$ we have $\Delta_+ = \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j, \text{where } 1 \leqq i \leqq n-1, j \geqq i\}$. Since

$$(\lambda, \alpha_i + \ldots + \alpha_j) = 1/(2n)((k_i - k_{i+1}) + (k_{i+1} - k_{i+2}) + \ldots + (k_j - k_{j+1})) = \\ = \frac{k_i - k_{j+1}}{2n},$$

we have

$$\prod\nolimits_{\beta \in \Delta_+} \left(1 + \frac{(\lambda, \beta)}{(\rho, \beta)}\right) = \prod\nolimits_{i=1}^{n-1} \prod\nolimits_{j=i}^{n-1} \left(1 + \frac{\left(\frac{k_i - k_{j+1}}{2n}\right)}{\left(\frac{j-i+1}{2n}\right)}\right) = \\ = \prod\nolimits_{i=1}^{n-1} \prod\nolimits_{j=1}^{n-i} \left(1 + \frac{k_i - k_{i+j}}{j}\right).$$

Table 1

m	n	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
3	2	$2\varepsilon_1 - \varepsilon_3 + \delta_1 - 2\delta_2$	–
2	3	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_3$	–
$\geqq 4$	2	$2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_1 - \delta_2$
2	$\geqq 4$	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_n$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$
$\geqq 3$	$\geqq 3$	$2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_m + \delta_1 - \delta_{n-1} - \delta_n$ $\varepsilon_1 - \delta_n$ (if $m = n$)	–
2	2	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$	–

Table 2

$gl(m) \oplus gl(n)$-module	Highest weight (λ)	Highest vector
$(V \otimes V^*)/C \otimes (U^* \otimes V)$	$2\varepsilon_1 - \varepsilon_m - \delta_n$ $\varepsilon_1 - \delta_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$ (if $m \geqq 3$)	$(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1)$ $v_\lambda^1 = \sum_{i=1}^m (e_i \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_1)$ $- m \sum_{i=1}^m (e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i)$ $(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_2)$ $- (e_2 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1)$
$(U \otimes U^*)/C \otimes (U^* \otimes V)$	$\varepsilon_1 + \delta_1 - 2\delta_n$ $\varepsilon_1 - \delta_n$ $\varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$ (if $n \geqq 3$)	$(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1)$ $v_\lambda^2 = \sum_{i=1}^n (f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1)$ $- n \sum_{i=1}^n (f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1)$ $(f_1 \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_n \otimes e_1)$ $- (f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_{n-1} \otimes e_1)$
$C \otimes (U^* \otimes V)$	$\varepsilon_1 - \delta_n$	$v_\lambda^3 = (n \sum_{i=1}^m e_i \otimes \tilde{e}_i$ $+ m \sum_{i=1}^n f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1)$

Table 3

$gl(m) \oplus gl(n)$-module	Highest weight (λ)	Highest vector
$\Lambda^2 U^* \otimes S^2 V$	$2\varepsilon_1 - 2\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1)$
$S^2 U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_2)$ $- (\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_{n-1} \otimes e_1) -$ $(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_2)$ $+ (\tilde{f}_{n-1} \otimes e_2) \otimes (\tilde{f}_n \otimes e_1)$
$\Lambda^2 U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 - 2\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_2)$ $- (\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_n \otimes e_1)$
$S^2 U^* \otimes S^2 V$	$2\varepsilon_1 - \delta_{n-1} - \delta_n$	$(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1)$ $- (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1)$

Table 4

$\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module	Highest weight (λ)	Highest vector
$V \otimes V^*/\mathbf{C} \otimes \Lambda^2 U^* \otimes S^2 V$	$3\varepsilon_1 - \varepsilon_m - 2\delta_n$	$(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 - 2\delta_n$	$v_\lambda^1 = \sum_{i=1}^m (e_i \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) - m \sum_{i=1}^m (e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i) \wedge (\tilde{f}_n \otimes e_1)$
	$\varepsilon_1 + \varepsilon_2 - 2\delta_n$	$v_\lambda^1 = \sum_{i=1}^m ((e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i) \wedge (\tilde{f}_n \otimes e_2) - (e_2 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i) \wedge (\tilde{f}_n \otimes e_1))$
	$2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n (m \geqq 3)$	$(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) - (e_2 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
$V \otimes V^*/\mathbf{C} \otimes S^2 U^* \otimes \Lambda^2 V$	$2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n$	$(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) - (e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2)$
	$2\varepsilon_1 - \delta_{n-1} - \delta_n (m \geqq 3)$	$v_\lambda^1 = \sum_{i=1}^m ((e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_{n-1} \otimes e_i) \wedge (\tilde{f}_n \otimes e_1) - (e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i) \wedge (\tilde{f}_{n-1} \otimes e_1))$
	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$	$v_\lambda^1 = \sum_{i=1}^m ((e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_{n-1} \otimes e_i) \wedge (\tilde{f}_n \otimes e_2) - (e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i) \wedge (\tilde{f}_{n-1} \otimes e_2) - (e_2 \otimes \tilde{e}_i) \wedge (\tilde{f}_{n-1} \otimes e_i) \wedge (\tilde{f}_n \otimes e_1) + (e_2 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i) \wedge (\tilde{f}_{n-1} \otimes e_1))$
	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n$ $(m \geqq 4)$	$\sum_{j=0}^2 ((e_{s^j(1)} \otimes \tilde{e}_m) \otimes (\tilde{f}_{n-1} \otimes e_{s^j(2)}) \wedge (\tilde{f}_n \otimes e_{s^j(3)}) - (e_{s^j(1)} \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_{s^j(2)}) \wedge (\tilde{f}_{n-1} \otimes e_{s^j(3)}))$
$U \otimes U^*/\mathbf{C} \otimes \Lambda^2 U^* \otimes S^2 V$	$2\varepsilon_1 + \delta_1 - 3\delta_n$	$(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 - 2\delta_n$	$v_\lambda^2 = \sum_{i=1}^n (f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) - n \sum_{i=1}^n (f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 - \delta_{n-1} - \delta_n$	$v_\lambda^2 = \sum_{i=1}^n ((f_i \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) - (f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_1))$
	$2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n$ $(n \geqq 3)$	$(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) - (f_1 \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$

Table 4 (cont.)

$gl(m) \oplus gl(n)$-module	Highest weight (λ)	Highest vector
$U \otimes U^*/C \otimes S^2U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n$	$(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) - (f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2)$
	$\varepsilon_1 + \varepsilon_2 - 2\delta_n$ $(n \geqq 3)$	$v_\lambda^2 = \sum_{i=1}^n ((f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) - (f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_i \otimes e_2))$
	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n (n \geqq 3)$	$v_\lambda^2 = \sum_{i=1}^n ((f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2) - (f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_2) \wedge (\tilde{f}_{n-1} \otimes e_1) - (f_i \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) + (f_i \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_i \otimes e_2) \wedge (\tilde{f}_n \otimes e_1))$
	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geqq 4)$	$\sum_{j=0}^2 ((f_1 \otimes \tilde{f}_{t^j(n-2)}) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_1) \wedge (\tilde{f}_{t^j(n)} \otimes e_2) - (f_1 \otimes \tilde{f}_{t^j(n-2)}) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_2) \wedge (\tilde{f}_{t^j(n)} \otimes e_1))$
$C \otimes \Lambda^2 U^* \otimes S^2 V$	$2\varepsilon_1 - 2\delta_n$	$v_\lambda^3 = (n\sum_{i=1}^m e_i \otimes \tilde{e}_i + m\sum_{i=1}^n f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
$C \otimes S^2 U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$	$v_\lambda^3 = (n\sum_{i=1}^m e_i \otimes \tilde{e}_i + m\sum_{i=1}^n f_i \otimes \tilde{f}_i) \otimes ((\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) - (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2))$

Table 5

$gl(m) \oplus gl(n)$-module	Highest weight (λ)	Highest vector
$(\Lambda^2 U^* \otimes U^*) \otimes (S^2 V \otimes V)$	$3\varepsilon_1 - 3\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 + \varepsilon_2 - 3\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_2) \wedge (\tilde{f}_n \otimes e_1) - (\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1)$
	$3\varepsilon_1 - \delta_{n-1} - 2\delta_n$	$(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) - (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$	$v_\lambda^1 = (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (f_n \otimes e_2) - (\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (f_n \otimes e_1) - (\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) + (\tilde{f}_{n-1} \otimes e_2) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$

Table 5 (cont.)

$\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$-module	Highest weight (λ)	Highest vector
$(S^2U^*\otimes U^*)\otimes(\Lambda^2V\otimes V)$	$2\varepsilon_2+\varepsilon_2-\delta_{n-1}-2\delta_n$	$v_\lambda^2=(\tilde{f}_n\otimes e_1)\otimes(\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-(\tilde{f}_n\otimes e_1)\otimes(\tilde{f}_{n-1}\otimes e_2)\wedge(\tilde{f}_n\otimes e_1)$
	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n$ $(m\geqq 3)$	$\sum_{j=0}^2((\tilde{f}_n\otimes e_{s^j(1)})\otimes(\tilde{f}_{n-1}\otimes e_{s^j(2)})\wedge(\tilde{f}_n\otimes e_{s^j(3)})-(\tilde{f}_n\otimes e_{s^j(1)})\otimes(\tilde{f}_n\otimes e_{s^j(2)})\wedge(\tilde{f}_{n-1}\otimes e_{s^j(3)}))$
	$2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n$ $(n\geqq 3)$	$\sum_{j=0}^2((\tilde{f}_{t^j(n-2)}\otimes e_1)\otimes(\tilde{f}_{t^j(n-1)}\otimes e_2)\wedge(\tilde{f}_{t^j(n)}\otimes e_1)-(\tilde{f}_{t^j(n-2)}\otimes e_1)\otimes(\tilde{f}_{t^j(n-1)}\otimes e_1)\wedge(\tilde{f}_{t^j(n)}\otimes e_2))$
	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-2}-\delta_{n-1}-\delta_n$ $(m,n\geqq 3)$	$\sum_{i=0}^2\sum_{j=0}^2((\tilde{f}_{t^j(n-2)}\otimes e_{s^i(1)})\otimes(\tilde{f}_{t^j(n-1)}\otimes e_{s^i(2)})\wedge(\tilde{f}_{t^j(n)}\otimes e_{s^i(3)})-(\tilde{f}_{t^j(n-2)}\otimes e_{s^i(1)})\otimes(\tilde{f}_{t^j(n)}\otimes e_{s^i(2)})\wedge(\tilde{f}_{t^j(n-1)}\otimes e_{s^i(3)}))$

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