MATH. SCAND. 71 (1992), 301–319

THE EXPONENTIAL RANK OF INDUCTIVE LIMIT C^*-ALGEBRAS

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Abstract.

Let A be a simple C^*-algebra of real rank zero and be an inductive limit of C^*-algebras of the form $C(X, M_n)$, where X is a fixed finite CW complex. We prove that the exponential rank of A is at most $1 + \varepsilon$. We also show that the exponential ranks of the C^*-algebras of real rank zero considered by Goodearl recently are at most $1 + \varepsilon$. Other simple C^*-algebras are also proved to have exponential rank at most $1 + \varepsilon$.

0. Introduction.

C^*-algebras A that are inductive limits of direct sums of C^*-algebras of the form $C(X_k, M_{n_k})$ have been studied for a long time. The theory has been revived with the recent successful work of G. A. Elliott's classification [Ell 2] of the algebras A that have real rank zero in the case where the base space X have a special form. We notice that all Elliott's algebras have stable rank one. It is shown recently in [DNNP], [BBEK], [BDR] and [G] that both stable rank and real rank of A can be reduced to one and zero, respectively, even if the base spaces have large (or even infinite) dimension.

Let A be a unital C^*-algebra. It is well known that if u is a unitary in the connected component of the identity, then u is a product of finitely many exponentials of self-adjoint elements in A. Is one exponential enough? Is u a limit of exponentials? Exponential rank of C^*-algebras has recently been studied by N. C. Phillips ([Ph1], [Ph 2] and [Ph 3]), J. R. Ringrose ([PR]) and Zhang ([Zh 2] and [Zh 3]).

It is shown by N. C. Phillips [Ph 1] that all Elliott's algebras have exponential rank (see (2) below) no more than $1 + \varepsilon$. It based on the fact proved in [Ph 1] that $\operatorname{cer}(C(X) \otimes M_n) \leqq 1 + \varepsilon$ if X is a compact metric space of dimension at most 2. N. C. Phillips shows that the exponential rank of $C(X, M_n)$ with high dimensional base space X can be large, The purpose of this paper is to show that under certain situations, the exponential rank of A can be reduced to at most $1 + \varepsilon$. In section

Received May 20, 1991; in revised form November 27, 1991.

1 we show that C^*-algebras A considered in [G] with real rank zero has exponential rank at most $1 + \varepsilon$. Our section 2 deals with general simple C^*-algebras. In section 3 we show that how our results in section 2 work for inductive limits of $C(X, M_n)$, where X is a finite CW complex. In particular, we show that if X is a finite CW complex, A has exponential rank at most $1 + \varepsilon$ provided A is simple and projections of A separate the traces on A. Consequently, all these C^*-algebras have weak (FU), i.e. unitaries in the connected component of the identity can be approximated by unitaries with finite spectra. The following are some notations used in this paper.

(1) Let A be a unital C^*-algebra. We denote by $U(A)$ the unitary group of A and $U_0(A)$ the connected component of $U(A)$ containing the identity.

(2) The exponential rank cer(A) of a unital C^*-algebra A is the smallest $k \in \{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\}$ such that each $u \in U_0(A)$ can be expressed as the product of at most k exponentials $\exp(ih)$ with $h \in A_{\mathrm{s.a.}}$, if k is an integer, or u can be approximated by the product of at most m exponentials, if $k = m + \varepsilon$, where m is an integer. For nonunital A, we define $\operatorname{cer}(A) = \operatorname{cer}(\tilde{A})$ (See [Ph1, 1.2]).

(3) Suppose that p is an open projection of A (in A^{**}). We will denote by $\operatorname{Her}(p)$ the hereditary C^*-algebra of A corresponding to p.

Finally, recall that a C^*-algebra A is said to have real rank zero if the set of self-adjoint elements of finite spectra is dense in $A_{\mathrm{s.a.}}$.

This work was done while the first author was a postdoctor fellow at University of Toronto and the second author was visiting the University of Toronto. Both authors are grateful to George A. Elliott and Man-Deun Choi for their support and hospitality. They were supported by a grant from the Natural Sciences and Engineering Research Council of Canada. They benefited from conversations with George A. Elliott and Man-Deun Choi.

1. Inductive limit C^*-algebras considered by Goodearl.

In [G], K. R. Goodearl studied a family of simple C^*-algebras of direct sums of C^*-algebras of the form $C(X, M_n)$, where X is a nonempty separable compact Hausdorff space. Following Goodearl's notations, we give the following list:

1) $\{x_1, x_2, \ldots\}$: elements of X such that $\{x_n, x_{n+1}, \ldots\}$ is dense in X for each n;
2) δ_n: $M_k(C(X)) \to M_k(\mathbb{C}) \subseteq M_k(C(X))$: evaluation at x_n;
3) $v(1), v(2), \ldots$: positive integers such that $v(n) \mid v(n+1)$ for all n;
4) A_n: the C^*-algebra $M_{v(n)}(C(X))$, $n = 1, 2. \ldots$;
5) ϕ_n: $A_n \to A_{n+1}$: unital block diagonal homomorphism of the form

$$\operatorname{diag}(\text{identity}, \text{identity}, \ldots, \delta_n, \ldots, \delta_n),$$

i.e.

$$\phi(a) = \operatorname{diag}(a, \ldots, a, \delta_n(a), \ldots, \delta_n(a))$$

for $a \in A_n$;

6) α_n: the number of identity maps appearing in ϕ_n and $\alpha_0 = 1$;

7) β_n: the number of δ_n appearing in ϕ_n;

8) $\phi_{s,n}$: the map $\phi_{s-1}\phi_{s-2} \ldots \phi_n$: $A_n \to A_s$ for $s > n$;

9) A: the C^*-inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \ldots;$$

10) η_n: $A_n \to A$: the natural map induced from the inductive limit;

11) In each of the maps ϕ_n, at least one identity map and at least one δ_n is used. In other words,

$$0 < \alpha_n < v(n+1)/v(n);$$

12) $\omega_{s,n}$: the number $\alpha_n\alpha_{n+1} \ldots \alpha_{s-1} v(n)/v(s)$.

Goodearl showed in [G] that A is a simple unital C^*-algebra with stable rank one. Moreover, if X is not totally disconnected, then

(1) if

$$\lim_{t\to\infty} \omega_{t,1} = \varepsilon > 0,$$

then $\mathrm{RR}(A) = 1$;

(2) if

$$\lim_{t\to\infty} \omega_{t,1} = 0,$$

then $\mathrm{RR}(A) = 0$. Notice that if X is totally disconnected, then A is AF, therefore $\mathrm{RR}(A) = 0$.

In this section we will show that $\mathrm{cer}(A) \leqq 1 + \varepsilon$, whenever $\mathrm{RR}(A) = 0$.

Let $m_{s,n}^{(k)} (n \leqq k \leqq s-1)$ be the number of $u(x_k)$ appearing in $\phi_{s,n}(u)$ and $m_{s,n}^{(0)}$ be the number of u appearing in $\phi_{s,n} = \alpha_n, \alpha_{n+1} \ldots \alpha_{s-1}$. Moreover,

$$v(s)/v(n) = m_{s,n}^{(0)} + \textstyle\sum_{k=n}^{s-1} m_{s,n}^{(k)}.$$

It is clear, by the definition,

$$m_{s+1,n}^{(k)} = m_{s,n}^{(k)} \cdot \alpha_s + m_{s,n}^{(k)} \cdot \beta_s = m_{s,n}^{(k)} \frac{v(s+1)}{v(s)}.$$

By induction, one sees easily that

$$m_{s,n}^{(k)} = \alpha_n\alpha_{n+1} \ldots \alpha_{k-1}\beta_k \frac{v(s)}{v(k+1)}.$$

Hence

$$m_{s,k}^{(k)} = \beta_k \frac{v(s)}{v(k+1)} = \frac{m_{s,n}^{(k)}}{\alpha_n \alpha_{n+1} \dots \alpha_{k-1}}.$$

LEMMA 1.1. *If* $\lim_{t\to\infty} \omega_{t,1} = 0$, *for any fixed integers* $0 < n < m$ *and* $M > 0$ *there is an integer* N *such that whenever* $s \geqq N\,(\geqq n, m)$

$$m_{s,n}^{(k)} \geqq M \cdot \alpha_n \alpha_{n+1} \dots \alpha_{s-1},$$

where $n \leqq k \leqq m$.

PROOF. Since $m_{s,k}^{(k)} = \beta_k \dfrac{v(s)}{v(k+1)}$ and $\lim_{t\to\infty} \omega_{t,n} = 0$ (see [G]),

$$\lim_{s\to\infty} \frac{m_{s,k}^{(0)}}{m_{s,k}^{k}} = \lim_{s\to\infty} \frac{\alpha_k}{\beta_k} \alpha_{k+1} \dots \alpha_{s-1} \frac{v(k+1)}{v(s)} = \frac{\alpha_k}{\beta_k} \lim_{s\to\infty} \omega_{s,k} = 0.$$

Therefore, for $n \leqq k \leqq m$, there is N, when $s \geqq N$,

$$m_{s,k}^{k} \geqq M(\alpha_n \dots \alpha_{k-1}) \cdot m_{s,k}^{(0)} = M\alpha_n \dots \alpha_{s-1}$$

for $n \leqq k \leqq m$.

Since

$$m_{s,k}^{(k)} = m_{s,n}^{(k)} \frac{1}{\alpha_n \dots \alpha_{(k-1)}} \leqq m_{s,n}^{(k)},$$

$$m_{s,n}^{(k)} \geqq M\alpha_n \alpha_{n+1} \dots \alpha_{s-1}.$$

LEMMA 1.2. *Let* $u \in A_n$ *for some* n. *Suppose that* $\mathrm{Sp}(u) = S^1$ *and* $S^1 = \bigcup_{i=1}^k I_i$, *where each* I_i *has the form* $\{e^{i\theta}\colon \theta_i \leqq \theta < \theta_{i+1}\}$ *and*

$$0 = \theta_0 < \theta_1 < \dots < \theta_k = 2\pi.$$

Then for any $M > 0$ *there is an integer* N *such that the number of eigenvalues of* $\phi_{s,n}(u)$, *counting multiplicities, which are in* I_i *is larger than* $M\alpha_n \alpha_{n+1} \dots \alpha_{s-1}$, *for* $i = 1, 2, \dots, k$, *whenever* $s \geqq N$.

PROOF. Let λ_i be the center of I_i, $i = 1, 2, \dots, k$. Since X is compact, there is $p_i \in X$ such that $\lambda_i \in \mathrm{Sp}(u(p_i))$, $i = 1, 2, \dots, k$. There is $x_{n_i} \in \{x_1, x_2, \dots\}$ such that x_{n_i} is close to p_i so that there is $\lambda_i' \in I_i$ and $\lambda_i' \in \mathrm{Sp}(u(x_{n_i}))$. Therefore, by Lemma 1.1, the multiplicity of λ_i' is larger than $M\alpha_n \dots \alpha_{s-1}$ if s is large enough.

THEOREM 1.3. *If* A *has real rank zero, then*

$$\mathrm{cer}(A) \leqq 1 + \varepsilon.$$

Consequently, A *has weak* (FU) (*see* [Ph 1]).

PROOF. Fix $u \in U_0(A)$ with $\mathrm{Sp}(u) = S^1$. There are unitaries

$$u = u_0, u_1, u_2, \ldots, u_L, u_{L+1} = 1$$

along a path connecting u to 1 such that

$$\|u_i - u_{i+1}\| < \varepsilon/8, \qquad i = 0, 1, 2, \ldots, L.$$

Without loss of generality, we may assume that $u_i \in U(A_1)$, $i = 0, 1, 2, \ldots, L$.

For any n, $\phi_{n,1}(u) = \operatorname{diag}(\bar{u}, \omega(n))$, where $\bar{u} = \operatorname{diag}(u, u, \ldots, u)$ with $\alpha_1\alpha_2\ldots\alpha_{n-1}$ repeating u's on the diagonal and $\omega(n)$ is a constant block diagonal unitary matrix with each block having size $v(1) \times v(1)$. Let

$$0 = \theta_1 < \theta_2 < \ldots < \theta_s = 2\pi$$

be a partition of the interval $[0, 2\pi]$, where $s = [16\pi/\varepsilon] + 1$, such that

$$\|\theta_{k+1} - \theta_k\| < \varepsilon/8,$$

and let $I_k = \{e^{i\theta}: \theta_k \leqq \theta \leqq \theta_{k+1}\}$. Denote by $M(n, k)$ the number of eigenvalues of $\omega(n)$, counting multiplicities, which are in I_k, $k = 1, 2,,,,, [16\pi/\varepsilon] + 1$. By Lemma 1.2, there is n_1 such that for any $n \geqq n_1$,

$$M(n, k) \geqq 4L\alpha_1\alpha_2\ldots\alpha_{n-1}\left(\left[\frac{16\pi}{\varepsilon}\right] + 1\right)v(1).$$

Set $\omega_0 = \operatorname{diag}(v_1, v_1, \ldots, v_1) \in M_{2Lv(1)\alpha_1\alpha_2\ldots\alpha_{n-1}}(C(X))$ with $\alpha_1\alpha_2\ldots\alpha_{n-1}$ many v_1's, where

$$v_1 = \operatorname{diag}(u_1^*, u_1, u_2^*, u_2, \ldots, u_L^*, u_L)$$

(notice that the size of v_1 is $2Lv(1) \times 2Lv(1)$). The element ω_0 can be regarded as an element in a corner of A_n since $2Lv(1)\alpha_1\alpha_2\ldots\alpha_{n-1} < v(n)$. And let $p_{\omega_0} = \omega_0\omega_0^*$ which is a projection of A_n of size $2Lv(1)\alpha_1\alpha_2\ldots\alpha_{n-1}$. Furthermore, we may regard p_{ω_0} as a projection in A and ω_0 as an element in the corresponding corner of A by identifying $\eta_n(p_{\omega_0})$ and p_{ω_0}, $\eta_n(\omega_0)$ and ω_0, respectively.

By [Ph 2, Corollary 5], there is $h \in A_{\text{s.a.}}$ such that

$$\|\omega_0 - e^{ih}p_{\omega_0}\| < \varepsilon/8.$$

Since A has real rank zero, there are mutually orthogonal projections $p_1, p_2, \ldots, p_{s_1}$ in A such that

$$\left\|\omega_0 - \sum_{k=1}^{s_1} \lambda_k p_k\right\| < \varepsilon/4,$$

where $|\lambda_k| = 1$, $k = 1, 2, \ldots, s_1$. Without loss of generality, we may assume that $p_k \in A_n$ for some $n \geqq n_1$, and we may further assume that $\lambda_k \in I_k$, $k = 1, 2, \ldots, s_1$, and $s_1 \leqq \left[\dfrac{16\pi}{\varepsilon}\right] + 1$. Since $M(n, k) \geqq 4L\alpha_1\alpha_2\ldots\alpha_{n-1}\left(\left[\dfrac{16\pi}{\varepsilon}\right] + 1\right)v(1)$, for $n \geqq n_1$ we may write

$$\omega(n) = \sum_{k=1}^{s}\left(\sum_{i=1}^{M(n,k)} \lambda_k^{(i)} e_{ik}\right),$$

where e_{ik} are mutually orthogonal, rank one constant projections (in A_n), $\lambda_k^{(i)} \in I_k$, and $[\sum_{i=1}^{M(n,k)} e_{ik}] \geqq [p_{\omega_0}] \geqq [p_j], j = 1, 2, \ldots, s$.

Set

$$\bar{\omega} = \sum_{k=1}^{s} \lambda_k\left(\sum_{i=1}^{M(n,k)} e_{ik}\right).$$

(For $k > s_1$, let λ_k be one of $\lambda_k^{(i)}$.)

Then $\|\omega(n) - \bar{\omega}\| < \varepsilon/8$. Let $\bar{V} = \mathrm{diag}(\bar{u}, \bar{\omega})$, then

$$\|\phi_{n1}(u) - \bar{V}\| < \varepsilon/8.$$

We may write

$$\bar{V} = \mathrm{diag}(\bar{u}, \bar{v}_0, \omega'),$$

where $\bar{v}_0 = \sum_{k=1}^{s_1} \lambda_k q_k$, $\omega' = \sum_{k=1}^{s_1} \lambda_k (\sum_{i=1}^{M(n,k)} e_{ik} - q_k) + \sum_{k=s_1+1}^{s} \lambda_k (\sum_{i=1}^{M(n,k)} e_{ik})$ and $[q_k] = [p_k]$. Therefore there is a unitary $\bar{W}$ such that

$$\bar{W}^* \bar{V} \bar{W} = \mathrm{diag}\left(\bar{u}, \sum_{k=1}^{s_1} \lambda_k p_k, \omega'\right).$$

So

$$\|\bar{W}^* \bar{V} \bar{W} - \mathrm{diag}(\bar{u}, \omega_0, \omega')\| < \varepsilon/4.$$

There is a unitary $\bar{W}_1$ such that

$$\bar{W}_1^* \,\mathrm{diag}(\bar{u}, \omega_0, \omega') \bar{W}_1 = \mathrm{diag}(\bar{u}, \bar{u}_1^*, \bar{u}_1, \bar{u}_2^*, \bar{u}_2, \ldots, \bar{u}_L^*, \bar{u}_L, \omega'),$$

where $\bar{u}_i = \mathrm{diag}(u_i, u_i, \ldots, u_i)$ with $\alpha_1 \alpha_2 \ldots \alpha_{n-1}$ many u_i's. Set

$$\tilde{V} = \mathrm{diag}(\bar{u}, \bar{u}^*, \bar{u}_1, \bar{u}_1^*, \bar{u}_2, \bar{u}_2^*, \ldots, \bar{u}_{L-1}^*, 1, \omega').$$

Then

$$\|\bar{W}_1^* \,\mathrm{diag}(\bar{u}, \omega_0, \omega') \bar{W}_1 - \tilde{V}\| < \varepsilon/8.$$

By [Ph 1, Corollary 5], there is an element $a \in (A_n)_{\mathrm{s.a.}}$ such that

$$\|\tilde{V} - \exp(ia)\| < \varepsilon/2.$$

Hence

$$\|\phi_{n,1}(u) - \exp(i\bar{W}\bar{W}_1 a \bar{W}_1^* \bar{W}^*)\| < \varepsilon/8 + \varepsilon/4 + \varepsilon/8 + \varepsilon/2 = \varepsilon.$$

2. Simple C^*-algebras.

A version of the following lemma first appeared in [Cu].

LEMMA 2.1. *Let A be a unital C^*-algebra and $u \in U(A)$. Then for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following condition:*

If $\lambda_1, \lambda_2, \dots, \lambda_n \in S^1$ are finitely many points and $I_k = \{\zeta \in \mathrm{Sp}(u) \colon |\zeta - \lambda_k| < \delta_k\}$ $(k = 1, 2, \dots, n)$ are finitely many subsets of S^1 with the properties $\delta_k \leqq \delta$ and $I_k \cap I_{k'} = \emptyset$ when $k \neq k'$, furthermore, if q_k are the spectral projections of u corresponding to I_k and if projections $p_k \in \mathrm{Her}(q_k)$, then there exists a unitary $v \in (1 - \sum_{k=1}^n p_k)A(1 - \sum_{k=1}^n p_k)$ such that

$$\left\| u - \left(v + \sum_{k=1}^{n} \lambda_k p_k \right) \right\| < \varepsilon.$$

PROOF. Since $p_k q_k = q_k p_k = p_k$ and $u q_k = q_k u$, we have

$$\begin{aligned}\left\| u\left(\sum_{k=1}^{n} p_k\right) - \left(\sum_{k=1}^{n} \lambda_k p_k\right) \right\| &= \left\| \sum_{k=1}^{n} (q_k u q_k p_k - \lambda_k p_k) \right\| \\ &= \left\| \sum_{k=1}^{n} q_k (u - \lambda_k) q_k p_k \right\| < \delta.\end{aligned}$$

Similarly,

$$\left\| \left(\sum_{k=1}^{n} p_k\right) u - \left(\sum_{k=1}^{n} \lambda_k p_k\right) \right\| < \delta,$$

$$\begin{aligned}\left\| \left(1 - \sum_{k=1}^{n} p_k\right) u \left(1 - \sum_{k=1}^{n} p_k\right) - u\left(1 - \sum_{k=1}^{n} p_k\right) \right\| &= \left\| \left(\sum_{k=1}^{n} p_k\right) u \left(1 - \sum_{k=1}^{n} p_k\right) \right\| \\ &= \left\| \left[\left(\sum_{k-1}^{n} p_k\right) u - \left(\sum_{k=1}^{n} \lambda_k p_k\right) \right] \left(1 - \sum_{k=1}^{n} p_k\right) \right\| < \delta.\end{aligned}$$

Hence

$$\left\| u - \left(1 - \sum_{k=1}^{n} p_k\right) u \left(1 - \sum_{k=1}^{n} p_k\right) - \left(\sum_{k=1}^{n} \lambda_k p_k\right) \right\| < 2\delta.$$

So, if δ is small enough (depends on ε only), then the unitary part of the polar decomposition of

$$\left(1 - \sum_{k=1}^{n} p_k\right) u \left(1 - \sum_{k=1}^{n} p_k\right) + \left(\sum_{k=1}^{n} \lambda_k p_k\right)$$

has the form $v + \sum_{k=1}^n \lambda_k p_k$ and

$$\left\| u - \left(v + \sum_{k=1}^{n} \lambda_k p_k \right) \right\| < \varepsilon.$$

Recall that an ideal J of an ordered group G is a subgroup of G satisfying: $J = J^+ - J^+ (J^+ = J \cap G^+)$ and $0 \leqq a \leqq b \in J$ implies $a \in J$.

Notice that if A is stably finite, then $(K_0(A), K_0(A)_+)$ is an ordered group ([Bl, 6.33]).

PROPOSITION 2.2. *Let A be a C^*-algebra of real rank zero and stable rank one. Then A is simple if and only if $K_0(A)$ is simple.*

PROOF. Assume that $K_0(A)$ is simple. Let I be a (closed) ideal of A. Let p, q be two projections in I with $[p] = [q]$ in $K_0(A)$. Since A has stable rank one, $p \sim q$ in A, i.e. there is a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* = q$. Therefore $u \in I$. Consequently, $p \sim q$ in I. This implies that $K_0(I)$ is an ordered subgroup of $K_0(A)$. If $[p] \leqq [q]$, $q \in I$, there is $u \in A$ such that $u^*u = p$, $uu^* \leqq q$, Hence $u \in I$, so $p \in I$. Consequently, $[p] \in K_0(I)$. So $K_0(I)$ is an ideal of $K_0(A)$. Hence $K_0(A) = K_0(I)$. By the above argument, each projection of A *is in* I. Since A has real rank zero, this implies that $A = I$.

If A is simple, it follows from [Bl 6.3.6] that $K_0(A)$ is simple.

The following is a known result in measure theory.

LEMMA 2.3. *Let μ be a positive Borel measure on interval I. For any $\varepsilon > 0$, and any $\delta > 0$, there are finitely many disjoint open intervals $I_1, \ldots, I_k$ on I such that*

$$mI_i < \delta$$

and

$$\mu(I \backslash \textstyle\bigcup_{i=1}^{k} I_i) < \varepsilon.$$

(where m is the Lebesque measure)

In [Ell 4], Elliott extends the notion of unperforated ordered groups to ordered groups with torsion.

DEFINITION 2.4. An ordered group G is said to be *unperforated* (in the sense of Elliott) if

(i) *G/G_{tor} is unperforated (see [EHS]);*

(ii) if $g \in G^+$ *and* $t \in G_{\text{tor}}$, then $g + t \geqq 0$ *if and only if* t belongs to the ideal of G generated by g;

(iii) any ideal of G is a relatively divisible subgroup ($H \subseteq G$ is said to be relatively divisible if an element of H is divisible by n in H if it is divisible by n in G.)

It is shown in [G] that C^*-algebra A considered in section 1 has $K_0(A)$ unperforated (in the sense of Elliott). All the C^*-algebras with real rank zero classified in [Ell 1] and [Ell 2] have $K_0(A)$ unperforated. We notice that if $K_0(A)$ is simple, this notion of unperforated ordered groups coincides with the notion of weakly unperforarted ordered groups (see [Bl, 6.7.1]).

LEMMA 2.5. (see [P, 1.2]). *Let A be a unital simple C^*-algebra and p a projection in A. Suppose that v is a unitary in pAp. If $[v+(1-p)]=0$ in $K_1(A)$, then $[v]=0$ in $K_1(pAp)$.*

2.6. Let A be a separable unital simple C^*-algebra with real rank zero and stable rank one. Suppose that $K_0(A)$ is unperforated (in the sense of Elliott) and of finite rank. It follows from 2.2 and 2.4 that $K_0(A)/K_0(A)_{\mathrm{tor}} \cong G$ is a simple ordered group with finite rank. By [Zh 1, 1.3] and [EHS], G is a simple dimension group. Let $\Delta=\{\tau\in S\colon \tau(1)=1\}$, where S is the set of positive homomorphisms from G into $\mathbb{R}$ (See [Eff. Chapter 4]). Then the map

$$\theta\colon\quad G\to A=\mathit{Aff}\Delta$$

determines the order on G in the sense that

$$G^+=\{a\in G\colon \theta(a)>>0\}\cup\{0\}.$$

Moreover, $\ker\theta=$ the set of infinitesimal elements. As in [Eff. 4.7], if $G\neq\mathbb{Z}$, $\theta(G)$ is order isomorphic to a dense subgroup of $\mathbb{R}^r$ for some integer r, provided with the relative strict order. Since $\ker\theta=$ the set of infinitesimal elements and $K_0(A)$ is simple, by (ii) of 2.4, $a<b$ in $K_0(A)$ if and only if $\pi(a)<\pi(b)$, where π is the composition map:

$$K_0(A)\to K_0(A)/K_0(A)_{\mathrm{tor}}\to\theta(G).$$

Furthermore, by [Bl, 6.9.2], if $[p]<[q]$, then $p\lesssim q$. *If* $a\in K_0(A)$, then $\pi(a)=(a_1,a_2,\ldots,a_r)$. Set

$$\pi_k(a)=a_k,\qquad k=1,2,\ldots,r.$$

THEOREM 2.7. *Let A be a separable unital simple C^*-algebra with real rank zero and stable rank one. If*

(1) $K_0(A)$ is unperforated (in the sense of Elliott) and of rank n,

(2) there is an integer $K>0$ such that for any finitely many mutually orthogonal projections $p_1,p_2,\ldots,p_m\in A$ and $\varepsilon>0$, there are projections $q_1,q_2,\ldots,q_m\in A$ such that $q_i\leqq p_i$ and $\pi_k(q_i)>\pi_k(p_i)-\varepsilon/m$, $i=1,2,\ldots,m; k=1,2,\ldots,r$, and

$$\mathrm{cer}\left[\left(1-\sum_{i=1}^m q_i\right)A\left(1-\sum_{i=1}^m q_i\right)\right]\leqq K,$$

then $\mathrm{cer}(A)\leqq 1+\varepsilon$. *Moreover, A has weak* (FU).

PROOF. Since A has real rank zero and stable rank one, by [Zh 1, 1.6], $K_0(A)$ has the Riesz interpolation property. If $K_0(A)/K_0(A)_{\text{tor}} = G \cong \mathbf{Z}$, then, by the Riesz interpolation property, $K_0(A) \cong \mathbf{Z}$. It follows from [Li 1, 2.9] that A has a minimal projection p. Since A has real rank zero, we obtain that $pAp \cong \mathbf{C}$. Since A is simple, unital and separable, by [Bn], $A \cong M_n$ for some n. It is well known that $\operatorname{cer}(M_n) = 1$. Therefore, without loss of gnerality, we may assume that $G \neq \mathbf{Z}$.

Step one. For any $1 > \varepsilon > 0$, let $L = (K+1)\left(\left[\frac{8\pi}{\varepsilon}\right] + 1\right)$. For any $v \in U_0(pAp)$, where p is a projection in A, if $\operatorname{cer}(pAp) \leqq K$, there are unitaries

$$v_0 = v, v_1, v_2, \ldots, v_L, v_{L+1} = 1,$$

in A such that

$$\|v_i - v_{i+1}\| < \varepsilon/8, \qquad i = 1, 2, \ldots, L.$$

In fact, there are $h_i \in (pAp)_{\text{s.a.}}$, $i = 1, 2, \ldots, k$ such that

$$v = \exp(ih_1)\exp(ih_2)\ldots\exp(ih_k). \qquad (k \leqq K)$$

Since A has real rank zero, so does pAp (see [BP, 2.8]). Therefore, there are $h_i' \in A_{\text{s.a.}}$ with finite spectra, $i = 1, 2, \ldots, k$, such that

$$\left\| v - \prod_{j=1}^{k} \exp(ih_j') \right\| < \varepsilon/16.$$

Since $\operatorname{Sp}(h_j')$ is finite, $\operatorname{Sp}(\exp(ih_j'))$ is finite. Hence there are $a_j \in A$, $0 \leqq a_j \leqq 2\pi$ such that

$$\exp(ih_j') = \exp(ia_j), \qquad j = 1, 2, \ldots, k.$$

So $\|v - \prod_{j=1}^{k} \exp(ia_j)\| < \varepsilon/16$.

There is $a_{k+1} \in A$ with $0 \leqq a_{k+1} \leqq 2\pi$ such that

$$v = \prod_{j=1}^{k+1} \exp(ia_j).$$

Thus there are unitaries

$$v_0 = v, v_1, v_2, \ldots, v_L, v_{L+1} = 1,$$

such that

$$\|v_i = v_{i+1}\| < \varepsilon/8, \qquad i = 1, 2, \ldots, L.$$

Step two. Fix $u \in U_0(A)$ with $\operatorname{Sp}(u) = S^1$. We will construct mutually orthogonal projections $\{p_k\}$ and $\{p_i^{(1)}\}$ such that

$$\left\| u - v - \sum_{k=1}^{l} \lambda_k p_k - \sum_{i=1}^{m} \alpha_i p_i^{(1)} \right\| < \varepsilon/8,$$

where v is a unitary in $(1 - \sum_{k=1}^{l} p_k - \sum_{i=1}^{m} p_i^{(1)})A(1 - \sum_{k=1}^{l} p_k - \sum_{i=1}^{m} p_i^{(1)})$ and λ_k and α_i are on the unit circle. Furthermore,

$$2L\left[1 - \sum_{k=1}^{l} p_k - \sum_{i=1}^{m} p_i^{(1)}\right] < [p_k]$$

for $k = 1, 2, \ldots, l$.

For each open subset Ω of S^1, let p_Ω be the spectral projection corresponding to Ω. Then p_Ω is an open projection in A^{**}. Let $\{p_n\}$ be an approximate identity for $\mathrm{Her}(p_\Omega)$ consisting of projections ([BP, 2.6 (iii)]).

Define $\mu_k(\Omega) = \lim_{n\to\infty} \pi_k(p_n)$; and if $\Omega = \emptyset$, $\mu_k(\Omega) = 0$. Clearly,

(i) $\mu_k(\Omega) \geqq 0$;

(ii) if $\{\Omega_j\}_{j=1}^{\infty}$ is a sequence of mutually disjoint open sets, then $\mu_k(\bigcup_{j=1}^{\infty} \Omega_j) = \sum_{j=1}^{\infty} \mu_k(\Omega_j)$.

Hence from measure theory, μ_k, defined by

$$\mu_k(S) = \inf\{\mu_k(\Omega) \colon S \subset \Omega, \Omega \text{ is open}\} \qquad \text{for any } S \subset S^1,$$

gives a (positive) Borel measure on S^1. Since $p \leqq 1$ for every projection $p \in A$, we may assume that $\mu_k(S^1) = 1$. For $\varepsilon/8$, choose $\delta > 0$ as in lemma 2.1 with additional restriction that $\delta < \varepsilon/16$. Let

$$0 = \theta_0 < \theta_1 < \ldots < \theta_l < \theta_{l+1} = 2\pi$$

such that $|\theta_{i+1} - \theta_i| < \dfrac{2\pi}{l+1}$, $\dfrac{2\pi}{l+1} < \delta/2$. Put $\lambda_k = e^{i\theta_k}$, $k = 1, 2, \ldots, l$ and $\Omega_k = \{\lambda \in S^1 \colon |\lambda - \lambda_k| < \delta_k\}$, where $\delta_k < \delta/4$ and $\Omega_k \cap \Omega_i = \emptyset$ if $k \neq i$. For any $\eta > 0$, take nonzero projections p_k in $\mathrm{Her}(p_{\Omega_k})$ (Notice that $\mathrm{Her}(p_{\Omega_k})$ has real rank zero) such that

$$\pi_j(p_k) > (1 - \eta)\mu_j(\Omega_k), \qquad j = 1, 2, \ldots, r,$$

where p_{Ω_k} is the spectral projection of u corresponding to Ω_k. (So p_{Ω_k} is an open projection in A^{**}). Set

$$\sigma_j = \min\{\pi_j(p_k) \colon k = 1, 2, \ldots, l\}.$$

Then $\sigma_j > 0$ for each j (Since $S^1 = \mathrm{Sp}(u)$, $p_{\Omega_k} \neq 0$). We may assume that

$$\pi_j(p_k) > \left(1 - \frac{\sigma_j}{4L}\right)\mu_j(\Omega_k), \qquad j = 1, 2, \ldots, r.$$

Set $J_1 = S^1 \setminus \bigcup_{k=1}^{l} \Omega_k$ by applying Lemma 2.3 repeatedly, one obtain a finitely many disjoint open subsegments $\Omega_j^{(1)}, j = 1, 2, \ldots, m$ such that

$$m(\Omega_i^{(1)}) < \delta/4, \qquad \mu_j(\bigcup_{i=1}^m \Omega_i^{(1)}) > \mu_j(J_1) - \frac{1}{4L}\sigma_j\mu_j(J_1)$$

for all $1 \leqq j \leqq d$.

Take projections $p_i^{(1)}$ in $\mathrm{Her}(p_{\Omega_i^{(1)}})$ such that

$$\pi_j(p_i^{(1)}) > \left(1 - \frac{\sigma_j}{4L}\right)\cdot \mu_j(\Omega_i^{(1)})$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$.

Put

$$\Omega = (\bigcup_{k=1}^l \Omega_k) \cup (\bigcup_{i=1}^m \Omega_i^{(1)}).$$

Then $\mu_j(\Omega) > 1 - \dfrac{\sigma_j}{4L}$ for $j = 1, 2, \ldots, r$.

Set $e = \sum_{k=1}^l p_k + \sum_{i=1}^m p_i^{(1)}$, then

$$\begin{aligned}
\pi_j(e) &\geqq \sum_{k=1}^l \left(1 - \frac{\sigma_j}{4L}\right)\mu_j(\Omega_k) + \sum_{i=1}^m \left(1 - \frac{\sigma_j}{4L}\right)\mu_j(\Omega_i^{(1)}) \\
&= \left(1 - \frac{\sigma_j}{4L}\right)\mu_j(\Omega) \\
&\geqq \left(1 - \frac{\sigma_j}{4L}\right)\left(1 - \frac{\sigma_j}{4L}\right) \\
&\geqq 1 - \frac{\sigma_j}{2L}.
\end{aligned}$$

Therefore,

$$\pi_j(1 - e) < \frac{1}{2L}\sigma_j, \qquad j = 1, 2, \ldots, r.$$

Thus by 2.6,

$$2L\left[1 - \sum_{k=1}^{l} p_k - \sum_{i=1}^{m} p_i^{(1)}\right] < [p_k]$$

for $k = 1, 2, \ldots, l$.

By Lemma 2.1, we have

$$\left\| u - v - \sum_{k=1}^{l} \lambda_k p_k - \sum_{i=1}^{m} \alpha_i p_i^{(1)} \right\| < \varepsilon/8,$$

where α_i is the center of $\Omega_i^{(1)}$, $i = 1, 2, \ldots, m$ and v is a unitary in $(1 - e)A(1 - e)$.

Step three. Let $u \in U_0(A)$. We will show that u is close to an exponential. We may assume that $\mathrm{Sp}(u) = S^1$. By condition (2) and the construction of e in step two, it is eady to see that one may choose e with

$$\operatorname{cer}[(1-e)A(1-e)] \leqq K.$$

Since A is simple, by Lemma 2.5, $[v] = 0$ in $K_1((1-e)A(1-e))$. By [Rff, 2.10], $v \in U_0((1-e)A(1-e))$. There are mutually orthogonal projections $q_1, q_2, \ldots, q_{2L}$ in eAe such that

$$q_k \sim 1-e \qquad \text{in A}, \qquad k = 1, 2, \ldots, 2L,$$

Let ω_k be the partial isometry such that

$$\omega_k^* \omega_k = 1-e, \qquad \omega_k \omega_k^* = q_k,$$

$k = 1, 2, \ldots, 2L$. Let $p = 1-e$ as in step one. Define

$$(*) \qquad \bar{v}_1 = \omega_1 v_1^* \omega_1^*,\ \bar{v}_2 = \omega_2 v_1 \omega_2^*,\ \bar{v}_3 = \omega_3 v_2^* \omega_3^*, \ldots, \bar{v}_{2L} = \omega_{2L} v_L \omega_{2L}^*.$$

Then $\sum_{k=1}^{2L} \bar{v}_k$ is a unitary in $(\sum_{k=1}^{2L} q_k)A(\sum_{k=1}^{2L} q_k)$. By [Ph 2, Corollary 5], there exists $h \in (\sum_{k=1}^{2L} q_k)A(\sum_{k=1}^{2L} q_k)_{\text{s.a.}}$ such that

$$\left\| \sum_{k=1}^{2L} \bar{v}_k - \exp(ih) \right\| < \varepsilon/32.$$

Since $(\sum_{k=1}^{2L} q_k)A(\sum_{k=1}^{2L} q_k)$ has real rank zero (see [BP, 2.8]), there are $\beta_1, \ldots, \beta_s$ in S^1 and mutually orthogonal projections $e_1, \ldots, e_s$ in $(\sum_{k=1}^{2L} q_k)A(\sum_{k=1}^{2L} q_k)$ such that

$$\left\| \sum_{k=1}^{2L} \bar{v}_k - \sum_{j=1}^{s} \beta_j e_j \right\| < \varepsilon/8.$$

We may assume that $\beta_i \in I_i = \{e^{i\theta}: \theta_i \leqq \theta < \theta_{i+1}\}$ and $s \leqq l$.

Since $e_j \leqq \sum_{k=1}^{2L} q_k$ and $[\sum_{k=1}^{2L} q_k] = 2L[1-e] < [p_k]$, there are $e_j' \leqq p_j$ such that

$$e_j' \sim e_j \qquad \text{in A}, \qquad j = 1, 2, \ldots, s.$$

So

$$v + \sum_{k=1}^{l} \lambda_k p_k + \sum_{i=1}^{m} \alpha_i p_i^{(1)} = v + \sum_{j=1}^{s} \lambda_j e_j' + \sum_{j=1}^{s} \lambda_j (p_j - e_j') + \sum_{j=s+1}^{l} \lambda_j p_j + \sum_{i=1}^{m} \alpha_i p_i^{(1)}.$$

Therefore, there is a unitary $W \in U(A)$ such that

$$\left\| W^* \left(v + \sum_{k=1}^{l} \lambda_k p_k + \sum_{i=1}^{m} \alpha_i p_i^{(1)} \right) W - \left(v + \sum_{k=1}^{2L} \bar{v}_k + v_0 \right) \right\| < \varepsilon/4,$$

where

$$v_0 = W^*\left(\sum_{j=1}^{s} \lambda_j(p_j - e_j') + \sum_{j=s+1}^{l} \lambda_j p_j + \sum_{i=1}^{m} \alpha_i p_i^{(1)}\right) W.$$

Let $\tilde{v}_1 = w_1 v^* w_1^*$, $\tilde{v}_2 = \bar{v}_2$, $\tilde{v}_3 = w_3 v_1^* w_3^*$, $\tilde{v}_4 = \bar{v}_4, \ldots, \tilde{v}_{2L} = q_{2L}$. (See (*)) Then

$$\left\|\left(v + \sum_{k=1}^{2L} \bar{v}_k + v_0\right) - \left(v + \sum_{k=1}^{2L} \tilde{v}_k + v_0\right)\right\| < \varepsilon/8.$$

Since v_0 has finite spectrum, by [Ph 2, Corollary 5], there is $h_0 \in A_{\mathrm{s.a.}}$ such that

$$\left\|\left(v + \sum_{k=1}^{2L} \tilde{v}_k + v_0\right) - \exp(ih_0)\right\| < \varepsilon/2.$$

And

$$\begin{aligned}
&\left\|W^*uW - \left(v + \sum_{k=1}^{2L} \tilde{v}_k + v_0\right)\right\| \\
&\quad \leqq \left\|W^*uW - W^*\left(v + \sum_{k=1}^{l} \lambda_k p_k + \sum_{i=1}^{m} \beta_i p_i^{(1)}\right) W\right\| \\
&\quad + \left\|W^*\left(v + \sum_{k=1}^{l} \lambda_k p_k + \sum_{i=1}^{m} \beta_i p_i^{(1)}\right) W - \left(v + \sum_{k=1}^{2L} \tilde{v}_k + v_0\right)\right\| \\
&\quad < \varepsilon/8 + \varepsilon/4 + \varepsilon/8 = \varepsilon/2.
\end{aligned}$$

Hence

$$\|u - \exp(iWh_0W^*)\| < \varepsilon.$$

COROLLARY 2.8. *Let A be a separable simple C^*-algebra with real rank zero and stable rank one. If*

(1) Sup{cer(pAp): *p is a projection in A*} $\leqq K$ *for some integer $K > 0$; and*

(2) *$K_0(A)$ is unperforated (in the sense of Elliott) and of finite rank, then* cer$(A) \leqq 1 + \varepsilon$. *Moreover, A has weak* (FU).

3. Inductive Limits of $C(X, M_n)$ with X being a Finite CW Complex.

In this section we will give examples of C^*-algebras satisfying the conditions in 2.7. Other related results will also be given.

THEOREM 3.1. *Let $A = \lim_{\to}(A_n, \phi_n)$ be the C^*-algebraic inductive limit of C^*-algebras A_n of the form $C(X, M_{k(n)})$, where ϕ_n are unital homomorphisms and X is a finite dimensional, connected, compact metric space. If A is simple and of real rank zero, then $K_0(A)$ is unperforated (in the sense of Elliott).*

PROOF. Suppose that $x \in X$, define a map σ_n: $C(X, M_{k(n)}) \to M_{k(n)}$ by

$$\sigma_n(f) = f(x), \qquad f \in C(X, M_{k(n)}).$$

Let $\phi_{1n} = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$ and $\sigma = \sigma_n \circ \phi_{1n}$. Then σ is a homomorphism from $C(X, M_{k(n)})$ into $M_{k(n)}$.

Define τ: $M_{k(n)} \to C(X, M_{k(n)})$ by

$$\tau(a)(x) = a, \qquad \text{for all} \qquad x \in X.$$

Then the composition map $\psi = \sigma \circ \tau$: $M_{k(1)} \to M_{k(n)}$ is unital. Hence $k(1) \mid k(n)$.

It follows from [DNNP] that A has stable rank one. Since A has real rank zero. A has cancellation of projections (See [Bl, 6.5.1]). By proposition 2.2, $K_0(A)$ is simple. Therefore, it suffices to show that $K_0(A)$ is weakly unperforated (see [Bl, 6.7.1]). So we only need to show that if p, q are projections in $M_\infty(A)$ with $mp \prec mq$ for some integer $m \geqq 2$, then $p \prec q$. We may assume that $p, q \in M_N(C(X, M_{k(1)}))$. The relation $mp \prec mq$ implies $\dim(p) < \dim(q)$. Since A is simple, $k(n) \to \infty$, unless $A \cong M_r$ for some integer r. Therefore, we may assume that $k(n)/k(1) > \dim X$. Set $\Omega = \{x \in X, \dim(p(x)) \geqq 1\}$. So

$$\dim \phi_{1n}(q(x)) - \dim \phi_{1n}(p(x)) > \dim X$$

for $x \in \Omega$. By [BDR, Lemma D, (iii)], $p \prec q$.

We believe that the following Lemma is a known result in algebra. We provide a proof since we failed to find a reference in literature.

LEMMA 3.2. *Let G be an inductive limit of finitely generated abelian groups*

$$G_1 \to G_2 \to G_3 \to \ldots$$

and $\operatorname{rank}(G_i) \leqq n$ *for some integer* n (i.e. *for any* i, *there is an injective homomorphism* φ_i *from* $G_i/\operatorname{tor} G_i$ *into* $\mathbf{Z}^n$). *Then* $G/\operatorname{tor} G$ *is a subgroup of* $\mathbf{Q}^n$ ($\mathbf{Q}$ *is the set of all rational numbers*).

PROOF. We say that k elements $x_1, x_2, \ldots, x_k$ in G are linearly independent if for any k integers $l_1, l_2, \ldots, l_k$, $l_1x_1 + l_2x_2 + \ldots + l_kx_k = 0$ implies $l_1 = l_2 = \ldots = l_k = 0$.

From the condition $\operatorname{rank}(G_i) \leqq n$, one can prove that any $n + 1$ elements in G are not linearly independent as follows. If $x_1, x_2, \ldots, x_{n+1} \in G$, then there exist a G_i and $n+1$ elements $y_1, y_2, \ldots, y_{n+1} \in G_i$ such that $\pi_i(y_j) = x_j$ ($j = 1, 2, \ldots, n + 1$), where π_i: $G_i \to G$ is the map induced by the inductive limit. Since $\operatorname{rank}(G_i) \leqq n$, there exist $l_1, l_2, \ldots, l_{n+1}$ (at least one of them is nonzero) such that

$$l_1y_1 + l_2y_2 + \ldots + l_{n+1}y_{n+1} = 0.$$

Hence

$$l_1x_1 + l_2x_2 + \ldots + l_{n+1}x_{n+1} = 0.$$

There is a maximum set $\{x_1, x_2, \ldots, x_k\}$ of linearly independent elements of G $(k \leqq n)$. Define a map $\psi\colon G \to Q^k$ as the follows.

For any element $y \in G$, there exists a set of integers $(l_0, l_1, l_2, \ldots, l_k)$ with $l_0 \neq 0$ such that

$$l_0y + l_1x_1 + l_2x_2 + \ldots + l_kx_k = 0. \tag{1}$$

Define

$$\psi(y) = \left(\frac{-l_1}{l_0}, \frac{-l_2}{l_0}, \ldots, \frac{-l_k}{l_0}\right) \in \mathbb{Q}^k.$$

If $(l'_0, l'_1, \ldots, l'_k)$ $(l'_0 \neq 0)$ is another set of integers such that

$$l'_0y + l'_1x_1 + l'_2x_2 + \ldots + l'_kx_k = 0. \tag{2}$$

Combining (1) and (2), we have

$$(l_1l'_0 - l'_1l_0)x_1 + (l_2l'_0 - l'_2l'_0)x_2 + \ldots + (l_kl'_0 - l'_kl_0)x_k = 0.$$

By linear independence of $\{x_1, x_2, \ldots, x_k\}$, one gets

$$\left(\frac{-l_1}{l_0}, \frac{-l_2}{l_0}, \ldots, \frac{-l_k}{l_0}\right) = \left(\frac{-l'_1}{l'_0}, \frac{-l'_2}{l'_0}, \ldots, \frac{-l'_k}{l'_0}\right).$$

So ψ is well defined. It is obvious that $\ker\psi = \operatorname{tor} G$.

THEOREM 3.3 *Let $A = \lim_{\to}(A_n, \phi_n)$ be the C^*-algebraic inductive limit of C^*-algebras A_n of the form $C(X, M_{k,(n)})$, where ϕ_n are unital homomorphisms and X is a finite* CW *complex. If A is simple and of real rank zero, then*

$$\operatorname{cer}(A) \leqq 1 + \varepsilon.$$

Moreover, A has weak (FU).

PROOF. It follows from 2.7 that it is enough to show that A satisfies conditions (1) and (2) in Theorem 2.7. By [DNNP], A has stable rank one. It follows from 3.1 and 3.2, A satisfies the condition (1) in 2.7. We then show that A satisfies the condition (2) with $K = 4$ *in* 2.7.

We will keep the notations in 2.6. For any $\varepsilon > 0$, since $\theta(G)$ is dense in R^r, there is a projection $e \in A$ such that

$$\pi_k(e) < \varepsilon/2, \qquad k = 1, 2, \ldots, r.$$

We may assume that $e \in A_n = C(X, M_{k(n)})$ for some n. As in 3.1, $\dim \phi_{n,m}(e) \to \infty$

as $m \to \infty$. Therefore we may assume that in A_n, $\dim(e) > 2d$, where d is the dimension of X. Hence, for any projection $e' \in A$ with $\dim(e') \leqq \frac{1}{2}d$, we have

$$\dim(e) - \dim(e') \geqq d.$$

It follows from [BDR, Lemma D (iii)] that e' is equivalent to a subprojection of e. Therefore,

$$\pi_k(e') < \pi_k(e) < \varepsilon/2, \qquad k = 1, 2, \ldots, r.$$

Now let $p_1, p_2, \ldots, p_m \in A$ be mutually orthogonal projections. We may assume that $p_k \in A_n$, $k = 1, 2, \ldots, m$. By [BDR, Lemma D (i)], there are trivial subprojections q_k of p_k in A_n such that

$$\dim(q_k) \geqq \dim(p_k) - \tfrac{1}{2}d.$$

Hence

$$\dim(p_k - q_k) \leqq \tfrac{1}{2}d.$$

By what has been established above, we obtain

$$\pi_k(q_k) > \pi_k(p_k) - \varepsilon.$$

Set $e = \sum_{k=1}^m q_k$. Then $1 - e$ is a trivial projection in A_n. For any unitary $u \in (1 - e)A(1 - e)$, we may assume that $u \in (1 - e)A_L(1 - e)$ for some integer L. Since $1 - e$ is trivial in A_n, there are integers i and j such that $i[1 - e] = j[1]$ in $K_0(A_n)$; so $i[1 - e] = j[1]$ in $K_0(A)$. Therefore, $1 - e$ is also trivial in A_L when L is large enough. So $(1 - e)A_L(1 - e) \cong C(X, M_N)$ for some N. Since $\dim \phi_{n,m}(1 - e) \to \infty$, as we see in 3.1, we may assume that N is as large as we wish. Therefore, by [Ph 3, 3.4], we may assume that $\mathrm{cer}(C(X, M_N)) \leqq 4$. This implies that there are $h_1, h_2, h_3, h_4 \in (1 - e)A_L(1 - e)_{\mathrm{s.a.}}$ such that

$$u = \exp(ih_1) \cdot \exp(ih_2) \cdot \exp(ih_3) \cdot \exp(ih_4).$$

Thus we conclude that A satisfies the condition (2) in 2.7. This completes the proof.

REMARK 3.4. It is shown in [Li 3] that if A is a σ-unital simple C^*-algebra with stable rank one, real rank zero and satisfies the condition (1) in 2.8, then A has trivial K_1-flow; i.e. $K_1(B) = 0$ for every hereditary C^*-algebra B of $M(A)$ which contains A properly. If B is a (non-unital) hereditary C^*-subalgebra of the C^*-algebra A in section 1 or the inductive limit

$$A = \lim_{\to} (A_n, \phi_n)$$

in 3.3, then, by [Li 3], $\operatorname{cer}(B) \leqq 1 + \varepsilon$, and B has trivial K_1-flow. It then follows from [Li 2] that $M(B)/B$ has real rank zero. If we further assume that $K_1(B) = 0$, then $M(B)$ has real rank zero.

ADDED IN PROOF. After this note was revised, we noticed that we could use Theorem 4.7 in the new revision of [Ph 3] instead of Theorem 3.4 in [Ph 3] to simplify the proof of 3.3.

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