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COMPLEX COBORDISM AND FINITE SOLVABLE GROUPS WITH PERIODIC COHOMOLOGY

ABDESLAM MESNAOUI

The complex cobordism ring $U^*(BG)$ is calculated in [2] when G is a finite abelian group (BG denotes a classifying space of G). Such rings are studied in [3] for generalized quaternion groups and in [4] for some particular metacyclic groups.

The purpose of this paper is to calculate the cobordism ring $U^*(BG)$ G being a finite solvable group with periodic cohomology using results contained in [3].

I wish to express my sincere thanks to Professor C. B. Thomas who suggested the subject to me some years ago.

0. Statement of results and preliminaries.

We have the following classification of all finite solvable groups with periodic cohomology (see [7] page 179).

I. $G = \langle A, B \rangle$, $A^m = B^n = 1$, $BAB^{-1} = A^r$, $m \geqq 1$, $n \geqq 1$, $(n(r-1), m) = 1$, $r^n \equiv 1(m)$, $\operatorname{ord} G = mn$.

II. $G = \langle A, B, R \rangle$ with $\langle A, B \rangle$ as in 1, $R^2 = B^{n/2}$, $RAR^{-1} = A^s$, $RBR^{-1} = B^k$, $n = 2^u v$, $u \geqq 2$, $(2, v) = 1$, $s^2 \equiv r^{k-1} \equiv 1(m)$, $k \equiv -1(2^u)$, $k^2 \equiv 1(n)$, $\operatorname{ord} G = 2mn$.

III. $G = \langle A, B, P, Q \rangle$ with $\langle A, B \rangle$ as is 1 and $P^4 = 1$, $P^2 = Q^2 = (PQ)^2$, $AP = PA$, $AQ = QA$, $BPB^{-1} = Q$, $PQB^{-1} = PQ$, $n \equiv 1(2)$, $n \equiv 0(3)$, $\operatorname{ord} G = 8mn$.

IV. $G = \langle A, B, P, Q, R \rangle$ with $\langle A, B, P, Q \rangle$ as in III and $R^2 = P^2$, $RPR^{-1} = QP$, $PQP^{-1} = Q^{-1}$, $RAR^{-1} = A^s$, $RBR^{-1} = B^k$, $k^2 \equiv 1(n)$, $k \equiv -1(3)$, $r^{k-1} \equiv s^2 \equiv 1(m)$, $\operatorname{ord} G = 16mn$.

In the type II condition $(2, v) = 1$ does not appear in the classification of [7] but is a consequence of the proof of 6-11 in the same book.

This paper contains four sections corresponding to the four types described above.

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In sections II, III, IV the generalized quaternien group Γ_k $(k \geqq 3)$ plays a central rôle. We recall that Γ_k is generated by u, v with $u^t = v^2$, $uvu = v$, $t = 2^{k-2}$. We have $U^*(B\Gamma_k) = \Lambda_*/I_*$, $\Lambda = U^*(\mathrm{pt})\,[[X, Y, Z]]$ the $U^*(\mathrm{pt})$-algebra of formal power series in X, Y, Z with coefficients in $U^*(\mathrm{pt})$ graded by taking $\dim X = \dim Y = 2$, $\dim Z = 4$ and I_* a graded ideal generated by six homogeneous formal power series (see [3]). We will also use the fact that $U^*(BZ_m) = U^*(\mathrm{pt})\ [[X]]/([m](X))$, $\dim X = 2$, $[m](X) = F([m-1](X), (X))$, $[1](X) = X$, F being the formal group law. If $P(Y)$ is a homogeneous element of $U^*(\mathrm{pt})\ [[Y]]$, $\dim Y = k$, $P = \Sigma a_i Y^i, a_p \neq 0$, then we shall denote $v(P) = 2kp$. The notation $U^*(\mathrm{pt})\,[[Y]]^+$ will be used for the $U^*(\mathrm{pt})$-algebra of formal power series $Q(Y)$, $Q(0) = 0$.

a) G of type I. We will show that it is sufficient to suppose $m = p^\alpha$, p prime. Let d be the order of r in $(\mathbf{Z}_{p^\alpha})^*$ the multiplicative group of integers q, $\mathrm{mod}(p^\alpha)$, $(p, q) = 1$. There are $X_1 \in U^*(\mathrm{pt})\ [[X]]$, $M(X_1) \in U^*(\mathrm{pt})\ [[X_1]]$, $\dim X = 2$, $\dim X_1 = \dim M(X_1) = 2d$ such that:

THEOREM. $\tilde{U}^*(\mathrm{pt})\quad [[X_1]]^+/I] \times [U^*(\mathrm{pt})\,[[X]]^+/([n](X))]$, $\quad I = U^*(\mathrm{pt})\ [[X_1]] \cap ([p^\alpha](X)) = M(X_1) \cdot U^*(\mathrm{pt})\ [[X_1]]$. (See theorem 1.3 and proposition 1.4).

b) G of type II. This case may be reduced to the following: $\langle A \rangle = \mathbf{Z}_m = \mathbf{Z}_{q_1^\gamma} \times \mathbf{Z}_{q_2^\gamma}$, q_1, q_2 primes, $q_1 \neq q_2$; the conjugation by R on $\mathbf{Z}_{q_1^\gamma}$ is trivial and on $\mathbf{Z}_{q_2^\gamma}$ is the inversion: $x \to -x$; $\langle B^{2^u} \rangle = \mathbf{Z}_v = \mathbf{Z}_{p_1^\alpha} \times \mathbf{Z}_{p_2^\beta}$, p_1, p_2 primes, $p_1 \neq p_2$; the conjugation by R on $\mathbf{Z}_{p_1^\alpha}$ is trivial and on $\mathbf{Z}_{p_2^\beta}$ is the inversion $x \to -x$. Then there are homogeneous elements Y_k of $U^*(\mathrm{pt})\ [[X]]$ and homogeneous formal power series $M_k(Y_k) \in U^*(\mathrm{pt})\ [[Y_k]]$, such that:

THEOREM. $\tilde{U}^*(BG) = \tilde{U}^*(B\Gamma_{u+1}) \times \prod_{K=1}^{4} U^*(\mathrm{pt})\ [[Y_k]]^+/I_k$,

$$I_k = M_k(Y_k) \cdot U^*(\mathrm{pt})\ [[Y_k]]. \quad \text{(see theorem 2.4).}$$

c) G of type III. Let $U^*(\mathrm{pt})\,[[Z]]$ be the $U^*(\mathrm{pt})$-algebra of formal power series graded by taking $\dim Z = 4$. The group $\langle A, B \rangle = G_1$ is of type I.

THEOREM. *There is* $T(Z) \in U^*(\mathrm{pt})\ [[Z]]$ *such that:*

$$\tilde{U}^*(BG) = \tilde{U}^*(BG_1) \times U^*(\mathrm{pt})\ [[Z]]^+/(T(Z)).$$

d) G of type IV. It is enough to suppose that $\langle A \rangle = \mathbf{Z}_{p_1^\alpha} \times \mathbf{Z}_{p_2^\beta}$, $\langle B^{3^u} \rangle = \mathbf{Z}_{q_1^\gamma} \times \mathbf{Z}_{q_2^\delta}$, p_1, p_2, q_1, q_2 primes, $p_1 \neq p_2$, $q_2 \neq q_2$ with the following property: the conjugation by R restricted to $\mathbf{Z}_{p_1^\alpha}, \mathbf{Z}_{q_1^\gamma}$ is trivial and on $\mathbf{Z}_{p_2^\beta}, \mathbf{Z}_{q_2^\delta}$ is the inversion $x \to -x$. The following $U^*(\mathrm{pt})$-algebras $\tilde{U}^*(BZ_{q_2^\delta})^{\langle R\rangle}$, $\tilde{U}(BZ_{p_2^\beta})^{\langle R,B\rangle}$ are calculated respectively in proposition 1.2, lemma 2.1 and lemma 2.3. We shall denote $U^*(\mathrm{pt})\,[[Y,Z]]^+ = \{Q(Y,Z) \in U^*(\mathrm{pt})\,[[Y,Z]],\ Q(0,0)=0\}$, $\dim Y = 2$, $\dim Z = 4$.

THEOREM. $\tilde{U}^*(BG) = (U^*(\mathrm{pt})\,[[X]]^+/([q_1^\gamma](X)) \times \tilde{U}(BZ_{p_1^\alpha})^{\langle B\rangle} \times \tilde{U}^*(BZ_{q_2^\delta})^{\langle R\rangle} \times \tilde{U}^*(BZ_{p_2^\beta}{}^{\langle R,B\rangle} \times U^*(\mathrm{pt})\,[[Y,Z]]^+/L_*, L_*$ *being a graded ideal generated by three homogenous formel power series* $Y(2+J(Z)) + E(Z)$, $Y^2 - YS(Z) - F(Z)$, $G(Z)$.

In the sequel we shall consider exact sequences of groups of the form:

$$1 \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ f\ } S \longrightarrow 1,$$

H being normal in G, S a subgroup de G, $\operatorname{ord} H$, $\operatorname{ord} S$ coprime and G finite, solvable, with periodic cohomology. There is a homomorphism $g\colon S \to G$ such that $f \circ g = 1$, $g^* \circ f^* = 1\colon \tilde{U}^*(BS) \to \tilde{U}^*(BS)$. We shall make use of the topologies on $\tilde{U}^*(BG)$, $\tilde{U}^*(BS)$, $\tilde{U}^*(BH)$ defined by the subgroups $J^{*,*}$ associated to the U^*-Atiyah-Hirzebruch spectral sequences for BG, BS, BH; these topological groups are complete, Hausdorff (see [3]). If μ denotes the edge homomorphism we get a commutative diagram:

$$\begin{array}{ccccc}
\tilde{U}^*(BS) & \longrightarrow & \tilde{U}^*(BG) & \longrightarrow & \tilde{U}^*(BH) \\
{\scriptstyle\mu}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\mu}\downarrow \\
\tilde{H}^*(BS) & \longrightarrow & \tilde{H}^*(BG) & \longrightarrow & \tilde{H}^*(BH)
\end{array}$$

The maps μ are surjective. Moreover S acts on H by inner automorphisms and we have the following exact sequence:

$$0 \longrightarrow \tilde{H}^*(BS) \xrightarrow{\ f^*\ } \tilde{H}^*(BG) \xrightarrow{\ i^*\ } \tilde{H}^*(BH)^S \longrightarrow 0$$

PROPOSITION 0.1. *The short sequence*

$$0 \longrightarrow \tilde{U}^*(BS) \xrightarrow{\ f^*\ } \tilde{U}^*(BG) \xrightarrow{\ i^*\ } \tilde{U}^*(BH)^S \longrightarrow 0$$

is exact.

PROOF. Since $g^* \circ f^* = 1$, $i^* \circ f^* = (f \circ i)^* = 0$ it follows that f^* is injective and $\operatorname{im} f^* \subset \ker i^*$. Let $x \in \tilde{U}^{2n}(BG)$ with $i^*(x) = 0$ and $J^{*,*}$, $J_1^{*,*}$, $J_2^{*,*}$ the filtrations on respectively $\tilde{U}^*(BG)$, $\tilde{U}^*(BH)$, $\tilde{U}(BS)$ associated to the U^*-Atiyah-Hirzebruch spectral sequences for BG, BH, BS. Suppose $x \neq 0$; then there is $p \geqq 0$ such that $x \in J^{2p,2n-2p}$, $x \notin J^{2p+1,2n-2p-1}$. Let φ be the quotient map:
$J^{2p,2n-2p} \to J^{2p,2n-2p}/J^{2p+1,2n-2p-1} = H^{2p}(BG, U^{2n-2p}(\mathrm{pt})$ and φ_1, φ_2 the

similar maps for BH, BS. We have the following commutative diagram, the bottom line being exact:

$$\begin{array}{ccccccccc} & & J_2^{2p,2n-2p} & \xrightarrow{f^*} & J^{2p,2n-2p} & \xrightarrow{i^*} & J^{2p,2n-2p} & & \\ & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_1} & & \\ 0 & \to & \tilde{H}^{2p}(BS, U^{2n-2p}(\mathrm{pt})) & \xrightarrow{f^*} & \tilde{H}^{2p}(BG, U^{2n-2p}(\mathrm{pt})) & \xrightarrow{i^*} & \tilde{H}^{2p}(BH, U^{2n-2p}(\mathrm{pt})))^S & \to & 0 \end{array}$$

There is $y_1 \in J_2^{2p,2n-2p}$ such that $x - f^*(y_1) \in J^{2p+1,2n-(2p+1)} = J^{2p+2,2n-(2p+2)}$. Similar argument shows that there are $y_2 \in J_2^{2p+4,2n-2n-(2p+4)}, \ldots, y_m \in J_2^{2p+2m,2n-(2p+2m)}$ such that $x - f^*(y_1 + y_2 + \ldots + y_m) \in J^{2p+2m+2,2n-(2p+2m+2)}$. Consequently $y = \sum y_i = U^{2n}(BS)$ satisfy $f^*(y) = x$ since f^* is a continuous map. It follows that $\operatorname{im} f^* = \ker i^*$. Now it is clear that $\operatorname{im} i^* \subset U^*(BH)^S$. Let $x \in U^{2n}(BH)^S$, x different from 0. There is $p \geqq 0$, $x \in J_1^{2p,2n-2p} \cap U^{2n}(BH)^S$, $x \notin J_1^{2p+1,2n-2p-1}$. It follows that $\varphi_1(x) \in H^{2p}(BH, U^{2n-2p}(\mathrm{pt}))^S$ and there is $y_1 \in J^{2p,2n-2p}$ such that $x - i^*(y_1) = J_1^{2p+2,2n-(2p+2)} \cap U^*(BH)^S$. As above there are $y_2, \ldots, y_m$ belonging to $U^{2n}(BG)$ with $x - i^*(y_1 + y_2 + \ldots + y_m) \in J_1^{2p+2m+2,2n-(2m+2)}$. As i^* is continuous we get $x = i^*(y)$, $y = \sum_{m \geqq 1} y_m$.

As a direct consequence we obtain:

COROLLARY 0.2. *The map* (g^*, i^*): $\tilde{U}^*(BG) \to \tilde{U}^*(BS) \times \tilde{U}^*(BH)^S$ *is an isomorphism of* $U^*(\mathrm{pt})$*-algebras.*

Section 1.

Suppose G of type I: $G = \langle A, B \rangle$, $A^m = B^n = 1$, $BAB^{-1} = A^r$, $m \geqq 1$, $n \geqq 1$ $(n(r-1), m) = 1$, $r^n \equiv 1(m)$, $\operatorname{ord} G = mn$. We have an exact split sequence: $1 \to H \to G \to S \to 1$ with $H = \langle A \rangle$, $S = \langle B \rangle$. By corollary 0.2 we get: $\tilde{U}^*(BG) = \tilde{U}(BS) \times \tilde{U}^*(BH)^S$. If $p_1^{\alpha_1} \ldots p_n^{\alpha_k}$ is the factorisation of $m = \operatorname{ord} A$, into primes and $H_i = \mathbf{Z}_{p_i^{\alpha_i}}$ then we have: $U^*(BH)^S = \prod_{i=1}^{k} U^*(BH_i)^S$. Hence it is enough to consider the case $m = p^\alpha$, p prime. We need to calculate $H^*(B\mathbf{Z}_{p^\alpha})^S$. Let d be the order of r in the multiplicative group $(\mathbf{Z}_{p^\alpha})^*$ of residues mod p^α of integers prime to p.

PROPOSITION 1.1. *We have* $\tilde{H}^*(B\mathbf{Z}_{p^\alpha})^S = Z_{p^\alpha}[a]^+$, $\dim a = 2d$.

PROOF. We have $\tilde{H}(B\mathbf{Z}_{p^\alpha}) = \mathbf{Z}_{p^\alpha}[b]^+$, $\dim b = 2$ and $H^0(B\mathbf{Z}_{p^\alpha})^S = Z$. As $r^n \equiv 1(p^\alpha)$, we have $d \mid n$. The conjugation by B sends A to A^r and if λ^*: $H(B\mathbf{Z}_{p^\alpha}) \to H^*(B\mathbf{Z}_{p^\alpha})$ denotes the map induced by this conjugation we get $\lambda^*(b) = rb$ and $\lambda^*(b^q) = r^q b^q$. If $k \in \mathbf{Z}$, we have $kb^q \in H^{2q}(B\mathbf{Z}_{p^\alpha})^S$, $q \geqq 1$, iff $k(r^q - 1) \equiv 0(p^\alpha)$. Three cases may occur:

i) $(p, r^q - 1) = 1$. The relation $k(r^q - 1) \equiv 0(p^\alpha)$ implies $k \equiv 0(p^\alpha)$ and $\tilde{H}^{2q}(BZ_{p^\alpha}) = \{0\}$.

ii) $r^q - 1 = p^{\alpha_1}p_1$, $(p, p_1) = 1, 0 < \alpha_1 < \alpha$. If $k(r^q - 1) \equiv 0(p^\alpha)$ then $kp_1 \equiv 0(p^{\alpha - \alpha_1})$ and $k \equiv 0(p)$. Take $c = (1 + r^q + \ldots + r^{(d-1)q})b^q \in H^{2q}(BZ_{p^\alpha})^S$. Then $d \equiv 1 + r^q + \ldots + r^{(d-1)q}(p)$ and $d \equiv 0(p)$ which is impossible since $d \mid n$ and $(p, n) = 1$.

iii) $r^q - 1 = p^{\alpha_1}p_1\alpha_1 \geqq \alpha$. Then $r^q \equiv 1(p^\alpha)$ and $q = kd \geqq 1$. Then $\tilde{H}^{2q}(BZ_{p^\alpha})^S = Z_{p^\alpha}b^q$ Finally $\tilde{H}^*(BZ_{p^\alpha})^S = Z_{p^\alpha}[a]^+$, $a = b^d$.

Let $X_1 = X^d + ([r](X))^d + \ldots + ([r^{d-1}](X))^d$, $x = h(X)$, $x_1 = h(X_1)$, h being the projection map $U^*(\mathrm{pt})\,[[X]] \to U^*(BZ_{p^\alpha})$. We have $\dim x = \dim X = 2$, $\dim x_1 = \dim X_1 = 2d$. Let $U^*(\mathrm{pt})\,[[x_1]] = \{P(x_1),\ P \in U^*(\mathrm{pt})\,[[X_1]]\}$ and $I = U^*(\mathrm{pt})\,[[X_1]] \cap ([p^\alpha](X))$.

PROPOSITION 1.2. *We have* $\tilde{U}^*(BZ_{p^\alpha})^S = U^*(\mathrm{pt})\,[[x_1]]^+ = U^*(\mathrm{pt})\,[[X_1]]^+/I$.

PROOF. We shall use the topology on $\tilde{U}^*(BZ_{p^\alpha})$ defined by the filtration corresponding to the Atiyah-Hirzebruch spectral sequence for BZ_{p^α} (see [3], section 1). The map λ^*: $U^*(BZ_{p^\alpha}) \to U^*BZ_{p^\alpha})$ induced by the conjugation map λ: $A \to BAB^{-1} = A^r$ sends $[r^i](x)$ to $[r^{i+1}](x)$ because x may be taken as the Euler class $e(\eta)$ of a classifying vector bundle η for Z_{p^α} over BZ_{p^α} and obviously we have $[r^{i+1}](x) = e(\eta^r) = e(\lambda^*(\eta^r)) = \lambda^*(e(\eta^r)) = \lambda^*([r^i](x))$, λ being the map: $BZ_{p^\alpha} \to BZ_{p^\alpha}$ corresponding to the conjugation by B. As $[r^d](x) = x$, we get $\lambda^*(x_1) = x_1$ and $x_1 \in U^*(BZ_p)^S$. We observe that λ^*: $\tilde{U}^*(BZ_{p^\alpha}) \to \tilde{U}^*(BZ_{p^\alpha})$ is continuous for the topology defined above and consequently $\lambda^*(P(x_1)) = P(\lambda^*(x_1)) = P(x_1)$ if $P(X_1) \in U^*(\mathrm{pt})\,[[X_1]]$, that is $P(x_1) \in U^*(BZ_{p^\alpha})^S$ if $P(X_1) \in U^*(\mathrm{pt})\,[[X_1]]$. Conversely let $y \in \tilde{U}^*(BZ_{p^\alpha})^S$, $y \neq 0$, $\dim y = 2k$. We may suppose that $y \in U^*(BZ_{p^\alpha})$, $y = \alpha_t x^t + \alpha_{t+1}x^{t+1} + \ldots, \alpha_t \notin p^\alpha U^*(\mathrm{pt})$ because there is $t \geqq 0$ such that $y \in J^{2t, 2k-2t}$, $y \notin J^{2t+1, 2k-2t-1}$ and if φ: $J^{2t, 2k-2t} \to J^{2t, 2k-2t}/J^{2t+1, 2k-2t-1} = H^{2t}(BZ_{p^\alpha}, U^{2k-2t}(\mathrm{pt})) = H^{2t}(BZ_{p^\alpha}) \otimes U^{2k-2t}(\mathrm{pt})$ denotes the quotient map then $\varphi(y) = b \otimes \alpha \neq 0$ and consequently $\alpha_t \notin p^\alpha U^*(\mathrm{pt})$. From the following commutative diagram:

$$\begin{array}{ccc} J^{2t, 2k-2t} & \xrightarrow{\lambda^*} & J^{2t, 2t-2t} \\ \varphi \downarrow & & \downarrow \varphi \\ H^{2t}(BZ_{p^\alpha} \otimes U^{2k-2t}(\mathrm{pt}) & \xrightarrow{\lambda^* \otimes 1} & H^{2t}(BZ_{p^\alpha}) \otimes U^{2k-2t}(\mathrm{pt}) \end{array}$$

it follows that $\varphi(y) = b^t \otimes \alpha_t = \lambda^*(b^t) \otimes \alpha$ and $\lambda^*(b^t) = b^t$. By proposition 1.1 we get $t = qd$, $q \geqq 1$. We denote $x_1 = (1 + r^d + r^{2d} + \ldots r^{d(d-1)})x^d + \beta_1 x^{d+1} + \ldots = (d + h \cdot p^\alpha)x^d + \beta_1 x^{d+1} + \ldots, \beta_i \in U^*(\mathrm{pt})$. As $(d, p) = 1$ there is $s \in Z$ with: $sx_1 = (1 + h'p^\alpha)x^d + \beta'_1 x^{d+1} + \ldots = x^d + \mu_1 x^{d+1} + \ldots$ by using the relation $[p^\alpha](x) = 0$. Hence $y - \alpha_t(sx_1)^q = \alpha'_{t_1}x^{t_1} + \sum_{i > t_1} \alpha'_i x^i, t_1 > t$. We have

$y - \alpha_t(sx_1)^q \in U^*(BZ_{p^\alpha})^S$ and $y - \alpha_t(sx_1)^q \in J^{2t_1, 2k-2t_1}$. By iteration of the same process we obtain a formula of the form:

$$y - [\alpha_t(sx_1)^q + \alpha'_{t_1}(sx_1)^{q_1} + \ldots + \alpha^{(i)}_{t_i}(sx_1^{q_i})] \in J^{2t_{i+1}, 2k-2t_{i+1}}$$

$t_{i+1} > t_i > \ldots > t$, $q_i > q_{i-1} > \ldots > q$. As $\tilde{U}^*(BZ_{p^\alpha})$ is complete Hausdorff it follows that:

$$y = \alpha_t(sx_1)^q + \sum_{i \geqq 1} \alpha^{(i)}_{t_i}(sx_1)^{q_i} \in U^*(\mathrm{pt})\,[[x_1]].$$

The second isomorphism is evident.

From corollary 0.2 and proposition 1.2 we have

THEOREM 1.3. *$\tilde{U}^*(BG) = U^*(\mathrm{pt})\,[[X_1]]^+/I \times U^*(\mathrm{pt})\,[[X]]^+/([n](X))$ as $U^*(\mathrm{pt})$-algebras, $I = U^*(\mathrm{pt})\,[[X_1]] \cap ([p^\alpha](X))$.*

REMARK. K. Shibata in [4] has found a similar result in the case $\alpha = 1$ by using different methods.

The $U^*(\mathrm{pt})$-algebra $U^*(\mathrm{pt})\,[[X_1]]$ is graded by taking $\dim X_1 = 2d$. If $PX_1) = \alpha_t X_1^t + \ldots \alpha_t \neq 0$, we define $v(P)$ as $2dt$. In the next proposition we shall consider the topology on $U^*(\mathrm{pt})\,[[X_1]]$ defined be the filtration corresponding to v.

PROPOSITION 1.4. *There is $M(X_1) \in U^*(\mathrm{pt})\,[[X_1]]$, $M(X_1) = p^\alpha X_1 + \sum_{i \geqq 2} \alpha_i X_1^i$, $\dim M = 2d$, such that: $U^*(BZ_{p^\alpha})^S = U^*(\mathrm{pt})\,[[X_1]]/(M(X_1))$.*

PROOF. Let $\sigma_i(Y_1, \ldots, Y_d)$, $1 \leqq i \leqq d$, be the elementary symmetric polynomials in d variables $Y_1, \ldots, Y_d$. There is $M_i(X_1) \in U^*(\mathrm{pt})\,[[X_1]]$, $\dim M_i(X_1) = 2i$, such that $\sigma_i(x, [r](x), \ldots, [r^{d-1}](x)) = M_i(x_1)$. We observe that $M_i(X_1)$ is well determined mod the ideal $([p^\alpha](X))$ of $U^*(\mathrm{pt})\,[[X]]$. Now $x^k + ([r](x))^k + \ldots + ([r^{d-1}](x))^k = s_k(M_1(x_1), \ldots, M_d(x_1))$, $s_k \in Z[Y_1, \ldots, Y_d]$ being such then $s_k(\sigma_1, \ldots, \sigma_d) = Y_1^k + \ldots + Y_d^k$. If $R_k(X_1) = s_k(M_1(X_1), \ldots, M_d(X_1))$ then $\dim R_k(X_1) = 2k$ and $v(R_k) \geqq 2k$.
Consider the formal power series:

$$N(X) = X^{d-1} \cdot [p^\alpha](X) + ([r](X))^{d-1} \cdot [p^\alpha]([r](X)) + \ldots + ([r^{d-1}](X))^{d-1}[p^\alpha]([r^{d-1}](X))$$

If $[p^\alpha](X) = p^\alpha X + \sum \lambda_i X_i$, then $N(X) = p^\alpha X_1 + \sum \lambda_i R_{d+i-1}(X_1) + [p^\alpha](X) \cdot G(X) = M(X_1) + [p^\alpha](X) \cdot G(X)$. As $\lim_k v(R_k(X_1)) = +\infty$ it follows that that $M(X_1)$ is a well defined formal power series $\mathrm{mod}([p^\alpha](X))$. Moreover $N(x) = M(x_1) = 0$. We observe that $v(R_{d+i-1}) > 2d$, $i \geqq 2$. Hence $M(X_1)$ has the form: $p^\alpha X_1 + \sum_{i \geqq 2} \alpha_i X_1^i$, $\alpha_i \in U^*(\mathrm{pt})$. From this remark it follows that $Q(X_1) \in M(X_1) U^*(\mathrm{pt})\ [[X_1]]$ iff $Q(x_1) = 0$ by a proof similar to that of [3], Theorem 2.12. Furthermore $\dim N(X) = \dim M(X_1) = 2d$.

From corollary 0.2 and proposition 1.4 we have:

THEOREM 1.5. *As* $U^*(\mathrm{pt})$-graded algebras we have:

$$\tilde{U}^*(BG) = U^*(\mathrm{pt})\ [[X]]^+/([n](X)) \times (U^*(\mathrm{pt})\ [[X_1]]^+/((M(X_1)).$$

Section 2.

Suppose G of type II: $G = \langle A, B, R\rangle$, A, B as in section 1, $R^2 = B^{n/2}$, $RAR^{-1} = A^s$, $RBR^{-1} = B^k$, $s^2 \equiv r^{k-1} \equiv 1(m) n = 2^u v$, $(2, v) = 1$, $u \geqq 2$, $k \equiv -1(2^u)$, $k^2 \equiv 1(n)$, $\operatorname{ord} G = 2mn$.

The subgroup $\langle B^{2^u}\rangle = \mathbf{Z}_v$ is normal in $\langle R, B\rangle$ and $\langle R, B\rangle/\langle B^{2^u}\rangle = \langle R, B^v\rangle = \Gamma_{u+1}$ is the generalized quaternion group of order 2^{u+1} (see [7], page 179). As $(2, v) = 1$ we have the split exact sequence:

$$1) \qquad 1 \to \langle B^{2^k}\rangle \to \langle {}^{R,\,B}\rangle \to \langle {}^{R,\,B_v}\rangle = \Gamma_{u+1} \to 1.$$

The conjugation by B^v on $\langle B^{2^u}\rangle = \mathbf{Z}_v$ is trivial and the one by R is of square 1. Hence we have a canonical decomposition $\mathbf{Z}_v = \mathbf{Z}_{v_1} \times \mathbf{Z}_{v_2}$, $v = v_1 v_2$, $(v_1, v_2) = 1$, $R \times R^{-1} = x$ if $x \in \mathbf{Z}_{v_1}$, $R \times R^{-1} = -x$ if $x \in Z_{v_2}$. It is enough to consider the case $v_1 = p_1^\alpha$, $v_2 = p_2^\beta$, p_1, p_2 prime, $p_1 \neq p_2$. Let $G_1 = \langle R, B\rangle$.

LEMMA 2.1. a) $\tilde{U}^*(BG_1) = \tilde{U}^*(B\Gamma_{u+1}) \times \tilde{U}^*BZ_{p_1^\alpha}) \times \tilde{U}^*(BZ_{p_2^\beta})^{\langle R\rangle}$

b) If $X_1 = X + ([p_2^\beta - 1](X))^2$ *then we have*

$$\tilde{U}^*(B\mathbf{Z}_{p_2^\beta})^{\langle R\rangle} = U^*(\mathrm{pt})\ [[X_1]]^+/I_1,\ I_1 = U^*(\mathrm{pt})\ [[X_1]] \cap ([p_2^\beta](X))$$

c) *There is* $M_1(X_1) = p_2^\beta X_1 + \sum_{k \geqq 2} \mu_{1,k} X_1^k$ *such that*

$$\tilde{U}^*(B\mathbf{Z}_{p_2^\beta})^{\langle R\rangle} = U^*(\mathrm{pt})\ [[X_1]]^+/(M(X_1)), \dim M(X_1) = 4.$$

PROOF. The assertion a) is direct consequence of the exact split sequence (1). The proofs of b), c) are identical to that of propositions 1.2 and 1.4.

The order m of A is odd because $(m, r) = (m, r-1) = 1$. Moreover since $\operatorname{ord}\langle R, B\rangle = 2n$ we obtain the following exact split sequence:

$$2) \qquad 1 \to \langle A\rangle \to \langle A, R, B\rangle = G \to \langle R, B\rangle \to 1.$$

There is a decomposition of $\langle A\rangle = \mathbf{Z}_m = \mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$ such that the conjugation by R on $\mathbf{Z}_{m_1}$ is trivial and on $\mathbf{Z}_{m_2}$ is the inversion $x \to -x$. It is sufficient to consider the case $m_1 = q_1^\gamma$, $m_2 = q_2^\delta$, q_1, q_2 prime, $q_1 \neq q_2$. Let $q = q_2^\delta$, $G_1 = \langle R, B\rangle$, d the order of r in the multiplicative group $(\mathbf{Z}_q)^*$. We have $H^*(BZ_q) = Z_q[b]^+$, $\dim b = 2$.

LEMMA 2.2. *We have* $\tilde{H}^*(BZ_q)^{G_1} = \mathbf{Z}_q[b^d]^+$ *if* d *is even and* $\tilde{H}^*(BZ_q)^{G_1} = \mathbf{Z}_q[b^{2d}]^+$ *if* d *is odd.*

PROOF. We have $\tilde{H}^*(BZ_q)^{\langle R\rangle} = \mathbf{Z}_q[b^2]^+$ and $\tilde{H}^*(BZ_q)^{\langle B\rangle} = \mathbf{Z}_q[b^d]^+$ (see proposition 1.1), then $\tilde{H}^*(BZ_q)^{G_1} = \mathbf{Z}_q[b^\alpha]^+$, α being the least common multiple of 2 and d.

We recall that $G_1 = \langle R, B\rangle$.

LEMMA 2.3. a)) d *even* $d = 2d_1$.
If $X_2 = \sum_{i=0}^{d_1-1} ([r^i](X))^d + \sum_{i=0}^{d_1-1} ([q-1)r^i](X))^d$ *then we get:* $\tilde{U}^*(BZ_q)^{G_1} = U^*(\mathrm{pt})[[X_2]]^+/I_2, I_2 = U^*(\mathrm{pt})[[X_2]] \cap ([q](X)), \dim X_2 = 2d.$

b) d *odd if* $X_3 = \sum_{i=0}^{d-1} ([r^i](X))^{2d} + \sum_{i=0}^{d-1} ([(q-1)r^i](X))^{2d}$ *then we have:* $\tilde{U}^*(BZ_q)^{G_1} = U^*(\mathrm{pt})[[X_3]]^+/I_3, I_3 = U^*(\mathrm{pt})[[X_3]] \cap ([q](X)) \dim X_3 = 4d.$

c) *There are* $M_i(X_i) \in U^*(\mathrm{pt})\ [[X_i]]$, $i = 2, 3$ *such that* $M_i(X_i) = qX_i + \sum \mu_{i,k} X_i^k$, $I_i = M_i(X_i)U^*(\mathrm{pt})\ [[X_i]]$, $\dim M_2(X_2) = 2d$, $\dim M_3(X_3) = 4d$.

PROOF. The generator b of $H^*(BZ_q)$ may be taken as the Euler class of a universal vector bundle η over BZ_q. Let f, g be the maps: $BZ_q \to BZ_q$ induced by the conjugations on $\mathbf{Z}_q$ respectively by B and R. The elements x, x_2, x_3 will be the images of X, X_2, X_3 by the quotient map:

$$U^*(\mathrm{pt})\ [[X]] \to U^*(BZ_q).$$

a) d even. If μ: $U^*(BZ_q) \to H^*(BZ_q)$ denotes the edge homomorphism then we have $\mu(x_2) = \sum_{i=1}^{d_1-1} r^{id}b^d + \sum_{i=1}^{d_1-1} (q-1)^d r^{id} b^d = d_1(1 + (q-1)^d)b^d = db^d$. As $(d, q) = 1$, db^d is a generator of the ring $\tilde{H}^*(BZ_q)^{G_1}$. Furthermore $x_2 \in \tilde{U}^*(BZ_q)^{G_1}$ since $f^*(x_2) = g^*(x_2) = x_2$. Then the proof of a) is similar to that of proposition 1.2.

b) d odd. We have: $\mu(x_3) = d(1 + (q-1)^{2d})b^d = 2db^d$. As $(2d, q) = 1, 2db^d$ is a generator $\tilde{H}^*(BZ_q)^{G_1}$ as a ring. Moreover $x_3 \in \tilde{U}^*(BZ_q)^{G_1}$ and we conclude as in a).

c) Consider $N_1(X) = \sum_{i=1}^{d_1-1} ([r^i](X))^{d-1}[qr^i](X) + \sum_{i=1}^{d_1-1} ([r^i](X))^{d-1}[q(q-1)r^i](X)).$
As in proposition 1.4 there is $M_2(X_2) \in U^*(\mathrm{pt})\,[[X_2]]$, $M_2(X_2) = qX_2 + \sum_{k\geq 2} \mu_{2,k} x_2^k$, $\dim M_2(X_2) = 2d$, such that $N_1(X) = M_2(X_2) + [q](X)G(X)$ and $N_1(x) = M_2(x_2) = 0$. It follows that $I_2 = M_2(X_2)U^*(\mathrm{pt})\,[[X_2]]$. Similarly if

$N_2(X) = \sum_{i=0}^{d-1} ([r^i](X))^{2d-1}[qr^i](X)) + \sum_{i=0}^{d-1} ([r^i](X))^{2d-1}[q(q-1)r^i](X)$, then $N_2(X) = M_3(X_3) + [q](X)G''(X)$, $M_3(X_3) = qX_3 + \sum \mu_{3,k}X_3$, $\dim M_3(X_3) = 4d$, $N_2(x) = M_3(x_3) = 0$. We have: $I_3 = M_3(X_3)U^*(\mathrm{pt})\,[[X_3]]$ (see proposition 1.4).

We have seen that $\langle A\rangle = \mathbf{Z}_m = \mathbf{Z}_{q_1^\gamma} \times \mathbf{Z}_{q_2^\delta}$ and consequently: $\tilde{U}^*(\mathbf{Z}_m)^{G_1} = \tilde{U}^*(B\mathbf{Z}_{q_1^\gamma}) \times \tilde{U}^*(B\mathbf{Z}_{q_2^\delta})^{G_1}$. The $U^*(\mathrm{pt})$-agebra $\tilde{U}^*(B\mathbf{Z}_{q_1^\gamma})^{\langle B\rangle}$ is calculated in proposition 1.4.: $\tilde{U}^*(B\mathbf{Z}_{q_1^\gamma})^{\langle B\rangle} = U^*(\mathrm{pt})\,[[X_4]]/I_4$. $X_4 = \sum_{i=0}^{e-1} [r^i](X)^e$, e being the order of r in $(\mathbf{Z}_{q_1^\gamma})^*$, $I_4 = U^*(\mathrm{pt})\,[[X_4]] \cap [q_1^\gamma](X] = M_4(X_4)U^*(\mathrm{pt})\,[[X_4]]$, $M_4 = q_1X_4 + \sum \mu_{4,k}X_4^k$, $\dim M_4(X_4) = 2e$.

Now we can state the main result of section 2 using the above notation: $n = 2^u v$, $u \geqq 2, (2, v) = 1, v = p_1^\alpha p_2^\beta, p_1, p_2$ prime, $p_1 = p_2$, $\langle B^{2^u}\rangle = \mathbf{Z}_{p_1^\alpha} \times \mathbf{Z}_{p_2^\beta}$, $RxR^{-1} = x$ if $x \in \mathbf{Z}_{p_1^\alpha}$, $RxR^{-1} = -x$ if $x \in \mathbf{Z}_{p_2^\beta}$, $\langle A\rangle = \mathbf{Z}_{q_1^\gamma} \times \mathbf{Z}_{q_2^\delta}$, $RxR^{-1} = x$ if $x \in \mathbf{Z}_{q_1^\gamma}$, $RxR^{-1} = -x$ if $x \in \mathbf{Z}_{q_2^\delta}$, d the order of r in $(\mathbf{Z}_{q_2^\delta})^*$ and e its order in $(\mathbf{Z}_{q_1^\gamma})^*$. Then lemmas 2.1, 2.3, the split exact sequence 2) and the remarks after lemma 2.3 imply the following result:

THEOREM 2.4. *We have* $\tilde{U}^*(BG) = \tilde{U}^*(B_{u+1}) \times \tilde{U}^*(\mathrm{pt})\ [[X]]^+/I \times U^*(\mathrm{pt})\ [[X]]^+/I_1 \times U^*(\mathrm{pt})\ [[X_k]]^+/I_k \times U^*(\mathrm{pt})\ [[X_4]]^+/I_4$, *with* $k = 2$ *if* d *is even*, $k = 3$ *if* d *is odd*, $I_j = M_j(X_j) \cdot U^*(\mathrm{pt})\ [[X_j]]$, $j = 1, 2, 3, 4$, $I = ([p_1^\alpha](X)$.

Section 3.

Suppose G of type III: $G = \langle A, B, P, Q\rangle$ with $\langle A, B\rangle$ as in 1, $P^4 = 1$, $P^2 = Q^2 = (PQ)^2$, $AP = PA$, $AQ = QA$, $BPB^{-1} = Q$, $BQB^{-1} = PQ$, n odd, $n \equiv 0(3)$, $\operatorname{ord} G = 8mn$.

The quaternion group G consists of $\{\pm 1, \pm i, \pm j, \pm k\}$ subject to the relations $ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = -1$. The vector bundles $\xi_i, \xi_j, \xi_k, \eta$ over $B\Gamma_k$ are associated to the irreducible representations: ξ_i: $i \to 1$, $j \to 1$, ξ_j: $i \to -1$, $j \to 1$, ξ_k: $i \to -1, j \to -1$,

$$\eta\colon i \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have $\langle P, Q\rangle = \Gamma$ (see [7] page 180) and $\langle P, Q\rangle$ is normal in G. As $\operatorname{ord} \Gamma$ and $\operatorname{ord} G/\langle P, Q\rangle = \operatorname{ord} \langle A, B\rangle$ are coprime we have the split extension:

1) $$1 \to \Gamma = \langle P, Q\rangle \to G \to \langle A, B\rangle \to 1$$

Let $G_1 = \langle A, B\rangle$; then we have $\tilde{U}^*(BG) = \tilde{U}^*(BG_1) \times \tilde{U}^*(B\Gamma)^{G_1} = \tilde{U}^*(BG_1) \times \tilde{U}^*(B\Gamma)^{\langle B\rangle}$. Let A_3, B_3, C_3, D_3 denote the Euler classes of ξ_i, ξ_j, ξ_k,

η as explained in [3], section 2; A_3, B_3, C_3 play a symmetrical role (in order to avoid any confusion with the generators A, B of G_1 we have adopted the notation A_3, B_3 instead of A, B appearing in [3]). We denote $U^*(\mathrm{pt})\,[[D_3]] = \{P(D_3), P(Z) \in U^*(\mathrm{pt})\,[[Z]] = \Omega_*, \dim Z = 4\} \simeq \Omega_*/(T(Z))$ (see [3], theorem 2.18). If $u = PQ$, $v = P$ then $\langle P, Q\rangle = \{u^\alpha v^\beta, \beta = 0, 1, 0 \leqq \alpha \leqq 3, u^2 = v^2, uvu = v\}$. It is easy to see that if λ: $B\Gamma \to B\Gamma$ denotes the map induced by the conjugation by B then $\lambda^*(A_3) = C_3$, $\lambda^*(C_3) = B_3$, $\lambda^*(B_3) = A_3$, $\lambda^*(D_3) = D_3$.

LEMMA 3.1. *We have* $U^*(B\Gamma)^{\langle B\rangle} = U^*(\mathrm{pt})\,[[Z]]/(T(Z))$ *as graded* $U^*(\mathrm{pt})$*-algebras.*

PROOF. As $\lambda^*(D_3) = D_3$ we have $U^*(\mathrm{pt})\,[[D_3]] \subset U^*(B\Gamma)^{\langle B\rangle}$. It is enough to show that if $A_3 M(D_3 + B_3 N(D_3) \in U^*(B\Gamma)^{\langle B\rangle}$ then $A_3 M(D_3) = B_3 N(D_3) = 0$. As $\xi_k^2 = \xi_i \xi_j$ we have $C_3 = A_3 + B_3 + \sum a_{ij} A_3^i B_3^j$ with $F(X, Y) = X + Y + \sum a_{ij} X^i Y^j$ the formal group law. By using the relation $A_3^2 = A_3 S(D_3)$, $B_3^2 = B_3 S(D_3)$, $A_3 B_3 = (A_3 + B_3)(P(D_3) - S(D_3)) - Q(D_3)$ (see [3], proposition 2.10), we see that there are $H(Z) \in \Omega_0$, $H_1(Z) \in \Omega_2$ such that $C_3 = A_3 + B_3 + (A_3 + B_3)H(D_3) + H_1(D_3) = A_3(1 + H(D_3)) + B_3(1 + H(D_3)) + H_1(D_3)$, $v(H) \geqq 4$. We have $\lambda^*(A_3 M(D_3) + B_3 N(D_3)) = C_3 M(D_3) + A_3 N(D_3) = A_3 M(D_3) + B_3 N(D_3)$ and consequently:

$$A_3[M(D_3)H(D_3) + N(D_3))] + B_3[M(D_3)(1 + H(D_3)) - N(D_3)] + M(D_3)H_1(D_3) = 0.$$

Thus $M(Z)H(Z) + N(Z) \in (2 + J(Z))\Omega_*$, $M(Z)(1 + H(Z)) - N(Z) \in (2 + J(Z))\Omega_*$ and then $M(Z)(1 + 2H(Z)) \in (2 + J(Z))\Omega_*$ (see [3], lemma 2.15). It follows that $M(Z) \in (2 + J(Z))\Omega_*$ and $A_3 M(D_3) = B_3 N(D_3) = 0$.

The $U^*(\mathrm{pt})$-algebra $U^*(BG_1)$ is calculated in section 1. Then lemma 3.1 and the sequence 1) show that:

THEOREM 3.2. *We have* $\tilde{U}^*(BG) = \tilde{U}^*(BG_1) \times U^*(\mathrm{pt})\;[[Z]]^+/(T(Z))$ *as* $U^*(\mathrm{pt})$*-algebras.*

Section 4.

Suppose G of type IV: $G = \langle A, B, P, Q, R\rangle$ with $\langle A, B, P, Q\rangle$ as in 3, $R^2 = P^2$, $RPR^{-1} = QP$, $RQP^{-1} = Q^{-1}$, $RAR^{-1} = A^s$, $RBR^{-1} = B^k$, n odd, $n \equiv 0(3)$, $k^2 \equiv 1(n)$, $k \equiv -1(3)$, $r^{k-1} \equiv s^2 \equiv 1(m)$, $\operatorname{ord} G = 16mn$.

We have the following split exact sequence:

1) $$1 \to \langle A\rangle \to G \to \langle B, P, Q, R\rangle \to 1.$$

As the action of R on A is of square 1 we have a decomposition $\langle A\rangle = \mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$, the action of R on Z_{m_1} being the identity and on Z_{m_2} the inversion $x \to -x$. We may suppose $m_1 = p_1^{\alpha}$, $m_2 = p_2^{\beta}$, p_1, p_2 primes, $p_1 \neq p_2$. Let $G_1 = \langle B, P, Q, R\rangle$.

LEMMA 4.1. *We have* $\tilde{U}^*(BG) = \tilde{U}^*(BG_1) \times \tilde{U}^*(BZ_{p_1^{\alpha}})^{\langle B\rangle} \times \tilde{U}^*(BZ_{p_2^{\beta}})^{\langle R, B\rangle}$.

PROOF. This lemma is a consequence of the sequence 1) and the fact that the actions of P, Q on A are trivial.

The $U^*(\mathrm{pt})$-algebras $\tilde{U}^*(BZ_{p_1^{\alpha}})^{\langle B\rangle}$ and $\tilde{U}^*(BZ_{p_2^{\beta}})^{\langle R, B\rangle}$ are calculated respectively in section 1 and section 2. It remains to calculate $\tilde{U}^*(BG_1)$. We may write $n = \mathrm{ord}\langle B\rangle = 3^u v$, $(v, 6) = 1$. *We have* $B^3P = PB^3$, $B^3Q = QB^3$, $RB^3R^{-1} = B^{3k}$ and then $R^2B^3R^{-2} = B^3$ since $k^2 \equiv 1(n)$. Hence the action of R on $\langle B^{3^u}\rangle = Z_v$ is of square 1 and $Z_v = Z_{v_1} \times Z_{v_2}$, $RxR^{-1} = x, x \in Z_{v_1}$, $RxR^{-1} = -x$, $x \in Z_{v_2}$. Furthermore we may suppose $v_1 = q_1^{\gamma}$, $v_2 = q_2^{\delta}$, q_1, q_2 primes, $q_1 \neq q_2$. The sequence:

$$2) \qquad 1 \to \langle B^{3^u}\rangle \to G_1 \to \langle B^v, P, Q, R\rangle \to 1$$

is split exact. Hence with $G_2 = \langle B^v, P, Q, R\rangle$ we get:

LEMMA 4.2. $\tilde{U}^*(BG_1) = \tilde{U}^*(BG_2) \times \tilde{U}^*(BZ_{q_1^{\gamma}}) \times \tilde{U}^*(BZ_{q_2^{\delta}})^{\langle R\rangle ®}$

Now we give some information about $G_2 = \langle B_1, P, Q, R\rangle$ with $B_1 = B^v$. We have $\langle P, Q, R\rangle = \Gamma_4$ the generalized quaternion group of order 24. If $x = RP$, $y = R$ we obtain the classical representation of Γ_4: $\langle P, Q, R\rangle = \{x^{\alpha}y^{\beta}, \beta = 0, 1, 0 \leqq \alpha \leqq 7, x^4 = y^2, xyx = y\}$. We have the relations $\{B_1PB_1^{-1} = Q, B_1QB_1^{-1} = PQ, RB_1R^{-1} = B_1^{-1}\}$ or $\{B_1PB_1^{-1} = PQ, B_1QB_1^{-1} = P, RB_1R^{-1} = B_1^{-1}\}$ according as $v \equiv 1(3)$ or $v \equiv 2(3)$. We shall consider the first case only, the second one being similar. As $H^2(BG_2) = \mathrm{Hom}(G_2, U((1))$, it follows easily that $H^2(BG_2) = Z_2a$, $a = c_1(\rho)$ the first Chern-class of the unitary representation ρ of G_2: $x \to -1, y \to -1, B_1 \to 1$. Moreover well-known results of R. Swan (see [5]) show that $H^4(BG_2) = Z_q g$, $q = 3^u 2^4$, g a generator and $\tilde{H}^*(BG_2)$ is periodic of period 4.

From [3], section III, we recall the following facts. The element $D_4 \in U^*(BG_4)$ denotes the Conner-Floyd class $cf_2(\eta_1)$, where η_1 is the irreducible unitary representation of Γ_4: $x \to \begin{pmatrix} \omega & 0 \\ 0 & \omega - 1 \end{pmatrix} y \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ω a primitive 8th root of unity, for example $\omega = \dfrac{1}{\sqrt{2}}(1 + i)$, $i^2 = -1$. The irreducible unitary representations of Γ_4 of dimension 1 are 1: $x \to 1$, $y \to 1$, ξ_1: $x \to 1$, $y \to -1$, ξ_2: $x \to -1$, $y \to 1$, ξ_3: $x \to -1$, $y \to -1$. If $d_1 = c_1(\eta_1)$, $a_1 = c_1(\xi_1)$, $b_1 = c_1(\xi_3)$, then $\{d_1^p\}$, $\{a_1d_1^p, b_1d_1^p\}$ are generators of respectively $H^{4p}(B\Gamma_4)$ and $H^{4p+2}(B\Gamma_4)$, $p \geqq 0$. Moreover if $B_4 = cf_1(\xi_2)$, $C_4 = cf_1(\xi_3)$ then any element of $U^*(B\Gamma_4)$ can be written in the form $B_4H_1(D_4) + C_4H_2(D_4) + H_3(D_4)$, where H_1, H_2, H_3 are elements of $\Omega_* = U^*(\mathrm{pt})\,[[Z]]$, $\dim Z = 4$. If $H(Z) = a_pZ^p + a_{p+1}Z^{p+1} + \ldots \in \Omega_*$, $a_i \in U^*(\mathrm{pt})$, $a_p \neq 0$, then $v(H)$ denotes the integer $4p$. We shall use the elements $2 + J(Z) \in \Omega_0$, $G_4(Z) \in \Omega_2$, $S(Z) \in \Omega_2$, $L_4(Z) \in \Omega_4$, $T_4(Z) \in \Omega_4$ from [3], section 3, which satisfy the relations: $B_4(2 + J(D_4)) + G_4(D_4) = C_4(2 + J(D_4)) + G_4(D_4) = T_4(D_4) = 0$, $B_4^2 = B_4S(D_4) + L_4(D_4)$, $C_4^2 = C_4S(D_4) + L_4(D_4)$. Denote $E = \{C_4H(D_4) + K(D_4), H(Z), K(Z) \in \Omega_*\} \subset U^*(B\Gamma_4)$. It is easy to see by using the methods of [3] that E is isomorphic to $U^*(\mathrm{pt})\,[[Z]]/J$ as graded $U^*(\mathrm{pt})$-algebras. J_* being the graded ideal generated by $Y(2 + J(Z)) + G_4(Z)$, $Y^2S(Z) + L_4(Z)$, $T_4(Z)$, $\dim Y = 2$, $\dim Z = 4$. Let i be the inclusion $\Gamma_4 \subset G_2$.

LEMMA 4.3. *We have* $(Bi)^*(BG_2) = E$.

PROOF. We show first that $E \subset (B_i)^*(BG_2)$. There is a unitary representation θ of G_2 defined by $\theta(x) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$, $\theta(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\theta(B_1) = \dfrac{1}{\sqrt{2}}\begin{pmatrix} \omega^3 & 1 \\ 1 & \omega \end{pmatrix}$, $\omega = \dfrac{1}{\sqrt{2}}(1 + i)$, $i^2 = -1$. As $i^*(\theta) = \eta_1$ it follows that $(Bi)^*(cf_2(\theta)) = D_4$. Moreover the above definition of the representation ρ of G_2 shows that $i^*(\rho) = \xi_3$ and consequently $(Bi^*)(cf_1(\rho)) = C_4$. So $E \subset (Bi)^*(BG_2)$. Now the inclusion $(Bi)^*(BG_2) \subset E$ is a consequence of the following assertion: if $(Bi)^*(z) = B_4H(D_4)$, $H \in \Omega_*$, then we have $B_4H(D_4) \in E$. Suppose $H(Z) = \gamma_sZ^s + \gamma_{s+1}Z^{s+1} + \ldots, \gamma_s \in 2U^*(\mathrm{pt})$. Then $\gamma_s = 2\gamma'_s$, $H(Z) - \gamma'_sZ^s(2 + J(Z)) = H_1(Z) = \gamma''_{s_1}Z^{s_1} + \ldots$ is such that $v(H) = s \langle v(H_1) = s_1$ and $B_4H(D_4) = B_4H_1(D_4) - \gamma'_sD_4^sL_4(D_4)$ because $B_4(2 + J(D_4)) + L_4(D_4) = 0$. If $\gamma''_{s_1} \in 2U^*(\mathrm{pt})$ we continue the same process. Therefore two cases may occur:

a) $H(Z), H_1(Z), \ldots, H_n(Z), \ldots$ have their first coefficient in $2U^*(\mathrm{pt})$. In this case for each $n > 0$ we get: $B_4H(D_4) = B_4H_{n+1}(D_4) - (\gamma'_sD_4^s + \gamma'_{s_1}D_4^{s_1} + \ldots + \gamma'_{s_n}D_4^{s_n})L_4(D_4)$, $s < s_1 < \ldots < s_n$, $v(H) < v(H_1) < \ldots < v(H_{n+1})$. It follows that $\lim\limits_{\overrightarrow{n\infty}} B_4H_n(D_4) = 0$ and $B_4H(D_4) = -L_4(D_4)\left(\sum\limits_{i=0}^{\infty} \gamma'_{s_i}D_4^{s_i}\right) \in E(s_0 = s)$.

b) There is $n > 0$ such that the first coefficient of $H_n(Z)$ does not belong to $2U^*(\mathrm{pt})$. We will show that this case is impossible. It is enough to prove that we cannot have $(Bi)^*(z) = B_4H(D_4)$, $H(Z) = \gamma_sZ^s + \gamma_{s+1}Z^{s+1} + \ldots$, $\gamma_s \notin 2U^*(\mathrm{pt})$. Let $J^{*,*}$, $J_1^{*,*}$ be the filtrations corresponding to the Atiyah-Hirzebruch spectral sequences respectively for $B\Gamma_4$ and BG_2. As $z \notin 0$, there are k, k' such that $z \in J_1^{k,k'}$, $z \notin J_1^{k+1,k'-1}$. Let φ_1, φ be the quotient maps $J_1^{k,k'} \to J_1^{k,k'}/J_1^{k+1,k'-1} = H^k(BG_2, U^{k'}(\mathrm{pt}))$ and $J^{k,k'} \to J^{k,k'}/J^{k+1,k'-1} = H^k(B\Gamma_4, U^{k'}(\mathrm{pt}))$. It is clear that $k = 4h + 2$ or $k = 4h$. The generator g of $H^4(BG_2)$ may be chosen so that $(Bi)^*(g) = d_1 = c_2(\eta_1)$ and we have $(Bi)^*(a) = b_1$, with $a = c_1(p)$, $b_1 = c_1(\xi_3)$. The above element $B_4H(D_4)$ belongs to $J^{4s+2,|\gamma_s|}$, $|\gamma_s| = \dim \gamma_s$, $B_4H(D_4) \notin J^{4s+3,|\gamma_s|-1}$ because $\gamma_s \notin 2U^*(\mathrm{pt})$. Three cases only may take place.

1) $$z \in J_1^{4h+2,h'}, z \notin J_1^{4h+3,h'-1}$$

Consider the commutative diagram:

$$\begin{array}{ccc} J_1^{4h+2,h'} & \xrightarrow{(Bi)^*} & J^{4h+2,h'} \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} \\ Z_2ah^h \otimes U^{h'}(\mathrm{pt}) & \xrightarrow{(B_i)^* \otimes 1} & (Z_2a_1d_1^h \oplus Z_2b_1d_1^h) \otimes U^{h'}(\mathrm{pt}) \end{array}$$

We have $\varphi_1(z) = ag^h \otimes \lambda \neq 0$ and then $\varphi(B_4H(D_4)) \neq 0$. It follows that $h = s$. As $\varphi(B_4H(D_4)) = a_1d_1^h \otimes \gamma_s$ we get: $b_1d_1^h \otimes \lambda = a_1d_1^h \otimes \gamma_s \neq 0$ which is impossible.

2) $$z \in J_1^{4h,h'}, z \notin J_1^{4h+1,h'-1}, \varphi_1(y) = pg^h \otimes \lambda, 2^4 \nmid p.$$

We have by using a similar diagram $4s + 2 = 4h$ which is impossible.

3) $$z \in J_1^{4h,h'}, z \in J_1^{4h+1,h'-1}, \varphi_1(y) = pg^h \otimes \lambda, 2^4 \mid p.$$

Then $\varphi_1(3^uz) = 0$ and $3^uz \in J_1^{k,k'}$, $k > 4h$. But $3^uB_4H(D_4) \in J^{4s+2,|\gamma_s|}$, $3^uB_4H(D_4) \notin J^{4s+3,|\gamma_s|-1}$. If 3^uz falls in the case 1) or 2) we have a contradiction, otherwise we form $3^{2u}z$, $3^{2u}B_4H(D_4)$ and then $3^{2u}z \in J_1^{n,n'}$, $n > k > 4h$. Therefore we see that after a finite number of operations we have either a contradiction or there is $p > 0$ such that

$$3^{pu}z \in J_1^{t,t'}, t \geqq 4s + 3, t + t' = 4s + |\gamma_s| + 2.$$

Hence $(Bi)^*(3^{pu}z) = 3^{pu}B_4H(D_4) \in J^{t,t'} \subset J^{4s+3,|\gamma_s|-1}$ which is in contradiction with $3^{pu}B_4H(D_4) \notin J^{4s+3,|\gamma_s-1}$. This ends the proof of lemma 4.3.

LEMMA 4.4. *There is $\tilde{D} \in U^4(BG_2)$ such that $(Bi)^*(\tilde{D}) = D_4$, $\mu(Bj)^*(\tilde{D}))$ is a generator of $H^4(BZ_{3^u}) = Z_{3^u} \subset G_2$, μ being the edge homomorphism, i: $G_4 \subset G_2$, j: $Z_{3^u} \subset G_2$.*

PROOF. We have $H^4(BG_2) = Z_qg$, $q = 2^43^u$; g can be chosen so that $(Bi)^*(g) = d_1 = c_1(\eta_1)$, $(Bj)^*(g) = c_1^2(\rho)$, which are generators respectively of $H^4(BG_4)$ and $H^4(BZ_{3^u})$. Let $D_0 \in \tilde{U}^4(BG_2)$ be such that $\mu(D_0) = g$. By lemma 4.3 $(Bi)^*(D_0) = C_4Q(D_4) + R(D_4)$, $Q, R \in \Omega_*$. We have $\mu((Bi)^*(D_0)) = d_1 = \mu(R(D_4))$; so we may write: $R(Z) = Z + \sum_{i \geqq 2} \gamma_i Z^i$. Now we refer to the notation used in the proof of lemma 4.3 and take $D_1 = cf_2(\theta)$, $\tilde{C} = cf_1(\rho)$. We have $(Bi)^*(D_0) = (Bi)^*(\tilde{C}Q(D_1) + R(D_1))$. Consider $\tilde{D} = D_0 - \left(\tilde{C}Q(D_1) + \sum_{i \geqq 2} \gamma_i D^i\right) =$

$D_0 - (\tilde{C}Q(D_1) + R(D_1)) + D_1 \in \tilde{U}^4(BG_2)$. We have $(Bi)^*(\tilde{D}) = (Bi)^*(D_1) = D_4$ and $\mu(Bj)^*(\tilde{D})) = \mu(Bj)^*(D_0)) = (Bj)^*(g)$ which is a generator of $H^4(B\mathbf{Z}_{3^u})$.

THEOREM 4.5. *There are $E(Z) \in \Omega_2$, $F(Z) \in \Omega_4$, $G(Z) \in \Omega_4$ such that if L_* is the graded ideal of $U^*(\mathrm{pt})$ $[[Y, Z]]$ $(\dim Y = 2,\ \dim Z = 4)$ generated by $Y(2 + J(Z)) + E(Z)$, $Y^2 - YS(Z) - F(Z)$, $G(Z)$ then $U^*(BG_2) = U^*(\mathrm{pt})$ $[[Y, Z]]/L_*$ as graded $U^*(\mathrm{pt})$-algebras.*

PROOF. The element $\tilde{D} \in U^4(BG_2)$ of lemma 4.4 is such that $\mu(\tilde{D})$ is a generator of $H^4(BG_2)$, μ: $U^*(BG_2) \to H^*(BG_2)$ being the edge homomorphism. We recall that $\tilde{C} = cf_1(\rho)$ and $\mu(\tilde{C})$ is a generator of $H^2(BG_2)$. It follows that for any element $z \in U^*(BG_2)$ (there are $P(Z), Q(Z) \in \Omega_*$ such that $z = \tilde{C}P(\tilde{D}) + Q(\tilde{D})$ (see [3]). As $\rho^2 = 1$ we get $0 = [2](\tilde{C}) = 2\tilde{C} + a_{11}\tilde{C}^2 + \ldots$. Hence by using the relation $\tilde{C}^2 = \tilde{C}P(\tilde{D}) + Q(\tilde{D})$ for some $P \in \Omega_2, Q \in \Omega_4$ we see that there are $J_1 \in \Omega_0$, $E_1 \in \Omega_2$ satisfying the relation $\tilde{C}(2 + J_1(\tilde{D})) + E_1(\tilde{D}) = 0$, $v(J_1) \geqq 4$. We have $0 = (Bi)^*(\tilde{C}(2 + J_1(\tilde{D})) + E_1(\tilde{D}) = C_4(2 + J_1(D_4)) + E_1(D_4)$. Consequently we can find $H \in \Omega_0$, H a unit of Ω_*, such that $2 + J_1(Z) = (2 + J(Z))H(Z))$ (see [3], section III). If $H_1 \in \Omega_0$, $H_1 = H^{-1}$, then we have $\tilde{C}(2 + J(\tilde{D})) + H_1(\tilde{D})E_1(\tilde{D}) = 0$. Hence:

1) $$\tilde{C}(2 + J(\tilde{D})) + E(\tilde{D}) = 0,\ E(Z) = H_1(Z) \cdot E_1(Z) \in \Omega_2.$$

There are $P, Q \in \Omega_*$ such that $\tilde{C}^2 = \tilde{C}P(\tilde{D}) + Q(\tilde{D})$ and as a consequence $C_4^2 = C_4P(D_4) + Q(D_4)$. But $C_4^2 = C_4S(D_4) + L_4(D_4)$ (see [3], proposition 3.9). So: $C_4(S(D_4) - P(D_4)) + L_4(D_4) - Q(D_4) = 0$. There is $H(Z) \in \Omega_2$ such that $S(Z) - P(Z) = H(Z)(2 + J(Z))$ and by using 1) we get:

2) $$\tilde{C}^2 = \tilde{C}S(\tilde{D}) + F(\tilde{D}),\ F \in \Omega_4.$$

Let $J_1^{*,*}$ be the filtration corresponding to the reduced Atiyah-Hirzebruch spectral sequence for BG_2. We have $\tilde{D} \in U^4(BG_2) = J_1^{4,0}$, $q\tilde{D} \in J_1^{6,-2}$, $q = 2^4 3^u$. From the commutative diagram (χ is the canonical map):

$$\begin{tikzcd} J_1^{6,0} \otimes U^{-2}(\mathrm{pt}) \arrow[r, "\chi"] \arrow[d, "\varphi_1 \otimes 1"'] & J_1^{6,-2} \arrow[d, "\varphi_1"] \\ \mathbf{Z}_2 a \cdot g \otimes U^{-2}(\mathrm{pt}) \arrow[r, "\simeq"] & H^6(BG_2, U^{-2}(\mathrm{pt})), \end{tikzcd}$$

where φ_1 denotes te quotient map:

$$J_1^{k,k'} \to J_1^{k,k'}/J_1^{k+1,k'-1} = H^k(BG_2, U^{k'}(\mathrm{pt})),$$

we see that there is $\alpha_1 \in U^{-2}(\mathrm{pt})$ such that $q\tilde{D} - \alpha_1\tilde{C}\tilde{D} \in J_1^{8,-4}$.
We continue the same process and as $U^*(BG_2)$ is complete, Hausdorff for the topology defined by the filtration $J_1^{*,*}$ there are $Q_0(Z) = qZ + \gamma_2 Z^2 + \ldots$,

$P_0(Z) = \alpha_1 Z + \alpha_2 Z^2 + \ldots$ such that $\tilde{C}P_0(\tilde{D}) + Q_0(\tilde{D}) = 0$. Hence $C_4 P_0(D_4) + Q_0(D_4) = 0$ and lemma 3.10 of [3] shows that: $YP_0(Z) + Q_0(Z) = M_1(Z)Y(2 + J(Z)) + G_4(Z)] + M_2(Z)T_4(Z)$, $M_1 \in \Omega_2$, $M_2 \in \Omega_0$. So $Q_0(Z) = M_1(Z)G_4(Z) + M_2(Z)T_4(Z)$. From $C_4(2 + J(D_4)) + G_4(D_4) = 0 = C_4(2 + J(D_4)) + E(D_4)$ it follows that $G_4(Z) = E(Z) + M_3(Z)T_4(Z)$, $M_3 \in \Omega_2$. Then $Q_0(Z) = M_1(Z)E(Z) + N(Z)T_4(Z)$. As $v(M_1) \geqq 4$, $v(E) \geqq 4$, we have: $M_1(Z)E(Z) = \sum_{i \geqq 2} \beta_i Z^i$ and therefore $N(Z)T_4(Z) = qZ + \sum_{i \geqq 2} h_i Z^i = G(Z)$. We want to prove that $G(\tilde{D}) = 0$. It is enough to show that $(Bi)^*(G(\tilde{D})) = (Bj)^*(G(\tilde{D})) = 0$, i: $\Gamma_4 \subset G_2$, j: $Z_{3^u} \subset G_2$. We have $(Bi)^*(G(\tilde{D})) = T_4(D_4)N(D_4) = 0$. It is clear that $(Bj)^*(E(\tilde{D})) = (Bj)^*(Q_0(\tilde{D})) = 0$ because $(Bj)^*(\tilde{C}) = 0$. As $Q_0(\tilde{D}) = M_1(\tilde{D})E(\tilde{D}) + G(\tilde{D})$ we get $(Bj)^*(G(\tilde{D})) = 0$. Hence:

3) $$G(\tilde{D}) = 0,\ G(Z) = 2^4 3^u Z + h_2 Z^2 + \ldots \in \Omega_4.$$

Now the relations 1), 2), 3) show by using the methods of [3], section III, that $U^*(BG_2) = U^*(\mathrm{pt})\ [[Y, Z]]/L_*$, L_* being the graded ideal generated by $Y(2 + J(Z)) + E(Z)$, $Y^2 - YS(Z) - F(Z)$, $G(Z)$.

We recall the notation used in this section: $\langle A \rangle = \mathbf{Z}_{p_1^\alpha} \times \mathbf{Z}_{p_2^\beta}$, $\langle B^{3^u} \rangle = \mathbf{Z}_{q_1^\gamma} \times \mathbf{Z}_{q_2^\delta}$, p_1, p_2, q_1, q_2 primes, $p_1 \neq p_2$, $q_1 \neq q_2$; the conjugations by R on $\mathbf{Z}_{p_1^\alpha}$, $\mathbf{Z}_{q_1^\gamma}$ are trivial and on $\mathbf{Z}_{p_2^\beta}$, $\mathbf{Z}_{q_2^\delta}$ are the inversion $x \to -x$; $\tilde{U}^*(B\mathbf{Z}_{p_1^\alpha})^{\langle B \rangle}$, $\tilde{U}^*(B\mathbf{Z}_{q_2^\delta})^{\langle R \rangle}$, $\tilde{U}^*(B\mathbf{Z}_{p_2^\beta})^{\langle R, B \rangle}$ are calculated respectively in proposition 1.2, lemma 2.1 and lemma 2.3; $\tilde{U}^*(B\mathbf{Z}_{q_1^\gamma}) = U^*(\mathrm{pt})\ [[X]]^+/([q_1^\gamma](X))$. Finally $U^*(\mathrm{pt})\ [[Y, Z]]^+ = \{Q(Y, Z) \in U^*(\mathrm{pt})\,[[Y, Z]],\ Q(0, 0) = 0\}$. Then lemma 4.1, lemma 4.2 and theorem 4.5 give the following result.

THEOREM 4.6. *$\tilde{U}^*(BG) = U^*(\mathrm{pt})\ [[X]]^+/([q_1^\gamma](X)) \times U^*(B\mathbf{Z}_{p_1^\alpha})^{\langle B \rangle} \times \tilde{U}^*(B\mathbf{Z}_{q_2^\delta})^{\langle R \rangle} \times \tilde{U}^*(B\mathbf{Z}_{p_2^\beta})^{\langle R, B \rangle} \times U^*(\mathrm{pt})\,[[Y, Z]]^+/L_*$, L_* being a graded ideal generated by three homogeneous formal power series $Y\,(2 + J(Z)) + E(Z)$, $Y^2 - YS(Z) - F(Z)$, $G(Z)$.*

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DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ MOHAMMED V
FACULTÉ DES SCIENCES BP 1014
RABAT
MOROCCO