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ON THE REGULARITY OF GRADED k-ALGEBRAS OF KRULL DIMENSION $\leqq 1$

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Let $F_1, \ldots, F_p$ be homogeneous polynomials in $A = \mathbf{C}[X_1, \ldots, X_n]$, such that the Krull dimension of $R = A/(F_1, \ldots, F_p)$ is at most one. In his article [1] Briançon gives an upper bound for the degrees of a system of generators for the relations between $F_1, \ldots, F_p$, depending on the degrees of the polynomials (which is less than D, say). In this article we generalize Briançon's results in two ways, using homological algebra.

The relations can be measured by $\mathrm{Tor}_2^A(R, \mathbf{C})$. First we extend the results to cover also all higher syzygies $\mathrm{Tor}_i^A(R, \mathbf{C})$, $i \geqq 2$. We will show (especially) that

$$\mathrm{Tor}_{i,k}^A(R, \mathbf{C}) = 0 \quad \text{for} \quad k > nD - (n-1) + i - 1,$$

that is, I has Castelnuovo regularity $nD - (n-1)$.

Next, we relax a little on the condition $\dim R \leqq 1$ to cover e.g. the case R a two-dimensional domain. If this is the case, and, if in addition, R is Cohen-Macaulay, we compare our result with that of Gruson-Lazarsfeld-Peskine, which states that I has Castelnuovo regularity $s - n + 3$ ($s = \text{mult. } R$).

After this introduction, we will now be more precise.

Let $A = k[X_1, \ldots, X_n]$, k an infinite field (if k is finite, just extend with a transcendent), and let I be a homogeneous ideal in A, with $\dim A/I = n - c$. We begin by recalling some definitions from [1].

Firstly, for the definition of characteristic sequence for I, find a set of generators for I; say $F_1, \ldots, F_p$ with degrees $d_1, \ldots, d_p$ respectively ($d_i \leqq d_{i+1}$). Let $I_j = (F_1, \ldots, F_j)$, $1 \leqq j \leqq p$, and put $i_t = \inf\{j \mid \dim A/I_j = n - t\}$ for $1 \leqq t \leqq c$. Now, define the characteristic sequence for I as

$$\delta(I) = (d_{i_1}, \ldots, d_{i_c}).$$

This sequence is independent of the choice of generators for I.

Secondly, we'll define two numbers, α and β, for an ideal I, when A/I is of

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dimension 0 or 1. Let the characteristic sequence be $\delta_1, \dots, \delta_n$ in the 0-dimensional case, and $\delta_1, \dots, \delta_{n-1}$ in the 1-dimensional case.

DEFINITION 1. $\alpha(I) = \sum \delta_i - (n-1)$

DEFINITION 2. $\beta(I) = \inf\{s \mid (A/I)_s = 0\}$, in the 0-dimensional case, $\beta(I) = \inf\{s \mid \dim_k (A/I)_t = \text{mult.}\,(A/I) \text{ for all } t \geqq s\}$, in the 1-dimensional case.

Next, we list some results from [1] (I is here homogeneous with characteristic sequence $\delta_1, \dots, \delta_c$).

1) There is a regular sequence $G_1, \dots, G_c$ of forms in I with degrees $\delta_1, \dots, \delta_c$ resp.

2) If $\dim A/I = 0$, then $\beta(I) \leqq \alpha(I)$.

3) If $\dim A/I = 1$, then $\beta(I) \leqq \alpha(I) + D$, where D is a majorant for the degrees of a set of generators for I.

We now turn to the generalizations mentioned earlier.

THEOREM 1. *If* $\dim A/I = 0$, *then* $\mathrm{Tor}^A_{i,p}(A/I, k) = 0$ *for* $p \geqq \beta(I) + i$.

PROOF. $X_1, \dots, X_n$ is a regular sequence in A, so the Koszul complex $K(A; X_1, \dots, X_n)$ is acyclic, that is

$$\mathsf{K} \qquad 0 \to A[-n] \to \dots \to \bigoplus_{1}^{\binom{n}{2}} A[-2] \to \bigoplus_{1}^{n} A[-1] \to A \to 0$$

is exact, but in dimension 0, where the homology is k. Applying the functor $A/I \otimes_A \cdot$ on K gives us the complex

$$0 \to \bar{A}[-n] \to \dots \to \bigoplus_{1}^{n} \bar{A}[-1] \to \bar{A} \to 0 \qquad (\bar{A} = A/I)$$

Now, $\mathrm{Tor}^A_i(A/I, k) = H_i(A/I \otimes_A \mathsf{K})$, so it remains to show

$$H_{i,p}(A/I \otimes_A \mathsf{K}) = 0 \quad \text{for} \quad p \geqq \beta(I) + i.$$

But, if we let $T_{j_1} \dots T_{j_i}$, $1 \leqq j_1 < \dots < j_i \leqq n$ be a basis for $(A/I)^{\binom{n}{i}} = \bar{A}_i$, and r an arbitrary element in A/I, we get

$$\deg(rT_{j_1} \dots T_{j_i}) = \deg r + i \leqq \beta(I) - 1 + i,$$

so every element in $\bar{A}_i$ has degree $\leqq \beta(I) - 1 + i$, thus

$$(A/I \otimes_A \mathsf{K})_{i,p} = 0 \qquad \text{for} \quad p \geqq \beta(I) + i.$$

Done.

Let's invoke some notation. $Z(R) = \{\text{zero-divisors in } R\}$ (R any ring). Let

$R = \oplus_0^\infty R_i$ be a graded k-algebra, and $m = \oplus_{i>0} R_i$ it's graded maximal ideal. Define the socle of R as $\mathrm{Soc}(R) = 0 : m$.

It's easy to verify that the following statements are equivalent.

i) $\mathrm{Soc}(A/I) = 0$

ii) $m \notin \mathrm{Ass}(I)$

iii) There is a non-zero divisor of positive degree in A

Before stating and proving the next theorem we state this

LEMMA 1. $\bar{X} \notin Z(A/I)$, $\deg X = 1 \Rightarrow \mathrm{Tor}_i^A(A/I, k) \simeq \mathrm{Tor}_i^{A/(X)}(A/I + (X), k)$

PROOF. We use the well-known change of rings spectral sequence (see [4], p. 364)

$$\mathrm{Tor}_p^{A/(X)}(\mathrm{Tor}_q^A(A/I, A/(X)), k) \underset{p}{\Rightarrow} \mathrm{Tor}_n^A(A/I, k)$$

It's enough to show that this sequence collapses to the p-axis. Consider the exact sequence $0 \to A \xrightarrow{X} A \to A/(X) \to 0$, and tensor it with A/I; this gives $\mathrm{Tor}_q^A(A/I, A/(X)) = 0$ for $q > 0$.

THEOREM 2. *If* $\dim A/I = 1$, *then* $\mathrm{Tor}_{i,p}^A(A/I, k) = 0$ *for* $p > \alpha(I) + D + i - 1$ *(D is the majorant mentioned earlier).*

PROOF. We divide the proof in two parts, which together give the result. Let's write $A/I = \oplus_0^\infty R_i$. We will show

a) If $\mathrm{Soc}(A/I) \subset \oplus_0^N R_i$, then $\mathrm{Tor}_{i,p}^A(A/I, k) = 0$ for $p > \max(N, \alpha(I)) + i$

b) $\mathrm{Soc}(A/I) \subset \oplus_0^N R_i$, with $N = \alpha(I) + D - 1$

PROOF OF a). I. First assume $\mathrm{Soc}(A/I) = 0$. Choose a regular sequence $G_1, \ldots, G_{n-1}$ (according to 1) in the list above). Let $G = (G_1, \ldots, G_{n-1})$. Now choose an element X of degree 1 such that $\bar{X} \notin Z(A/I) \cup Z(A/G)$ (this is possible, since $m \notin \mathrm{Ass}(I) \cup \mathrm{Ass}(G)$ and k infinite). Applying the lemma and Theorem 1 we get

$$\mathrm{Tor}_{i,p}^A(A/I, k) \simeq \mathrm{Tor}_{i,p}^{A/(X)}(A/I + (X), k) = 0 \text{ for } p \geqq \beta(I + (X)) + i.$$

Now, $\beta(I + (X)) \leqq \alpha(I + (X)) \leqq \alpha(G + (X)) = \alpha(I) + 1$, where the first inequality is due to 2) above, and the second to the fact that $G + (X) \subset I + (X)$. Hence $\mathrm{Tor}_{i,p}^A(A/I, k) = 0$ for $p > \alpha(I) + i$.

II. Now assume $\mathrm{Soc}(A/I) \neq 0$. Choose $\sigma_1 \in \mathrm{Soc}(A/I)$ of maximal degree. Then, choose $\sigma_2 \in \mathrm{Soc}(A/I + (\sigma_1))$ of maximal degree. Note that $\deg \sigma_2 \leqq \deg \sigma_1$, and that factoring out with σ_1 hasn't changed the dimension (it's still 1). Continue this process, until it terminates (there are only finitely many monomials of degree $\leqq N$). This gives us elements $\sigma_1, \ldots, \sigma_t$, such that

$$\mathrm{Soc}(A/I') = 0, \text{ where } I' = I + (\sigma_1, \ldots, \sigma_t).$$

Let K be the Koszul complex $K(A; X_1, \ldots, X_n)$, which, as pointed out before, is a resolution of k. Put $\mathsf{K}_{A/J} = \mathsf{K} \otimes_A A/J$ for J any ideal.

CLAIM 1. *$H_{i,p}(\mathsf{K}_{A/I}) \simeq H_{i,p}(\mathsf{K}_{A/I'})$ for $p > N + i$*

Before proving the claim, let's finish the argument. The right-hand side is 0 for $p > \alpha(I) + i$, according to I above. It remains only to verify that $\alpha(I) \geqq \alpha(I')$; but this is clear, since $I \subset I'$.

PROOF OF CLAIM. Let $S_j = (\sigma_1, \ldots, \sigma_j)$, $1 \leqq j \leqq t$ $(S_0 = 0)$. We will show, by induction over j, that $H_{i,p}(\mathsf{K}_{A/I+S_j}) \simeq H_{i,p}(\mathsf{K}_{A/I})$ for $p > N + i$ (note $I + S_t = I'$). Consider the exact sequence

$$0 \to \sigma_j(A/I + S_{j-1}) \to A/I + S_{j-1} \to A/I + S_j \to 0$$

Tensoring the complex K with each term in this sequence gives us a short exact sequence of complexes, namely

$$\begin{array}{ccccccc} 0 \to \mathsf{K} \otimes_A \sigma_j(A/I + S_{j-1}) & \to & \mathsf{K} \otimes_A A/I + S_{j-1} & \to & \mathsf{K} \otimes_A A/I + S_j & \to & 0 \\ \| & & \| & & \| & & \\ \sigma_j \mathsf{K}_{A/I+S_{j-1}} & & \mathsf{K}_{A/I+S_{j-1}} & & \mathsf{K}_{A/I+S_j} & & \end{array}$$

This, in turn, yields a long exact sequence of graded A-modules

$$\begin{array}{llll} \ldots \to & H_i(\sigma_j \mathsf{K}_{A/I+S_{j-1}}) & \to H_i(\mathsf{K}_{A/I+S_{j-1}}) & \to \\ \to & H_i(\mathsf{K}_{A/I+S_j}) & \to H_{i-1}(\sigma_j \mathsf{K}_{A/I+S_{j-1}}) & \to \ldots \end{array}$$

Note here, that the boundary homomorphism, also, is of degree zero. Hence, we get exact sequences of k-spaces, one for each p, which read

$$\begin{array}{llll} \ldots \to & H_{i,p}(\sigma_j \mathsf{K}_{A/I+S_{j-1}}) & \to H_{i,p}(\mathsf{K}_{A/I+S_{j-1}}) & \to \\ \to & H_{i,p}(\mathsf{K}_{A/I+S_j}) & \to H_{i-1,p}(\sigma_j \mathsf{K}_{A/I+S_{j-1}}) & \to \ldots \end{array} \tag{1}$$

Now, since $(\sigma_j \mathsf{K}_{A/I+S_{j-1}})_i = \sigma_j(A/I + S_{j-1})^{\binom{n}{i}}$, and $\deg \sigma_j \leqq N$, we have that $(\sigma_j \mathsf{K}_{A/I+S_{j-1}})_{i,p} = 0$ for $p > N + i$.

Fix i, fix $p > N + i$, and use (1); we get

$$H_{i,p}(\mathsf{K}_{A/I+S_{j-1}}) \simeq H_{i,p}(\mathsf{K}_{A/I+S_j}).$$

The induction hypothesis now establishes the claim.

PROOF OF b). Choose a regular sequence $G_1, \ldots, G_{n-1}$ according to 1) above. Choose a linear X as in the proof of a) above. Just as there, we get $\beta(I + (X)) \leqq \alpha(I) + 1$. Multiplication by $\bar{X}\,(\in A/I)$ is a homomorphism of graded A-modules of degree 1, that is

$$\varphi_s : (A/I)_s \xrightarrow{\bar{X}} (A/I)_{s+1}$$

is a homorphism of k-spaces, for each s.

Now,

$$s \geqq \alpha(I) \Rightarrow s+1 \geqq \beta(I+(X)) \Rightarrow m^{s+1} \subset I+(X) \Rightarrow \varphi_s \text{ surjective},$$

and, since $s \geqq \alpha(I)+D$ according to 3) above implies $\dim_k(A/I)_s = \dim_k(A/I)_{s+1}$, we have φ_s injective for $s \geqq \alpha(I)+D$. Since $\mathrm{Soc}(A/I) \subset \mathrm{Ker}(\text{mult. by } \bar{X})$, the result follows.

We also have the result of Theorem 2 for A/I a 2-dimensional domain; or, more generally:

COROLLARY 1. *If* $\dim A/I - \operatorname{depth} A/I \leqq 1$, *then* $\mathrm{Tor}_{i,p}^A(A/I,k)=0$ *for* $p>(n-t)(D-1)+i$, *where* $t = \operatorname{depth} A/I$ ($\operatorname{depth} A/I$ = *length of a maximal regular sequence in* m/I).

PROOF. We can assume that $\dim A/I - \operatorname{depth} A/I = 1$. Let $\bar{Y}_1,\ldots,\bar{Y}_t$ be a regular sequence in A/I such that $\deg Y_i = 1$. Repeated use of the lemma above gives

$$\mathrm{Tor}_{i,p}^A(A/I,k) \simeq \mathrm{Tor}_{i,p}^{A/(Y_1,\ldots,Y_t)}(A/I+(Y_1,\ldots,Y_t),k).$$

For large p, the right-hand side is zero, according to Theorem 2.

One way to define the *Castelnuovo regularity* for an ideal I in A is as follows (cf. [2])

DEFINITION 3. $I \subset A$ is *t-regular* if $\mathrm{Tor}_{i,p}^A(A/I,k)=0$ for $p>t+i-1$, $\forall i$.

With this in mind, we can restate our two theorems:

$$\begin{aligned} &1)\ A/I \text{ 0-dimensional} \Rightarrow I\ \beta(I)\text{-regular}\\ &2)\ A/I \text{ 1-dimensional} \Rightarrow I\ \alpha(I)+D\text{-regular}\end{aligned}$$

But, since

$$\begin{aligned}&\beta(I) \leqq \alpha(I) \leqq nD-(n-1) \text{ in case 1), and}\\ &\alpha(I)+D \leqq (n-1)D-(n-1)+D = nD-(n-1) \text{ in case 2),}\end{aligned}$$

we can cover both cases by saying that I is $nD-n+1$-regular.

Now, finally, let's discuss a connection between our result and a result of Gruson-Lazarsfeld-Peskine [3], which states:

$$A/I \text{ 2-dimensional domain} \Rightarrow A/I\ s-n+3\text{-regular} \quad (s = \text{mult.}(A/I)).$$

To compare, we also need the additional condition, that A/I is Cohen-Macauley (C-M). Let Y_1, Y_2 be a regular sequence in A, $\deg Y_i = 1$.

$$\mathrm{Tor}_{i,p}^A(A/I,k) \simeq \mathrm{Tor}_{i,p}^{A/(Y_1,Y_2)}(A/I+(Y_1,Y_2),k) \simeq \mathrm{Tor}_{i,p}^{\bar{A}}(\bar{A}/\bar{I},k)$$

where $\bar{A} = k[X_1,\ldots,X_{n-2}]$.

Our result for $\bar{I}$ tells us that I is $(n-2)D-n+3$-regular. There is a theorem of Treger [5], which assumes that A/I is a 2-dimensional C-M domain. It reads $t(n-2) \geqq s \Rightarrow D \leqq t$. This is obviously equivalent to

$$D \leqq \lceil s/(n-2) \rceil, \text{ where } \lceil x \rceil = \min\{\text{integers } N \mid N \geqq x\}.$$

So, I has Castelnuovo regularity $(n-2)\lceil s/(n-2)\rceil - n + 3$. This is the same result as that of [3] if $n-2 \mid s$. With this assumption, we have the following string of inequalities:

$$\beta(\bar{I}) \leqq \alpha(\bar{I}) \leqq (n-2)D - n + 3 \leqq s - n + 3.$$

Now, what happens in the case of equality in the theorem of [3], that is, I is precisely $s-n+3$-regular (i.e. and not less)? Since we have shown that I is $\beta(\bar{I})$-regular, we must have equalities everywhere in the string above.

Equality in 1st step $\Rightarrow$ $\bar{I}$ (and hence I) is gen. by a reg. sequence [1]
Equality in 2nd step $\Rightarrow$ $\bar{I}$ (and hence I) is gen. in one degree, D.

So, $I = (f_1, \ldots, f_{n-2})$, where $f_1, \ldots, f_{n-2}$ is a regular sequence with $\deg f_i = D$; whence $s = D^{n-2}$.

$$\text{Equality in 3rd step} \Rightarrow (n-2)D = D^{n-2}.$$

But this is true only if $n = 3$ or $n = 4$, $D = 2$. Putting aside these cases leaves us with a contradiction.

Conclusion: If $n-2 \mid s$ and equality holds in the theorem of [3], and $n = 3$ or $n = 4$, $D = 2$ is *not* the case, then A/I can not be C-M.

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