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ON THE STRUCTURE OF LOCALLY COMPACT TOPOLOGICAL GROUPS

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Abstract.

We show that the normal nilpotent subgroups of certain solvable groups are compactly generated. A solvable group which satisfies a maximal condition on closed normal subgroups is compactly generated. We obtain several results on the existence of maximal compact normal subgroups of locally compact groups. For example, if G has a uniform solvable subgroup H which has compactly generated derived subgroups, then G has maximal compact subgroups and the resulting maximal compact normal subgroup of G has a Lie factor. If $P(G)$ denotes the subset of G of elements which are contained in compact subgroups, we show that if *G has a closed normal solvable subgroup F* such that $P(F) = F$ and $P(G/F) = G/F$, then $P(G) = G$. On the other hand, we also show that if G is a compactly generated locally compact solvable group and $\mathrm{P(G)} = \mathrm{G}$, then G is compact.

Introduction.

We continue our study of maximal compact subgroups and compatly generated subgroups of locally compact topological groups with the aim of contributing to the knowledge of the structure of locally compact groups. At present, very little has been done for the structure of locally compact groups in general. This is not surprising since general locally compact groups include all the discrete (abstract groups were the vast diversities of groups already have been demonstrated clearly. We do not gain any meaningful results (or insight) without severe restrictions being imposed on the groups. Looking from the direction of infinite abstract groups, there are certain basic notions whose importance can not be overestimated. Here are few of them: finite generations, existence of small normal subgroups, finite conjugacy classes of subgroups or elements, maximum and minimum conditions of subgroups, etc. The natural setting of corresponding notions in locally compact groups are: compact generation, existence of compact normal subgroups, bounded elements, Max and Min on closed normal subgroups, respectively. Our past experience already demonstrates the usefulness and importance of these notions. For example, we know precisely the structure of compactly generated locally compact abelian group. Without the condition of

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compactly generated, the structure of locally compact abelian group un-manageable in a nice way. Another good example is the the automorphism groups of Lie group. If a Lie group G is not compactly generated, then the automorphism group of G is not a Lie group in general. (On the other hand, if G is compactly generated, the automorphism group of G us a Lie group – a well known result due to G. Hochchild). As to the existence of small compact normal subgroup, as it is well known via the solution of Hilbert fifth problem, is the foundation of almost connected locally compact groups. The notion of bounded element has helped us to understand the fine structure of locally compact groups. The reader may look up the very informative survey article by T. W. Palmer, [3], for early results on this subject and related topics. It goes without saying all these notions will be important and crucial for our further understanding the structure of locally compact groups. This motivates present study. Now we briefly sketch the content of present article. In Section 1 we study compactly generated subgroups. We say that a locally compact group G is compactly generated if G is generated by a compact set, thus by a compact neighborhood of the identity. We show that the normal nilpotent subgroups of certain solvable groups are compactly generated, Theorem 1.4. We also obtain conditions under which every closed subgroup of a solvable locally compact groups is compactly generated, Theorem 1.8. In Section 2 we discuss open subgroups and irreducible subgroups of locally compact groups. We obtain several results on the existence of maximal compact normal subgroups of locally compact groups. For example, if G has a uniform solvable subgroup H which has compactly generated derived subgroups, then G has maximal compact subgroups and the resulting maximal compact normal subgroup of G has a Lie factor, Theorem 2.13. We also show that if G has a closed normal solvable subgroup F such that $P(F) = F$ and $P(G/F) = G/F$, then $P(G) = G$, Theorem 2.16. On the other hand, we show that if G is a compactly generated locally compact solvable group and $P(G) = G$, then G is compact, Theorem 2.17.

Notation and terminology. We restrict our attention to locally compact topological groups. We call G an H-group, $\{H(d)$-group$\}$, $\{H(c)$-grpup$\}$, [1], of G has a maximal compact normal subgroup with Lie factor, {if G has a compact normal subgroup with Lie factor and every subgroup of G/G_1 where G_1 is open normal, is finitely generated), {if G has a compact normal subgroup with Lie factor and every closed subgroup of G is compactly generated}. For agroup G, we denote its identity component by G_0. A subgroup H of G is uniform in G if H is closed and G/H is compact.

By $B(G)$ and $P(G)$ we denote the set of bounded elements and the set of periodic elements of G respectively. Thus $B(G)$ is the set of elements of G with relatively compact conjugacy classes and $P(G)$ consists of those elements which are contained in compact subgroups. It is easy to see that $B(G)$ is a characteristic subgroup but not necessarily closed.

We also use the following notation:

$A^G = \{gxg^{-1}: g \in G, x \in A\}$, where A is a subset of the group G.

$\langle A \rangle$ = the subgroup generated by the subset A of G.

$[A, B] = \{aba^{-1}b^{-1}: a \in A, b \in B\}$.

$D^{(1)}(G) = \langle [G, G] \rangle^{-}$.

$D^{(i+1)}(G) = \langle [D^{(i)}(G), D^{(i)}(G)] \rangle^{-}$.

For notations and terminology not defined here, the readers are referred to [1].

Section 1. Compactly generated subgroups.

In this section we study conditions under which a subgroup of a locally compact groups is compactly generated.

PROPOSITION 1.1. *Let G be a locally compact group generated by a compact neighborhood K of e. Let H be a closed normal subgroup and F a closed subgroup such that $G = HF$. If $K \subset DF$ for $D \subset H$, then $D^F = \{fdf^{-1}: f \in F, d \in D\}$ with $H \cap F$ generate H.*

PROOF. Let $h \in H$. Then $h = d_1 f_1 d_2 f_2 \ldots d_n f_n = d_1 f_1 d_2 f_1^{-1} f_1 f_2 d_3 \ldots f_1 \ldots f_{n-1} \ldots d_n f_{n-1}^{-1} \ldots f_1^{-1}(f_1 f_2 \ldots f_n)$. We note that $f_1 f_2 \ldots f_n$ is in H, hence in $H \cap F$. That is $h \in D^F (H \cap F)$; and the proposition is proved.

COROLLARY 1. *If in addition to the hypothesis of Proposition 1.1, $H \cap F$ is compactly generated and F acts on H as a group of automorphisms, then H and F are compactly generated.*

PROOF. Let U and V be compact with non-void interiors in H and F, respectively. Since H and F are σ-compact, $H \subset \cup h_i U$ and $F \subset UVf_i$ for $h_i \in H$, $f_j \in F$. Thus $G = \cup_{i,j} h_i UVf_j$. Since G is second category, the sets $h_i UVf_j$ have non-void interior. Consequently $K \subset \cup_{i,j=1}^{n} h_i UVf_j$ for some n. That is, $K \subset DF$, where $D = \cup_{i=1}^{n} h_i U$. Since D is compact and F acts on H as a group of bounded automorphisms, D^F is compact. It follows that H is compactly generated since $H \cap F$ is compactly generated and $D^F(H \cap F)$ generates H. Since $F/(H \cap F) \cong HF/H = G/H$, it follows that F is compactly generated.

COROLLARY 2. *If in addition to the hypothesis of Proposition 1.1, $G = H \times F$, a direct product, and $H \subset B(G)$, then H is compactly generated.*

PROOF. Corollary 2 follows immediately from Corollary 1.

THEOREM 1.2. *Let N be a locally compact nilpotent group. Then N is compactly generated if and only if $N/D^{(1)}(N)$ is compactly generated.*

PROOF. The "only if" part is obvious. Under the hypothesis that $N/D^{(1)}(N)$ is compactly generated we prove that N is compactly generated by induction on the length of the lower central series $N \supset N^{(1)} \supset N^{(2)} \supset \ldots \supset N^{(k)} = e$. If $k = 1$, the conclusion follows trivially. Assume the conclusion holds for any nilpotent group with $k = n$ and let N have a lower central series of length $k = n + 1$. Let $Z = N^{(n)}$. Then Z is contained in the center of N. Now $(N/Z)/\langle[N/Z, N/Z]\rangle^- \cong (N/Z)/(\langle[N,N]\rangle^-/Z) \cong N/\langle[N,N]\rangle^-$. Thus by the inductive hypothesis N/Z is compactly generated. Let X be a compact subset of N such that $\phi(X)$ generates N/Z, where ϕ is the canonical homomorphism $\phi\colon N \to N/Z$. Let L be the subgroup of N generated by X. Thus $N = LZ$. Since Z is central in N, it follows that $[N,N] = [L,L]$ and $Z \subset \langle[L,L]\rangle^-$. Consequently X generates N as desired.

LEMMA 1.3. *Let G be a locally compact compactly generated group. If $D^{(1)}(G)$ is abelian and contained in $B(G)$, then $D(1)(G)$ and $B(G)$ and $B(G)$ are compactly generated.*

PROOF. Let X be a compact set which generates G. Using the formulas $[gx, y] = [x; t]^g[g, y]$ and $[x, gy] = [x, y]^g$ we see that $\langle[G,G]\rangle$ is generated by $[X,X]^G$. Since $D^{(1)}(G) \subset B(G)$, $\langle[X,X]^G\rangle \subset B(G)$. Thus $\langle[X,X]^G\rangle$ is relatively compact and generates a dense subgroup of $D^{(1)}(G)$. Thus $D^{(1)}(G)$ is compactly generated. Since $G/D^{(1)}(G)$ is compactly generated and abelian, $\overline{B(G)}/D^{(1)}(G)$ is compactly generated. Thus $\overline{B(G)}$ is compactly generated. By Proposition 4, [4], $B(G) = \overline{B(G)}$, completing the proof.

THEOREM 1.4. *Let G be a compactly generated solvable group with a nilpotent normal subgroup N such that G/N is abelian. If $D^{(1)}(G)/D^{(2)}(G) \subset B(G/D^{(2)}(G))$, then N is compactly generated.*

PROOF. We prove that $N/D^{(1)}(N)$ is compactly generated and apply Theorem 1.2. Since G/N is abelian, $N \supset D^{(1)}(G)$. Thus $D^{(1)}(N) \supset D^{(2)}(G)$ and $D^{(1)}(N)$ is normal in $D^{(1)}(G)$. Now $D^{(1)}(G/D^{(2)}(G)) = D^{(1)}(G)/D^{(2)}(G) \subset B(G/D^{(2)}(G))$ and is abelian. Thus $D^{(1)}(G/D^{(2)}(G))$ is compactly generated by Lemma 1.3. Consequently $D^{(1)}(G)/D^{(1)}(N) \cong (D^{(=)}(G)/D^{(2)}(G))/(D^{(1)}(N)/D^{(2)}(G))$ is compactly generated. Since $G/D^{(1)}(G)$ is compactly generated abelian, $N/D^{(1)}(G)$ is compactly generated. Finally $N/D^{(1)}(N)$ is compactly generated as desired.

DEFINITION 1. A locally compact group G is said to satisfy the maximal condition on closed normal subgroups, or simply called an MCCN group, if every chain of closed normal subgroups contains a maximal element.

THEOREM 1.5. *Every* MCCN *solvable group is compactly generated.*

PROOF. We prove the theorem by induction on the length of the derived series for G. Assume $D^{(1)}(G) = e$, i.e. G is abelian. Let V be a compact neighborhood of the identity. Let $H = \langle V^G \rangle^-$. Then G/H is discrete and finitely generated. That G/H is finitely generated follows from the fact that G is an MCCN group. Since N is compactly generated, G is compactly generated.

Now suppose the conclusion holds for groups with derived series of length less than n and assume $G \supset D^{(1)}(G) \supset \dots \supset D^{(n)}(G) = e$. Let $D = D^{(n-1)}(G)$. By the inductive hypothesis G/D is compactly generated. Let H be a compactly generated subgroup such that $G = HD$. We order the collection $\{\langle K^H \rangle^-; K$ a compact subset of $D\}$ of subgroups of D, closed and normal in G, by inclusion. By Zorn's Lemma and the fact that G is an MCCN group it follows that $D = \langle K^H \rangle^-$ for some compact subset K, and consequently G is compactly generated.

The following is a consequence of Corollary 1, Theorem 1, [1].

PROPOSITION 1.6. *If every closed normal subgroup of a locally compact solvable group G is compactly generated, then every closed subgroup of G is compactly generated. If in addition G is totally disconneced, then the same conclusion holds if every open normal subgroup is compactly generated.*

PROPOSITION 1.7. *If a locally compact group G has an open normal subgroup which is not compactly generated, then G has an open normal subgroup maximal with respect to being not compactly generated.*

PROOF. The union $\cup F_\alpha$ of an ordered by inclusion chain of open normal subgroups which are not compactly generated is itself not compactly generated. Thus, by Zorn's Lemma, there is a maximal such subgroup.

It follows from Corollary 1, Theorem 1, [1], that a connected locally compact solvable group is an $H(c)$-group.

THEOREM 1.8. *If G is a locally compact compactly generated solvable group which does not contain a maximal non-compactly generated open normal subgroup F such that every subgroup of G/F is finitely generated, then every closed subgroup of G is compactly generated.*

PROOF. If G contains a closed subgroup which is not compactly generated, then G is not an $H(c)$-group. Thus, there is an open characteristic subgroup which is not compactly generated, by Corollary 2, Theorems 1, [1], and the remark preceding the statement of the theorem. By Proposition 1.7 there is a maximal non-compactly generated open normal subgroup F.

We show that every subgroup of G/F is finitely generated. If G/F has a subgroup which is not finitely generated, then G/F is not an $H(c)$-group; so there is a characteristic subgroup Q/F which is not finitely generated contradicting the fact that F is maximal with respect to being normal non-compactly generated.

Section 2. Open subgroups and irreducible subgroups.

In this section we study open subgroups and irreducible subgroups of locally compact totally disconnected groups. If K is a compact subgroup of the locally compact group G, we are also interested in general when $K \cap B(G)$ is dense in K, which is obviously the case for H-groups if K is a maximal compact normal subgroup of G. In certain cases the converse holds. For example, we have the following

PROPOSITION 2.1. *Let G be a totally disconnected locally compact group. Suppose that G has a torsionfree discrete uniform subgroup Γ. If there exists a compact open subgroup K of G such that $K \cap B(G)$ is dense in K, then G is an H-group.*

PROOF. Since G/Γ is compact and Γ is torsion free, G has a maximal compact open subgroup M which contains K by Proposition 1.9, [4]. Then $M \supset K \supset K \cap B(G)$. Now $K = \overline{B(G) \cap K} \subset \overline{B(G) \cap P(G)} = B(G) \cap P(G)$, and $B(G) \cap P(G)$ is a compact characteristic subgroup of G, Corollary 1.10, [4]. Hence $\langle K^G \rangle^- \subset B(G) \cap P(G)$ and $\langle K^G \rangle^-$ is a compact open normal subgroup pf G. Thus $G/\langle K^G \rangle^-$ is discrete. It follows that G is an H-group.

REMARK 2.2. The following example shows the necessity of the hypothesis that G contains the torsion free discrete subgroup Γ: Let $G = \Sigma_1{}^\infty H_i \times_\eta \Pi_i{}^\infty K_i$ where$H_i = Z_3$, $K_i = Z_2$, $\Sigma_1{}^\infty H_i$ has the discrete topology, and $\Pi_1{}^\infty K_i$ has the product topology. Each K_i acts as the cyclic group of automorphisms on the corresponding H_i. If $K = \Pi_1{}^\infty K_i$, then $K \cap B(G)$ is dense in K but G is not an H-group.

LEMMA 2.3. *Let G be a totally disconnected locally compact group, and let K be a compact open subgroup of G such that $K \cap B(G)$ is dense in K. If H is the smallest closed normal subgroup which contains $K \cap B(G)$, then H is an open normal subgroup of G, $P(H) = H$, $B(G) \cap H$ is dense in H, and $B(H)$ is dense in H.*

PROOF. It is clear that H is open and that $P(H) = H$. Since $B(G) \cap H$ is a normal subgroup containing $B(G) \cap K$, we have $H \supset \overline{B(G) \cap H} \supset H$ and $\overline{B(H)} = H$.

PROPOSITION 2.4. *Let G be a totally disconnected locally compact group and let P be a uniform subgroup of G in which every open normal subgroup is compactly generated. Then G has a compact-open normal subgroup if and only if there exists a compact open subgroup K such that $K \cap B(G)$ is dense in K.*

PROOF. Suppose that there exists a compact open subgroup K such that K $\cap$ B(G) is dense in K. Let H be the smallest closed normal subgroup of G that contains $K \cap B(G)$, i.e. $H = \langle (k \cap B(G))^G \rangle^-$. Then $HF/F \cong H/(H \cap F)$ is compact, hence H is compactly generated. Now $\overline{B(H)} = H$ by Lemma 2.3 above, hence H is an $\overline{\mathrm{FC}}$-group, [6, Proposition 2]. Thus, by Corollary 3 to Theorem 1 of [6], H has a compact open normal subgroup. Consequently H must be compact. The converse is clear.

REMARK 2.5. The above proposition, in fact, shows that if a totally disconnected locally compact group G contains a uniform subgroup in which every open normal subgroup is compactly generated, and if G contains a compact open subgroup K such that $K \cap B(G)$ is dense in K, then the smallest closed normal subgroup of G that contains $K \cap B(G)$ is a compact open normal subgroup of G.

LEMMA 2.5.1. *Let G be a totally disconnected locally compact group. Let G have a uniform subgroup Γ in which every open normal subgroup is compactly generated. If K is any compact open subgroup of G, then the normal closure $M = \langle K^G \rangle^-$ of K in G is compactly generated.*

PROOF. M is an open normal subgroup of G, hence ΓM is closed and contains Γ as a uniform subgroup. Thus $M/(M \cap \Gamma) \cong \Gamma M/\Gamma$ is compact. Hence M is compactly generated.

PROPOSITION 2.5..2. *Let G be a totally disconnected locally compact group, and let H be uniform subgroup in which every open normal subgroup is compactly generated. If K is a compact open subgroup of G such that $\overline{B(G) \cap K} = K$, then the normal closure M of K in G is compact.*

PROOF. By Proposition 2.4 and Remark 2.5, $F = \langle (B(G) \cap K)^G \rangle^-$ is a compact open normal subgroup of G, and $M = \bar{F}$. Since $M \supset B(M) \supset F$, we have $M = \overline{B(M)}$. Since M is compactly generated, Lemma 2.5.1, M is an $\overline{\mathrm{FC}}$-group. Thus M has a compact open normal subgroup. Hence M must be compact.

DEFINITION 2.1. A closed normal subgroup N of a totally disconnected locally compact group G is said to be a G-residual Lie group if there exists a family $\{N_\alpha\}$ of closed subgroups of N, normal in G, such that $\cap N_\alpha = e$ and each N/N_α is a Lie group.

It is clear that if N is open in G, then N is a G-residual Lie group if and only if G is a residual Lie group.

PROPOSITION 2.6. *Let G be a totally disconnected locally compact group, and let K be a compact open subgroup of G such that $K \cap B(G)$ is dense in K. If $N = \langle K^G \rangle^-$ and if N is G-resdidual Lie, then there exists a totally disconnected locally compact group F with a compact open normal subgroup Q and an isomorphism θ: $G \to F$ such that (1)$\theta(N)$ is dense in Q and $Q\theta(G) = F$, and (2) F is pro-Lie.*

PROOF. Let (N_α) be a family of closed normal subgroups of G such that $\cap N_\alpha = e$ and each N/N_α is Lie. For each α, $K/(N_\alpha \cap K)$ is dense in K, there exists

$E_\alpha \subset B(G)$ such that E_α is a compact normal subgroup of G, $N = E_\alpha N_\alpha$, and $E_\alpha/(E_\alpha \cap N_\alpha)$ is finite. Let $\tilde{Q} = \lim\limits_{\leftarrow} N/N_\alpha$. Then $\tilde{Q}$ is compact. If $\theta_1: N \to \lim\limits_{\leftarrow} N/N_\alpha$ denotes the canonical map, then $\theta_1(N)$ is dense in $\tilde{Q}$. Since G can be viewed as a group of automorphisms of $\tilde{Q}$, we may form the semidirect product $\tilde{Q} \times_\eta G$, and let $\Delta = \{(\theta_1(n), n^{-1}) \mid n \in N\}$. Then Δ is a closed normal subgroup of $\tilde{Q} \times_\eta G$. Let $F = (\tilde{Q} \times_\eta G)/R$, where R is the relation induced by Δ. Let $\pi_\alpha: \tilde{Q} \to N/N_\alpha$ be the αth projection map. Then $\pi_\alpha^{-1}(\{N_\alpha\})$ is an open normal subgroup of $\tilde{Q}$, hence it is open in $\tilde{Q} \times_\eta G_d$, where G_d denotes the group G endowed with the discrete topology. Since N is open in G, $(\tilde{Q} \times_\eta G_d)/R \cong (\tilde{Q} \times_\eta G)/R$. Now $\pi_\alpha^{-1}(\{N_\alpha\})$ is normal in $\tilde{Q} \times_\eta G_d$, hence it is normal in F and $F/\pi_\alpha^{-1}(\{N_\alpha\})$ is discrete, a fortiori, F is pro-Lie. Now let ϕ: $G \to \tilde{Q} \times_\eta G$ be the natural injection, and let p: $\tilde{Q} \times_\eta G \to F$ be the canonical homomorphism. Define $\theta(g) = p`\phi`g))$ for each g in G. Then it is clear that θ is an isomorphic of G into F. If we let $Q = p((\tilde{Q}, e))$, then $\theta(N)$ is dense in Q, and $Q\theta(G) = F$.

For a converse of the above proposition we have the following

PROPOSITION 2.7. *Let θ: $G \to F$ be an isomorphism of a locally compact group G into a pro-Lie group F. Then G is a residual Lie group. If $\{F_\alpha\}$ is a family of compact normal subgroups of F such that $\cap F_\alpha = e$ and F/F_α is Lie for each α, let $N_\alpha = \theta^{-1}(F_\alpha)$ for each α. Let K be a compact open subgroup of G such that, for each α, there exists a compact normal subgroup W_α of G such that $W_\alpha(N_\alpha \cap K) \supset K$. Then $K \cap B((G)$ is dense in K.*

PROOF. It is clear that G is residual Lie. If $k \in K$, then $k = n_\alpha x_\alpha$ for $n_\alpha \in W_\alpha$ and $x_\alpha \in N_\alpha \cap K$. Since $x_\alpha \to e$, we see that $n_\alpha \to k$. Thus $k \in \overline{B(G)}$. Since K is open, $\overline{B(G)} \cap K \subset \overline{B(G) \cap K}$. Hence $K \subset \overline{B(G)} \cap K \subset \overline{B(G) \cap K} \subset K$, and we have $\overline{B(G) \cap K} = K$.

THEOREM 2.8. *Let G be a totally disconnected locally compact group. Suppose that G contains a uniform subgroup F every closed subgroup of which is compactly generated. Then G has a maximal compact normal subgroup.*

PROOF. Let H be the smallest closed normal subgroup of G which contains all compact normal subgroups of G. Then $H = \overline{B(G) \cap P(G)}$, [4, Corollary 5.6]. Then $\overline{HF}/F$ is compact. Let K be any compact open subgroup of $\overline{HF}$. Then $\overline{HF} = KHF$, and KHF/F is compact. Since $KH/(KH \cap F) \cong KHF/F$, we see that KH is compactly generated. Hence H is compactly generated also. Since $H = \overline{B(G) \cap P(G)}$, we have $\overline{B(H)} = H$. Hence H is an $\overline{\mathrm{FC}}$-group, i.e. $H = B(H)$, so it has a compact open normal subgroup. Consequently H is compact.

THEOREM 2.9. *Let G be a totally disconnected locally compact group. Suppose that G contains a uniform normal subgroup F, every closed subgroup of which is* *compactly generated. Then G has maximal compact open subgroup, and $B(G)$ is closed.*

Proof. We only need to prove the first assertion. The second assertion will then follows from Proposition 1.8 of [4].

Let $\{H_\alpha\}$ be a chain of compact open subgroups, ordered by inclusion, and let $H = \cup H_\alpha$. Then H is an open subgroup of G, and $HF/F = H/(H \cap F)$ is compact. Hence H is compactly generated. Let K be a compact open subgroup of H, then there exists $h_1, h_2, \ldots, h_n$ in H such that $K \cup \{h_1, h_2,,,, h_n\}$ generates H. Since $H \supset K$, there exists α such that $H_\alpha \supset K$, and hence $H_c \supset K \cup \{h_1, h_2, \ldots, h_n\}$ for some H_c in the chain. This means that $H = H_c$, and H is compact. Now it is readily seen using Zorn's Lemma that G has maximal compact open subgroups.

Theorem 2.10. *Let G be a totally disconnected compactly generated locally compact group. If every closed subgroup of G is compactly generated, then G has maximal compact open subgroups, and $B(G)$ is closed.*

Proof. Once again we only need to show that G has maximal compact open subgroups.

Let $\{F_\alpha\}$ be a chain of compact open subgroups, ordered by inclusion, and let $F = \cup F_\alpha$. Then F is compactly generated. Let F be generated by the compact set X, and let $X_\alpha = X \cap F_\alpha$ for each α. Then $\{X_\alpha\}$ forms an open cover for X, hence $X = X_a$ for some a. This implies that $F = F_a$. Hence F is a compact open subgrop of G. Thus, by Zorn's Lemma, G has maximal compact open subgroups.

Corollary. *Let G be a totally disconnected locally compact group. If G is a compact extension of a closed normal subgroup H which is an $H(c)$-group, then G has maximal compact open subgroups, and $B(G)$ is closed.*

Proof. This follows from Theorem 1, [1], and Theorem 2.10.

Definition 2.2. Let G be a totally disconnected locally compact group. Define TFrat G to be the intersection of all maximal open subgroup of G.

Note that TFrat G is a characteristic subgroup of G. The following theorem is known for discrete groups.

Proposition 2.11. *Let G be a totally disconnected locally compact group. Then* TFrat $G = \{g \in G \mid G = \langle X \rangle$ whenever $G = \langle g, X \rangle\}$, i.e. TFrat G *is the set of all non-generators.*

Proof. We first show that if $g \in \mathrm{TFrat}\, G$ and $G = \langle g, X \rangle$, then $g \in \langle X \rangle$. Suppose on the contrary that $g \notin \langle X \rangle$. Then the interior of $\langle X \rangle$ in not null, i.e. $\langle X \rangle$ is an open subgroup of G. Consider all open subgroups M of G such that $g \notin M$ and $\langle X \rangle \subset M$. Let $\{M_\lambda\}$ be a chain of such open subgroups, and let

$M' = \cup M_\lambda$. Then M' is an open subgroup of G, $g \notin M'$, $\langle X \rangle \subset M'$, and $M' \neq G$. Thus the family of all subgroups M of G such that $g \notin M$ and $\langle X \rangle \subset M$ has a maximal element N. Then $g \notin N$ and $\langle X \rangle \subset N$. It is clear that N is a maximal open subgroup of G, which leads to a contradiction.

Suppose now that $g \in G$ such that $G = \langle g, X \rangle$ implies $G = \langle X \rangle$. Let M be a maximal open subgroup of G. Then $G = \langle g, M \rangle$, and hence $g \in M$. Thus $g \in \mathrm{TFrat}\ G$.

LEMMA 2.12. *Let G be a discrete solvable group. If G has a finite sequence of normal subgroups $G \supset G_1 \supset G_2 \supset \dots \supset G_n \supset G_{n+1} = e$ such that G_i/G_{i+1} is a finitely generated abelian group for each i. Then there exists a positive integer* 1 *which bounds the order of all the torsion subgroups.*

PROOF. We shall prove the assertion by induction on the length of the sequence. If G is abelian, the assertion is clear. Suppose that the assertion is true for discrete solvable groups with length $\leqq n$, and G is a discrete solvable group of length $n + 1$ with the stated property. Then we have the sequence $G/G_n \supset G_1/G_n \supset \dots \supset G_{n-1}/G_n \supset e$ which also satisfies the stated property. Let H be a torsion subgroup of G. Then $HG_n/G_n \cong H/(H \cap G_n)$, so $|H/(H \cap G_n)| < k_1$ for some positive integer k_1. Now $H \cap G_n$ is a torsion subgroup of G_n, $|H \cap G_n| < k_2$ for some k_2. Hence $|H| < k_1 k_2$, and the proof is complete.

The following theorem deals with the existence of maximal compact subgroups. For its proof we need the following notation: If G is a locally compact group with a (left) invariant measure dg then, for every measurable subset X of G, we denote $\int_X dg$ by $v(X)$.

THEOREM 2.13. *Let G be a locally compact group with a uniform solvable subgroup H. If the derived subgroups of H are compactly generated, then G has maximal compact subgroups, each compact subgroup is contained in a maximal one, and G is an $H(d)$-group. If in addition G is totally disconnected, then G is an $H(c)$-group and the maximal compact subgroups of G are open.*

PROOF. We first show that we can assume that G is totally disconnected. This follows since G/G_0 and $\overline{HG_0}/G_0$ satisfy the hypothesis on G and H. Also G has maximal compact subgroups if and only if G/G_0 has maximal compact open subgroups. By Proposition 4 [7] G is an $H(c)$-group. By Corollary 1 of Theorem 1 [1] H is an $H(c)$-group, hence H has a maximal compact open normal subgroup Q such that H/Q is discrete. Now we can assume that H is discrete and finitely generated since G has maximal compact open subgroups if and only if G/Q has. Thus H is unimodular, and, since G/H is compact, G is unimodular and G/H has finite invariant measure. Thus, by Lemma 2.3 [5], for any measurable subset X of G we have

$$\nu_G(X) \leqq \nu_{G/H}(XH/H)\nu_H(H \cap X^{-1}X)$$

In particular, when X us a compact subgroup, $H \cap X^{-1}X = H \cap X$ is a torsion group. Hence $\nu_H(H \cap X)$ is bounded by some number 1. By taking $\nu_{G/H}(G/H) = 1$ we have $\nu_G(X) \leqq 1$. This proves that all compact subgroups of G have measures bounded by 1. Thus G has maximal compact open subgroups. It follows readily that every compact subgroup is contained in a maximal one.

LEMMA 2.14. *Let G be locally compact group with a compact open subgroup K and let H be a uniform normal solvable subgroup of G. If $P(H) = H$, then $P(G) = G$.*

PROOF. Let $x \in G$. Then either $\langle x \rangle$ is discrete or $\overline{\langle x \rangle}$ is compact. If $\overline{\langle x \rangle}$ is compact, then $x \in P(G)$. Hence we may assume that $\langle x \rangle$ is discrete and not relatively compact. Then $\overline{H\langle x \rangle} \subset G$. We assume without loss of generality that $G = \overline{H\langle x \rangle}$. Thus we may write $G = HK\langle x \rangle$, where K is a compact open subgroup of G. Then we have either $x^n \in HK$ for some n, or $x^nHK \cap x^mHK = \phi$ whenever $n \neq m$. In the later case $G = \cup x^nHK$. But G/HK is compact, so $x^n \in HK$ for some n. Hence, in any case, $x^n \in HK$ for some n. Thus we may assume without loss of generality that $x \in HK$ for some n. Thus we may assume without loss of generality that $x \in HK$. We shall prove that $x \in P(G)$ by induction on the solvability of H.

If H is abelian, $H \subset B(G) \cap P(G)$. Hence each $h \in H$ is contained in some compact normal subgroup N of G. Thus $hk \in NK$ for every $k \in K$. In particular $x \in NK$, hence $x \in P(G)$. Suppose now that the lemma holds for solvable groups of class $\leqq n$, and let H be a solvable group of class $n + 1$. Then $D^{(1)}(H)$ is a characteristic subgroup of H, hence $D^{(1)}(H)$ is normal in G. $G/D^{(1)}(H)$ has $H/D^{(1)}(H)$ as a normal uniform subgroup. If $\pi: G \to G/D^{(1)}(H)$ denotes the canonical homomorphism, then $\pi(x)$ is a compact element. Hence we may assume that $D^{(1)}H\langle x \rangle/D^{(1)}(H)$ is compact. Since $P(D^{(1)}(H)) = D^{(1)}(H)$, applying the induction hypothesis we have $P(D^{(1)}(H)\langle x \rangle = D^{(1)}(H)\langle x \rangle$. Hence $x \in P(G)$.

LEMMA 2.15. *Let H be a connected locally compact group such that $P(H) = H$. Then H is compact.*

PROOF. Since H is pro-Lie, we may assume that H is a Lie group by factorizing out a compact normal subgroup. Then the radical $R(H)$ of H is a compact group. Since $H/R(H)$ is an analytic group with only compact elements, hence it has no non-compact simple factor, a fortiori, H is compact.

THEOREM 2.16. *Let G be a locally compact group. If G has a closed normal solvable subgroup F such that $P(F) = F$, and $P(G/F) = G/F$, then $P(G) = G$.*

PROOF. Note first that if K is a compact normal subgroup of G and if π: $G \to G/K$ is the canonical homomorphism, then $x \in G$ is a compact element if and only if $\pi(x)$ is compact in G/K.

By Lemma 2.15, F_0 is compact. By factoring out F_0 we may assume that F is totally disconnected. Since we wish to show that $P(G) = G$, we may assume that G/F is monothetic, i.e. it is generated by one element. Then G is solvable. We first assume that F is abelian. Let $G_1 = \overline{G_0 F}$, and let K be the maximal compact normal subgroup of G_0. Without loss of generality we may assume that K is trivial. Since $G_0 \cap F$ is totally disconnected and is normal in G_0, $G_0 \cap F$ is contained in the center of G_0 (in fact in K). Hence $G_0 \cap F = e$. Observe that the natural map $G_0 \to G_1/F$ is injective, and that G/F is abelian, a fortiori, G_1 is abelian. Factoring out a compact subgroup from G_1, we may assume that G_1 is a Lie group. Then $(G_1)_0 = G_0$ is open in G_1, hence $G_1 = G_0 F = G_0 \times F$ (direct product). Now $P(G/F) = G/F$, hence $P(G_0) = G_0$. Thus G_0 is compact by Lemma 2.15. Hence we may assume further that G is totally disconnected. Then, by Lemma 2.14, $P(G) = G$.

Now assume that the theorem holds for closed normal solvable subgroups of class $\leqq n$, and let $F \supset D^{(1)}(F) \supset \ldots \supset D^{(n)}(F) \supset D^{(n+1)}(F) = e$ be the topological derived series of F. Then $F/D^{(1)}(F)$ is an abelian subgroup of $P(G/D^{(1)}(F))$, hence $P(G/D^{(1)}(F)) = G/D^{(1)}(F)$ from the previous paragraph. Now the induction hypothesis readily implies that $P(G) = G$ and the proof is complete.

THEOREM 2.17. *Let G be a compactly generated locally compact solvable group. If $P(G) = G$, then G is compact.*

PROOF. Let $G \supset D^{(1)}(G) \supset \ldots \supset D^{(n)}(G) \supset D^{(n+1)}(G) = e$ be the topological derived series of G. If $n = 1$, G is abelian and the theorem is obvious from the structure theorem of locally compact compactly generated abelian groups. Hence we may assume that $n > 1$. By induction on n, we may assert that $G/D^{(1)}(G)$ *and* $D^{(i)}(G)/D^{(i+1)}(G)$ are compact for $1 \leqq i \leqq n$. Hence $D^{(n)}(G)$ is compact, and, a fortiori, G is compact as desired.

DEFINITION 2.3. A locally compact group G is said to be an almost linear solvable group if there exists a closed normal subgroup H such that G/H is compact and $D^{(1)}(H)$ is nilpotent.

THEOREM 2.18. *Let G be a totally disconnected locally compact compactly generated almost linear solvable group.* T hen *G has an open normal subgroup S consisting of compact elements if and only if $G/P(G)$ is an H-group where N is the nilradical of G.*

PROOF. Suppose that $G/P(N)$ is an H-group. Then there exists an open normal subgroup G_1 of G such that $G_1/P(N)$ is compact. Then, by Lemma 2.14,

$P(G_1) = G_1$. Hence G has an open normal subgroup consisting of compact elements.

Conversely suppose that G has an open normal subgroup S consisting of compact elements. Let F be a closed normal subgroup of G such that G/F is compact and $D^{(1)}(F)$ is nilpotent. Without loss of generality we may assume $N = D^{(1)}(F)$. Since $P(N)$ is an open characteristic subgroup of N, $P(N)$ is a closed normal subgroup of G. By factoring out $P(N)$ we may assume that N is a discrete torsion free group. We shall show that G is an H-group. Since F/N is a compactly generated abelian group, it is an $H(c)$-group. Now G/N is a compact extension of F/N, and hence is an $H(c)$-group, Lemma 5 [1]. Hence G is an H-group, Lemma 8 [1]. This shows that $G/P(G)$ is an H-group as desired.

For a subgroup S of the locally compact group G, we denote by $Z_G(S)$ the centralizer of S in G.

LEMMA 2.19. *Let G be a totally disconnected locally compact compactly generated almost linear solvable group, and let N be the nilradical of G. Assume that N is discrete torsion free. If $Z_G(N)$ is an open subgroup, then G has an open subgroup consisting of compact elements.*

PROOF. Let H be a closed normal subgroup of G such that G/H is compact and $D^{(1)}(H)$ is nilpotent. We may assume that $N = D^{(1)}(H)$. Then H/N is a compactly generated locally compact abelian group, and hence is an $H(c)$-group, a fortiori, G/N is an $H(c)$-group. Thus there exists an open normal subgroup G_1 of G such that G_1/N is compact. Hence $Z_G(N) \cap G_1$ and $(Z_G(N) \cap G_1)N$ are open, and thus $(Z_G(N) \cap G_1)N/N$ is compact. This implies that $(Z_G(N) \cap G_1)/(Z_G(N) \cap G_1) \cap N$ is compact. Since $(Z_G(N) \cap G_1) \cap N$ is discrete torsion free, $Z_G(N) \cap G_1$ has maximal compact open subgroups, Proposition 1.9 of [4], and the proof is complete.

PROPOSITION 2.20. *Let G be a totally disconnected locally compact compactly generated almost linear solvable group. Let N be the nilradical of G. If $Z_{G/P(N)}(N/P(N))$ is an open subgroup of $G/P(N)$, then $G/P(N)$ is an H-group.*

PROOF. This follows from Lemma 2.19.

The following proposition generalizes Theorem 1 of [6].

PROPOSITION 2.21. *Let G be a locally compact group and let H be a closed normal subgroup of G. Then H has a compact G-invariant neighborhood of e if and only if $B(G) \cap H$ is open in H.*

PROOF. Let G_d denote the group G with the discrete topology, and let $\tilde{G} = H \times_\eta G_d$ (semidirect product), where G_d acts on H by conjugation. Then $\tilde{G}$ has a compact invariant neighborhood of the identity e if and only if $B(\tilde{G})$ is open, Theorem 1, [6]. Since $\tilde{H} = H \times \{e\}$ is open in $\tilde{G}$, hence $B(\tilde{G})$ is open if and only if $B(\tilde{G}) \cap \tilde{H}$. Now for given (h,e) in $\tilde{H}$ and (h',g') in $\tilde{G}$, $(h',g')(h,e)(h',g')^{-1} = (h'g'hg'^{-1}h'^{-1}, e)$. Hence $(h,e) \in B(\tilde{G})$ if and only if $h \in B(G)$. Thus H has a compact G-invariant neighborhood of e if and only if $B(G) \cap H$ is open in H.

DEFINITION 2.4. Let H be a closed normal subgroup of a locally compact totally disconnected group G. Then H is said to be irreducible if for every neighborhood V of the identity e in G, the smallest closed normal subgroup of G which contains $V \cap H$ is H, i.e. $\langle (V \cap H)^G \rangle^- = H$ for each V.

THEOREM 2.22. *Let Ω be the collection of all irreducible compact normal subgroups of a locally compact totally disconnected group G. Let M be the normal subgroup of G generated by $\cup\Omega$, and let $Q = \bar{M}$. Then we have the following conditions:*

(1) *Q is contained in every open normal subgroup of G.*
(2) *If $B(G) \cap Q$ is dense in Q, then $Q \subset B(G) \cap P(G)$.*
(3) *Q is irreducible.*
(4) *If Q contains a compact open G-invariant subgroup P, then $Q = P = M$.*
(5) *Q is compact if and only if $Q \subset B(G)$.*

PROOF. First we note that Q is a closed normal subgroup of G.

Let F be any open normal subgroup of G, and let N be any compact irreducible subgroup of G. Then $F \cap N \subset N$, and $F \cap N \subset F$. Hence $N \subset F$ since N is the smallest closed normal subgroup containing $F \cap N$. Thusss $Q \subset F$, and (1) is proved.

Suppose $B(G) \cap 0$ is dense in Q. Then $M[B(G) \cap Q] = Q$ since M is dense in Q. Thus $Q \subset B(G)$. But $Q \subset P(G)$. Hence $Q \subset B(G) \cap P(G)$, and (2) is proved.

To show that (3) holds, let V be a neighborhood of e. Then for each $X \in \Omega$, the smallest closed normal subgroup that contains $X \cap V$ is $\langle (V \cap X)^G \rangle^-$. Hence $\langle (V \cap Q)^G \rangle^- \supset X$ for each $X \in \Omega$, thus $\langle (V \cap Q)^G \rangle \supset M$. Therefore $\langle (V \cap Q)^G \rangle^- = Q$.

For (4) suppose that Q has a compact open G-invariant subgroup P. Then, for each $X \in \Omega$, $\langle (X \cap P)^G \rangle^- = X$, hence $X \subset P$. Thus $Q = P = M$.

Finally to show that (5) holds. It is clear that $Q \subset B(G)$ if Q is compact. Conversely, if $Q \subset B(G)$, then, by Proposition 2.21, Q contains a compact open G-invariant subgroup P. Then $Q = P$ by (4) above. Hence Q is compact.

REMARK 2.23. (1) Any discrete compact (hence finite) normal subgroup is not irreducible.

(2) The following example shows that the normal subgroup M in Theorem 2.22 is not necessarily closed. For each integer i, let $G_i = \Pi_{-\infty}^{\infty}(Z_2)_j \times_\eta Z$, and let $H_i = \Pi_{-\infty, j \neq 0}^{\infty}(Z_2)_j \times \{0\}) \times_\eta Z$. Let G_L be the local direct product

$$G_L = (\{G_i\}, \{H_i\}) \qquad \text{(c.f. [2, p. 56])}$$

Then $M_i = \Pi_{j=-\beta,\ i\neq 0}^{\infty} (Z_2)_i \times \{0\}$ are the only irreducible compact normal subgroups, M is generated by $\{M_i\}_i$, and $\bar{M} = G_L$.

PROPOSITION 2.24. *Let G be a locally compact totally disconnected group and let M be the normal subgroup of G defined in Theorem 2.22. If M is compact, then G/M has no non-trivial irreducible compact normal subgroup.*

PROOF. Let K' be an irreducible compact normal subgroup of G/M, and let K be the complete inverse image of K' in G. Then $K \supset M$ and K is compact. If V is a neighborhood of the identity in G, it is immediate that $\langle (V \cap K)^G \rangle^- = K$. Hence K is an irreducible compact normal subgroup of G, and thus $K \subset M$. Therefore $K = M$ and K' must be trivial.

LEMMA 2.25. *Let G be a totally disconnected locally compact group, and let K be a compact normal subgroup of G. Then K contains a maximal compact normal dubgroup Q of G such that Q is irreducible and K/Q does not contain any irreducible compact normal subgroup.*

PROOF. Let Ω be the collection of all ireducible compact normal subgroups of G which are contained in K. Then $Q = \langle \cup \Omega \rangle^-$ is irreducible as in Theorem 2.23. Now if P' is an irreducible compact normal subgroup of K/Q, and if P is the complete inverse image of P' in K, then $K \supset P \supset Q$ and P is irreducible. Hence $P = Q$ and P' is trivial.

REMARK 2.26. A compact normal subgroup L of a locally compact totally disconnected group G is irreducible if and only if L contains no proper G-invariant open subgroup.

PROOF. Let L' be a G-invariant open subgroup of L, and let V be any neighborhood of e in G. Then $\langle (V \cap L')^G \rangle \subset L'$. Since L' is open in L, there exists a neighborhood W of e in G such that $L \cap W \subset L'$. Then $L = \langle (V \cap (W \cap L))^G \rangle^- \subset \langle (V \cap L')^G \rangle^-$. Hence $L = L'$. The converse is clear.

THEOREM 2.27. *Let G be a totally disconnected locally compact group and let K be a compact normal subgroup of G. Let Q be a normal subgroup of G contained in K such that K/Q is an irreducible compact normal subgroup of G/Q. Then there exists a compact normal subgroup L of K such that*

(1) $QL = K$,

(2) *L is the intersection of all G-invariant open subgroups of K, and*

(3) *L is irreducible.*

PROOF. Let Ω be the collection of all G-invariant open subgroups L' of K such that $QL' = K$. Then $\Omega = \phi$. Take a maximal chain Ω^* in Ω and let $L = \cap \Omega^*$.

Then L is a minimal compact normal subgroup of K and $QL = K$. To see that $K \subset QL$, let $g \in K$. Then $g = q_\alpha l_\alpha{}^\alpha$ where $l_\alpha \in L_\alpha \in \Omega^*$ and $q_\alpha \in Q$. We may assume that $q_\alpha \to q \in Q$. Then $l_\alpha \to l$. We claim that $l \in L$ which would imply that $g \in QL$. Let V be any neighborhood of e in G. Then VL is open, hence for large α, $l_\alpha \in VL$. Thus $l \in VL$. This implies that $l \in L$ as claimed.

It remains to show that L is irreducible. Let L' be a G-invariant open subgroup of L. Then $QL' \subset K$. Since L' is open in L, L/L' is finite, hence QL' has finite index in $K = QL$. Hence QL' is open in K, and QL'/Q is a G/Q-invariant open subgroup of K/Q. Then by assumption we have that $QL'/Q = K/Q$. Hence $QL' = KQL$. Thus $L' = L$.

In concluding this article, we observe the following.

REMARK 2.28. Let G be a totally disconnected locally compact group. Then $P(G) \cap B(G)$ is a characteristic subgroup of G generated by all compact normal subgroups of G. If M is the normal subgroup of G generated by all irreducible compact normal subgroups of G, $M \subset P(G) \cap B(G)$. Now let $x \in \overline{B(G) \cap P(G)}$ but $x \notin \bar{M}$. If $H(x) = \langle x^G \rangle^-$ denotes the smallest closed compact normal subgroup generated by x, then $H(x) \not\subset \bar{M}$ but $H(x)\bar{M}$ is closed. We claim that $H(x)\bar{M}/\bar{M} \cong H(x)/(H(x) \cap M)$ is not irreducible. For if it is, by Theorem 2.27, there exists an irreducible compact normal subgroup L of $H(x)$ such that $(H(x) \cap \bar{M})L = H(x)$. Since L is irreducible, $L \subset M$, and we must have $H(x) \subset \bar{M}$, a contradiction. Hence if $x \in \overline{B(G) \cap P(G)}$ but $x \notin \bar{M}$, then $H(X)\bar{M}/\bar{M}$ is not irreducible.

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REFERENCES

1. R. W. Bagley, T. S. Wu and J. S. Yang, *Compact and compactly generated subgroups of locally compact groups*, Proc. Amer. Math. Soc. 108 (1990), 1085–1094.
2. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Springer-Verlag, Berlin, 1963.
3. T. W. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. 8 (1978), 683–741.
4. S. P. Wang, *Compactness properties of topological groups*, Trans. Amer. Math. Soc. 154 (1971), 301–314.
5. S. P. Wang, *On density properties of certain subgroups of locally compact groups*, Duke Math. J. 43 (1976), 561–578.
6. T. S. Wu and Y. K. Yu, *Compactness properties of topological groups*, Michigan Math. J. 19 (1972), 299–313.
7. T. S. Wu, *On the structure of locally compact topological groups*, Trans. Amer. Math. Soc. 325 (1991), 413–434.

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