BOUNDARY CONVERGENCE IN
NON-NONTANGENTIAL AND NONADMISSIBLE
APPROACH REGIONS

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§0. Introduction.

Let \( u(x, y) = Pf(x, y) \) be the Poisson integral in the upper halfspace \( \mathbb{R}^{n+1}_+ = \{(x, y); x \in \mathbb{R}^n, y > 0\} \) of a function \( f \in L^p(\mathbb{R}^n) \), \( p \geq 1 \). The question whether \( u(x, y) \) has boundary limits has been extensively studied. To state this problem more precisely we let \( \Omega \) be an open set in \( \mathbb{R}^{n+1}_+ \) having \((0, 0)\) as a limit point and put \( \Omega^x = x + \Omega \) for \( x = (x, 0) \in \mathbb{R}^n \). Then the question is for which \( \Omega \),

\[
\lim_{\Omega^x \to (x_0, 0)} u(x_0, y) = f(x_0), \text{ a.e. } x_0 \in \mathbb{R}^n, f \in L^p(\mathbb{R}^n).
\]

The classical result of Fatou states that (1) holds if \( \Omega \) is the cone \( C_\alpha = \{(x, y) \in \mathbb{R}^{n+1}_+; |x| < \alpha y\} \); i.e. \( u \) has nontangential boundary values almost everywhere. It was proved by Littlewood [L] that (1) cannot hold if \( \Omega \) contains a tangential curve ending at the origin. However, Nagel and Stein showed in [NS] that there are many approach regions \( \Omega \) not contained in any \( C_\alpha \) for which (1) holds, i.e. we may have non-nontangential convergence. The conditions on \( \Omega \) are

a) \( \Omega + C_\alpha \subset \Omega \)

and

b) \( |\{x; (x, y) \in \Omega\}| \leq Cy^n \).

In §1 we provide an alternative proof of Nagel-Stein’s result; in fact we show that if \( \Omega \) satisfies a) and b), then the distribution functions of the maximal functions corresponding to \( \Omega \) and \( C_\alpha \), respectively, are equivalent, so everything is reduced to the nontangential case. Thus we also get e.g. a generalization to real \( H^p \)-spaces, \( p < 1 \).

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In §2 we discuss the same problem in bounded domains in \( \mathbb{R}^n \) with \( C^2 \)-boundary. It turns out that if \( \Omega = \Omega^{x_0} \) satisfies conditions as (a) and (b) for some \( x_0 \in \partial D \) and if \( \Omega^x \) is some smooth move of \( \Omega^{x_0} \) for \( x \) near \( x_0 \), then an analogous result holds for \( \{ \Omega^x \} \).

It follows from the theory of harmonic functions in \( \mathbb{R}^{2n} \) that holomorphic functions in \( H^p(D) \) for smooth domains \( D \) in \( \mathbb{C}^n \) and \( p \geq 1 \), have nontangential boundary limits a.e. However, convergence holds for all \( p > 0 \) and in larger regions, the so-called admissible approach regions, that allow parabolic tangential approach in the complex tangential directions. It is a natural problem to find an analogue of Nagel-Stein's theorem in the complex case. This problem has been considered by Sueiro [Su2] in the case of the generalized halfspace and the ball.

In §3 we treat the complex case and find an analogous generalization of admissible convergence, "nonadmissible" convergence, in strictly pseudoconvex smooth domains in \( \mathbb{C}^n \).

Finally in §4 we consider a specific example, the generalized halfspace in \( \mathbb{C}^n \), and deduce a local fact about strictly pseudoconvex boundaries which we use in §3.

\section*{1. Non-nontangential boundary values in the upper halfspace.}

As usual the existence of boundary limits follows from estimates for the corresponding maximal function. If \( \Omega \) is a region in \( \mathbb{E}^{n+1}_x \) having 0 as a limit point and \( \Omega^x = x + \Omega \) for \( x \in \mathbb{R}^n \), we define the maximal function by \( M_{\Omega} u(x) = \sup_{\Omega^x} |u| \), for any measurable function \( u \) in \( \mathbb{R}^{n+1}_x \). In particular, \( M_{\Omega} u \) is the nontangential maximal function of \( u \). Let \( \Omega^x(t) = \{ x' \in \mathbb{R}^n; (x', t) \in \Omega^x \} \), \( \Omega(t) = \Omega^0(t) \) and let \( Pf \) denote the Poisson integral of \( f \). In [NS] Nagel and Stein proved

\textbf{Theorem 1.} \textit{Suppose that for some} \( \alpha \), \( \Omega \) satisfies

\begin{enumerate}
\item \( \Omega + C_{\alpha} \subset \Omega \)
\item \( |\Omega(t)| \leq Ct^n \).
\end{enumerate}

\textit{Then the map} \( f \mapsto M_{\Omega} Pf \) \textit{is of weak type} \( (1, 1) \) \textit{and strong type} \( (p, p) \) \textit{if} \( 1 < p \leq +\infty \).

\textbf{Example 1.} Let \( \gamma \) be any tangential curve ending at 0. If \( p_j = (x_j, y_j) \) is a sequence of points on \( \gamma \) and \( x_j \to 0 \) rapidly enough, then \( \Omega = \bigcup_j p_j + C_{\alpha} \) satisfies

\begin{enumerate}
\item a) and b) but clearly \( \Omega \) is not contained in any cone. Thus we have non-nontangential convergence.
Remark 1. Nagel and Stein also proved a converse of Theorem 1; namely if \( \Omega \) is any set such that \( M_\Omega P \) is of weak type \((1, 1)\) and \( \tilde{\Omega} = \Omega + C_\alpha \) then \( |\tilde{\Omega}(t)| \leq C t^\alpha \).

Conditions a) and b) clearly implies

i) \( t \to \Omega(t) \) is increasing,

ii) if \( B_\rho(t) \cap \Omega(t) \neq \emptyset \) then \( \rho \in \Omega(Ct), \left( \text{with } C = 1 + \frac{1}{\alpha} \right) \),

and

iii) \( |\Omega(t)| \leq C t^\alpha \).

Here \( B_\rho(t) \) is the Euclidean ball with center \( \rho \) and radius \( t \). The conditions are in fact essentially equivalent since if \( \Omega \) satisfies i)–iii), then \( \tilde{\Omega} = \Omega + C_\alpha \) satisfies a) and b). Hence Theorem 1 is a consequence of the following proposition that states that the distribution functions of \( M_\Omega \) and \( M_{C_\alpha} \) are equivalent.

Proposition 1. Suppose that \( \Omega \) satisfies i)–iii). Then

\[
\| \{ x \in \mathbb{R}^n; M_\Omega u(x) > \lambda \} \| \leq C \| \{ x \in \mathbb{R}^n; M_{C_\alpha} u(x) > \lambda \} \|
\]

and hence

\[
\| M_\Omega u \|_{L^p(\mathbb{R}^n)} \leq C \| M_{C_\alpha} u \|_{L^p(\mathbb{R}^n)}, \ p > 0,
\]

for any measurable function \( u \) in \( \mathbb{R}^{n+1}_+ \)

Remark 2. The estimate (1) also follows from the proof of Theorem 1 given by Sueiro in [Su].

By Proposition 1 we may deduce estimates for \( M_\Omega \) from known estimates for \( M_{C_\alpha} \). For instance if we let \( u = Pf \), Theorem 1 follows from the classical estimates of the nontangential maximal function of Poisson integrals. As another example we may assume that \( u \in H^p(\mathbb{R}^n) \) and deduce that \( \| M_\Omega u \|_{L^p(\mathbb{R}^n)} \leq C \| u \|_{H^p(\mathbb{R}^n)}, \ p > 0 \) since the corresponding estimates holds for \( M_{C_\alpha} u \), see [FS, Theorem 9].

To prove (1) we will use the idea, suggested by Carleson in [C, p. 67] that estimates of maximal functions can be deduced from results about Carleson measures. In fact, if we let \( \mu_\Omega \) be the outer measure in \( \mathbb{R}^{n+1}_+ \) defined by \( \mu_\Omega(E) = \| \{ x \in \mathbb{R}^n; \Omega^x \cap E \neq \emptyset \} \| \), then \( \| \{ x \in \mathbb{R}^n; M_\Omega u(x) > \lambda \} \| = \mu_\Omega(\{ z \in \mathbb{R}^{n+1}_+; u(z) > \lambda \}) \). Thus Proposition 1 is a consequence of the following two results.

Lemma 1. If i)–iii) holds then \( \mu_\Omega \) is a Carleson measure.

Lemma 2. If \( \mu \) is a Carleson measure, then

\[
\mu(\{ z \in \mathbb{R}^{n+1}_+; |u(z)| > \lambda \}) \leq C \| \{ x \in \mathbb{R}^n; M_{C_\alpha} u(x) > \lambda \} \|.
\]

We recall that \( \mu \) is a Carleson measure if \( \mu(B_\rho(t) \times (0, t)) \leq C t^\alpha \). We use, somewhat incorrectly, the terminology “Carleson measure” also when \( \mu \) only is
an outer measure. Lemma 2 is well-known and is proved by a standard covering argument, for details see for instance [G, Ch.1, §5]. It also holds for outer measures since only subadditivity is needed in the proof. So it only remains to prove Lemma 1:
\[
\mu(B_p(t) \times (0, t)) = |\{x; \Omega^x \cap B_p(t) \times (0, t) \neq \emptyset\}| = \text{by i)}
\]
\[
= |\{x; \Omega^x(t) \cap B_p(t) \neq \emptyset\}| \leq \text{by ii)} \leq |\{x; p \in \Omega^x(Ct)\}|
\]
\[
= |\{x; -x \in \Omega^{-p}(Ct)\}| \leq \text{by iii)} \leq Ct^n.
\]

In the next two paragraphs we generalize this argument to domains in \(\mathbb{R}^n\) and \(C^n\). As the boundaries are homogeneous spaces, Lemma 2 immediately generalizes. The problem therefore reduces to give natural conditions on the approach regions \(\Omega^x\) that guarantee that \(\mu_\Omega\) is a Carleson measure.

§2. Non-nontangential limits for bounded domains in \(\mathbb{R}^n\).

Let \(D\) be a bounded domain in \(\mathbb{R}^n\) with \(C^2\) boundary. We fix a neighborhood \(U\) of \(\partial D\) in \(\partial \Omega\) so small that \(U\) can be identified with \(\partial D \times [0, \varepsilon]\). For instance, if \(n(x)\) is the inward normal vector field, \(x \in \partial D\), then by the inverse function theorem the map \(\partial D \times [0, \varepsilon) \ni (x, t) \mapsto x + tn(x) \in \partial \Omega\) is a \(C^2\)-diffeomorphism onto some neighborhood \(U\) of \(\partial D\), and hence realizes such an identification. For \(x \in \partial D\) we let \(B_x(t)\) be the intersection of \(D\) with the ball in \(\mathbb{R}^n\) of radius \(t\) and center \(x\). The cone \(C^x_s\) at \(x \in \partial D\) is \(C^x_s = \{(x', y) \in U; x' \in B_x(\varepsilon y)\}\). If \(Pf\) denotes the Poisson integral of a function \(f \in L^p(\partial D)\), then \(M_{C^x_s} Pf\) is of weak type \((1, 1)\) and bounded on \(L^p\), \(1 < p \leq +\infty\), see [S, §5]. Here of course \(M_{C^x_s} u(x) = \sup_{\mathcal{C}^x_s} |u|\).

As in §1, we want to extend this result to approach regions \(\Omega^x\), \(x \in \partial D\), with \(x \in \partial \Omega^x\) that are not contained in any cone \(C^x_s\). In view of §1, it is natural to assume that \(\Omega^x\) satisfies
\[
i) \quad t \mapsto \Omega^x(t) \text{ is increasing},
\]
\[
ii) \quad \text{if } B_p(t) \cap \Omega^x(t) \neq \emptyset, \text{ then } p \in \Omega^x(Ct),
\]
and
\[
iii) \quad |\Omega^x(t)| \leq Ct^n.
\]

Here \(\Omega^x(t) = \{x' \in \partial D; (x', t) \in \Omega^x\}\). Note that the cones \(C^x_s\) satisfy i–iii). However, in order to obtain an analogue of Theorem 1, we also need some continuity in \(x\). Suppose that to \(x, x' \in \partial D\) there are local diffeomorphisms \(T_{xx'}\) mapping a neighborhood of \(x\) in \(\partial D\) to a neighborhood of \(x'\). We also assume that \((x, x', x'') \mapsto T_{xx'}(x'')\) is smooth, \(T_{xx'}(x) = x'\) and that \(T_{xx}\) is the identity. We extend \(T_{xx'}\) to \(U\) by \(T_{xx'}(x'', t) = (T_{xx'}(x''), t)\) and assume that \(\Omega^x = T_{xx}(\Omega^x)\).

Remark 1. Such a family diffeomorphisms \(T_{xx'}\) always exists locally in \(x, x' \in \partial D\). One can for instance use the translation structure induced from some
local coordinate system on \( \partial D \), so that in these coordinates \( T_{xx'}(x'') = x'' + x' - x \). However global existence of \( T_{xx'} \) is impossible unless the tangent bundle \( T(\partial D) \) is trivial. Furthermore, to obtain \( \Omega^x \) and \( T_{xx'} \) as above it is enough to prescribe \( \Omega^{x_0} \) for some \( x_0 \), satisfying i)--iii), and local diffeomorphisms \( F_x \) on \( \partial D \) with \( F(x_0) = x \); if we extend \( F_x \) to \( U \) by \( F(x', t) = (F_x(x'), t) \) and put \( \Omega^x = F_x(\Omega^{x_0}) \), then \( \Omega^x \) also satisfies i)--iii) (perhaps with a larger \( C \)) and \( \Omega^x \) are connected via \( T_{xx'} = F_{x'} \circ F_x^{-1} \).

Under these assumptions we can prove

**Proposition 1.** If \( \Omega^x, x \in \partial, \) are connected via local diffeomorphisms as above and \( \Omega^x \) satisfies i)--iii), then

\[
|\{ x \in \partial; M_\Omega u(x) > \lambda \} | \leq C |\{ x \in \partial D; M_{C_x} u(x) > \lambda \} |
\]

and

\[
\| M_\Omega u \|_{L^p(\partial)} \leq C \| M_{C_x} u \|_{L^p(\partial D)}.
\]

As Lemma 2 in §1 applies in this setting. Proposition 1 follows from

**Lemma 1.** If \( \Omega^x \) is as in Proposition 1, then \( \mu_\Omega \) is a Carleson measure.

Here \( \mu_\Omega(E) = |\{ x \in \partial; \Omega \cap E \neq \emptyset \} | \). As in the proof of Lemma 2 in §1, we obtain

\[
\mu_\Omega(B_p(t) \times (0, t)) \leq |\{ x; p \in \Omega^x(ep) \} |.
\]

Now we cannot deduce that this equals \( |\{ x; -x \in \Omega^{-p}(Ct) \} | \). However, if we let \( F_x = T_{x_0x} \) and \( G(x, y) = F_x^{-1}(y) \), then for fixed \( y \), \( G(x, y) \) is defined for \( x \) near \( y \). Since \( G(y, y) = x_0, \partial G/\partial x|_{(p, p)} + \partial G/\partial y|_{(p, p)} = 0 \)

\[
\frac{\partial G(x, y)}{\partial x} \bigg|_{(p, p)} = - \left[ \frac{\partial F_x(y)}{\partial y} \right]^{-1}.
\]

Hence \( \partial G/\partial x|_{(p, p)} \) is non-singular and \( g(x) = g_p(x) = G(x, p) \) is a diffeomorphism. Thus

\[
\mu_\Omega(B_p(t) \times (0, t)) \leq |\{ x; p \in \Omega^x(ep) \} |
\]

\[
= |\{ x; p \in T_{x_0x}(\Omega^{x_0}(Ct)) \} | = |\{ x; g(x) \in \Omega^{x_0}(Ct) \} |
\]

\[
= |g^{-1}(\Omega^{x_0}(Ct)) | \leq C |\Omega^{x_0}(Ct) | \leq Ct^*.
\]

**§3. Boundary values of \( HP \)-functions.**

Let \( D \) be a bounded strictly pseudoconvex domain of class \( C^2 \) in \( C^* \). In this section we will study boundary values of \( HP \)-functions in \( D \). For a background to this problem we refer to [S]. The appropriate analogue of nontangential convergence in this situation is the so-called admissible convergence, i.e. convergence in
Korányi cones, which are larger than the standard cones. (We give the precise
definition of the Korányi cones and the associated maximal function $M_{K_s}$ in
a moment.) The basic theorem is, see [S, §9] or [H],

**Theorem A.** If $D$ is a bounded $C^2$-domain and $f \in H^p(D)$, then

$$\|M_{K_s}f\|_{L^p} \leq C_p \|f\|_{H^p}, \ 0 < p \leq +\infty,$$

and $f$ has admissible boundary values for almost all $x \in \partial D$.

In analogy to the non-nontangential approach regions in §1 and §2, we will
generalize this result to nonadmissible approach regions that are strictly larger
than the Korányi cones. As before we will show that the corresponding maximal
functions are comparable. As a consequence, we obtain a generalization of
Theorem A to these nonadmissible approach regions. As another corollary, we
obtain the result of Sueiro [Su2] about Poisson-Szegö integrals in the generaliz-
ed halfspace and the ball.

We start by briefly discussing the nonisotropic structure on $\partial D$, the associated
Korányi balls and the admissible approach regions. For each $p \in \partial D$ we have
a preferred $(2n-2)$-dimensional real subspace $M_p$, the complex tangent space, of
the real tangent space $T_p(\partial D)$. If $J$ is the complex structure, then

$$M_p = T_p(\partial D) \cap JT_p(\partial D).$$

If $D = \{\rho < 0\}$, then the real 1-form $\alpha = Jd\rho = d^\rho \rho =
\bar{\rho} d\rho$ defines $M_p$, i.e. $M_p = \{v \in T_p(\partial D); \alpha(v) = 0\}$. That $D$
is strictly pseudconvex means that the quadratic form (the Levi form) $d\alpha(v,w)
= dd^\rho(v,w)$ is positive definite on $M_p(\partial D)$ for each $p \in \partial D$. We define a basis of neighborhoods $B_p(t)$ in
$\partial D$, the Korányi balls at $p$, by requiring that $B_p(t)$ have length $\sim \sqrt{t}$ in the
$M_p$-directions and length $\sim t$ in the last one. Then $|B_p(t)| \sim t^n$ and $B_p(t)$ is
determined up to equivalence. $B_p(t)$ and $B'_p(t)$ are equivalent if $B_p(t/C) \subset
B'_p(t) \subset B_p(Ct)$ for small $t$. From now on we fix a choice of $B_p(t)$ at each $p$ continuously in $p$. (Any other such a choice will be equivalent to $B_p(t)$ uniformly for $p$ in
compact sets). The Korányi balls satisfy

\begin{align*}
(1) & \quad \text{if } s \leq t \text{ and } B_p(t) \cup B_q(s) \neq \emptyset, \text{ then } B_q(s) \subset B_p(Ct), \\
(2) & \quad |B_p(2t)| \leq C|B_p(t)|.
\end{align*}

Thus $\partial D$ equipped with these balls $B_p(t)$ and surface measure is a homogeneous
space, and the standard tools from harmonic analysis such as covering lemmas,
$L^p$-estimates of maximal functions, etc. can be applied.

As in §2 we identify a neighborhood $U$ of $\partial D$ with $\partial D \times [0,\varepsilon)$. The admissible
approach regions, or Korányi cones, are $K^x_s = \{x', t; x' \in B_x(at)\}$. We consider
approach regions $\Omega^x$, $x \in \partial D$, with $x \in \partial^x$ that satisfies
i) \( t \mapsto \Omega^x(t) \) is increasing,

ii) if \( \Omega^x(t) \cap B_p(t) = \emptyset \), then \( p \in \Omega^x(Ct) \)

and

iii) \( |\Omega^x(t)| \leq Ct^\alpha \).

Here of course \( \Omega^x(t) = \{ x' \in \partial D; (x', t) \in \Omega^x \} \). (Note that the Korányi cones satisfy these conditions.) We further assume that \( \Omega^x \) are connected via local diffeomorphisms \( T_{xx'} \) as in \( \S 2 \). Hence \( \Omega^x = T_{xx'}(\Omega^x) \). However an arbitrary diffeomorphismmmm \( T_{xx'} \) will destroy condition ii). Hence we further need to assume that \( T_{xx'} \) preserves the nonisotropic structure, that is \( T_{xx'}(B_x(t)) \) is equivalent to \( B_x(t) \). This is equivalent to that \( dT_{xx'}|_p \) maps \( M_p \) onto \( M_{T_{xx'}}(p) \). Then if \( \Omega^x \) satisfies i)–iii), so will \( \Omega^x \). We will show in \( \S 4 \) that such a family \( T_{xx'} \) always exists locally if \( D \) is strictly pseudoconvex.

**Remark 1.** The requirement is not that \( T_{xx'} \) are CR-mappings. For that one also requires that \( dT_{xx'}|_p \) commute with the complex structure \( J \). This implies that the Levi form is preserved and this cannot hold in general.

Let \( M_\Omega u(x) = \sup |u|, x \in \partial \Omega \). We have \( \Omega^x \)

**Theorem 1.** Suppose that \( D \) is a strictly pseudoconvex bounded domain with \( C^2 \) boundary. Further assume that we have a family of regions \( \Omega^x, x \in \partial \Omega \subset \partial D \), connected via Korányi ball preserving diffeomorphisms as above. If \( \Omega^x, x \in \partial \Omega \), satisfies i)–iii), then for \( f \in H^p(D) \),

\[
\| M_\Omega f \|_{L^p(\partial \Omega)} \leq C \| f \|_{H^p(D)}, \; 0 < p \leq +\infty,
\]

and \( f \) has \( \Omega \)-limits for almost all \( x \in \partial \Omega \).

**Example 1.** As in Example 1 in \( \S 1 \), for any tangential curve \( \gamma \) we can choose points \( p_j = (x_j, y_j) \) on \( \gamma \) such that \( x_j \to x_0 \) and such that \( \Omega^{x_0} = \bigcup \{ (x', y_j + y); x' \in B_x(y) \} \) satisfies i)–iii). Then if we put \( \Omega^x = T_{x_0}(\Omega^{x_0}) \), \( \Omega^x \) satisfies the conditions in Theorem 1. In particular, we may choose \( \gamma \) so that \( \Omega^x \) is not contained in any \( K^x \).

Theorem 1 is a consequence of

**Proposition 1.** If \( \Omega^x, x \in \partial \Omega \), satisfy the assumptions of Theorem 1, we have

\[
|\{ x \in \partial \Omega; M_\Omega u(x) > \lambda \}| \leq C |\{ x \in \partial D; M_\Omega u(x) > \lambda \}|
\]

and

\[
\| M_\Omega u \|_{L^p(\partial \Omega)} \leq C \| M_\Omega u \|_{L^p(\partial D)}, \; p > 0.
\]

To prove Proposition 1, we need two lemmas about Carleson measures as in
§1. The Carleson measure is defined with respect to the Korányi balls, 
\[ \mu(B_p(t) \times (0, t)) \leq C |B_p(t)|. \] We also define \( \mu_\Omega(E) = |\{ x \in \hat{\omega}; \Omega^x \cap E \neq \emptyset \}|. \)

**Lemma 1.** If \( \Omega^x, x \in \hat{\omega}, \) satisfy the conditions of Theorem 1, then \( \mu_\Omega \) is a Carleson measure.

This is proved exactly as Lemma 1 in §2. Finally we have

**Lemma 2.** If \( \mu \) is a Carleson measure, then

\[ \mu(\{ x \in D; |u(x)| > \lambda \}) \leq C |\{ x \in \partial D; M_{k_x} u(x) > \lambda \}|. \]

This follows by the same covering argument as proved Lemma 2 in §1; the covering argument works because of (1) and (2).


In this last paragraph we show that locally on a strictly pseudoconvex boundary, there always exists a family of diffeomorphisms as in §2 that also preserves the Korányi balls. As was noted in §3, the last requirement is equivalent to that the derivatives preserve the real complex tangent spaces \( M_x \).

**Proposition 1.** Let \( D \in \mathbb{C}^n \) be strictly pseudoconvex with \( C^\infty \)-boundary. For any point \( x_0 \in \partial D \) there is a compact neighborhood \( \hat{\omega} \in \partial D \) of \( x_0 \) and local diffeomorphisms \( F_x, x \in \hat{\omega}, \) such that \( F_x(x_0) = x, F_{x_0} \) is the identity, \( (x, x') \rightarrow F_x(x') \) is smooth and \( dF_x|_{x'} \) maps \( M_x \) onto \( M_{F_x(x')}. \)

Before the proof we study an important example.

**Example 1.** Let \( \Pi = \{(z_1, z') \in \mathbb{C}^n; |z|^2 - \text{Im} z_1 < 0\} \) be the generalized halfspace with its strictly pseudoconvex boundary \( \partial \Pi = \{(z_1, z') \in \mathbb{C}^n; |z|^2 = \text{Im} z_1\}. \) The Heisenberg group \( H \), with underlying manifold \( \mathbb{R} \times \mathbb{C}^n \) and noncommutative group structure \( (t, a) \cdot (t', a') = (t + t' + 2 \text{Im} a \cdot \overline{a'}, a + a') \) acts on \( \Pi \) from the left by \( (t, a) \cdot (z_1, z') = (z_1 + t + i |a|^2 + 2i \text{Im} z' \cdot \overline{a}, z' + a) \). Note that for fixed \( g \in H, (z_1, z') \rightarrow g \cdot (z_1, z') \) is a biholomorphic mapping of \( \Pi \) (and of \( \mathbb{C}^n \)) and hence a CR-mapping from \( \partial \Pi \) to \( \partial \Pi \), so in particular its derive maps \( M_p \) onto \( M_{g,p}. \) Since for each pair \( p_0, p \in \partial \Pi \) there is a unique \( g \in H \) such that \( p = g \cdot p_0 \), we can put \( F_p(x) = g \cdot x \). Then \( F_p \) is defined globally on \( \partial \Pi \) and we have proved Proposition 1 in this particular case.

By the linear fractional mapping \( \psi \), where \( w = \psi^{-1}(z) \) is given by

\[ z_1 = i \frac{1 - w_1}{1 + w_1} \text{ and } z_i = \frac{w_i}{1 + w_1} \text{ if } 2 \leq i \leq n, \]

\( \Pi \) is mapped biholomorphically onto the unit ball \( B \) and \( \psi|_{\partial \Pi} \) is a CR-mapping.
from \( \partial \Pi \) onto the boundary of \( B \) minus one single point. Together with Example 1, this shows Proposition 1 for the ball.

Now we consider the general strictly pseudoconvex case. We claim that there is locally a diffeomorphism \( \psi \) from \( \partial D \) to \( \partial \Pi \) such that \( d\psi|_x \) maps \( M_\psi(\partial D) \) onto \( M_{\psi(x)}(\partial \Pi) \). Then the proposition is a consequence of Example 1 above.

Contrary to the case of the ball, it is not possible in general to have \( \psi \) as the restriction to \( \partial D \) of a biholomorphic mapping, since then it also would be a CR-mapping.

However, what is relevant for us is that we have a real \((2n - 1)\)-dimensional manifold \( X (= \partial D) \) and a smooth distribution (subbundle) \( M \) of \( T(X) \). Such a pair \((X, M)\) is called a contact manifold if \( M \) is nondegenerate. For a discussion of contact manifolds see [A, appendix 4H]. The nondegeneracy conditions means that \( M \) is as far as possible from being integrable (in the sense of Frobenius theorem) and can be stated as follows: If \( \alpha \) is a 1-form that defines \( M_x \), i.e. \( \alpha_x \perp M_x \) for all \( x \), then for any \( 0 \neq v \in M_x \) there is a \( w \in M_x \) such that \( d\alpha_x(v, w) \neq 0 \).

If \( \partial D \) is strictly pseudoconvex, then the Levi form \( L(v, w)_x = \pm \text{id} \alpha_x(v, w) \) is positive definite, compare p. 9, so in particular \( d\alpha_x \) is nondegenerate on \( M_x \) and hence \((X, M)\) is a contact manifold. Now the claim follows from

**Lemma 1** (see [A, appendix 4H]). Any two contact manifolds are locally contact-diffeomorphic.

**REFERENCES**


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