A CHARACTERIZATION OF BALANCED RATIONAL NORMAL SURFACE SCROLLS IN TERMS OF THEIR OSCULATING SPACES II

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In this note we prove a result conjectured by the second and third author [3, Conjecture (i), p. 216]; the proof was found independently by the first author and by the second and third authors. The results needed in the proof are contained in [3]; it only remains to apply a theorem of Van de Ven [6, Theorem III, p. 406] to the adjunction divisor. We would like to take this opportunity to point out adjunction theory as a very powerful tool in projective geometry!

Let $X \subset \mathbb{P}^N$ be a smooth, complex, projective surface. Recall [3, p. 216] that the $m$th osculating space to $X$ at a point $x$ is the linear subspace $\text{Osc}_X^m(x)$ determined by the partial derivatives of order $\leq m$ of the coordinate functions, with respect to a system of local parameters for $X$ at $x$ and evaluated at $x$. More precisely, let $\mathcal{P}_X^m(1)$ denote the sheaf of principal parts of order $m$ of $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Then $\mathcal{P}_X^m(1)$ is locally free with rank $\binom{m+2}{2}$, and there are homomorphisms

$$a^m : \mathcal{O}_X^{N+1} \to \mathcal{P}_X^m(1)$$

such that $\text{Im} (a(x))$ defines the $m$th order osculating space to $X$ at $x$, i.e., such that

$$\text{Osc}_X^m(x) := \mathcal{P}(\text{Im}(a^m(x))) \subset \mathbb{P}^N.$$

For a general surface, one expects the dimension of $\text{Osc}_X^m(x)$ to be $\binom{m+2}{2} - 1$ for almost all points $x$ of $X$ and all $m$ such that $\binom{m+2}{2} - 1 \leq N$. Points where this dimension is smaller than expected are "flat" points of the surface – often called points of hyperosculation. If the surface $X$ contains a line through the

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point \( x \), then the dimension of \( \text{Osc}_{x}^{m}(x) \) is at most \( 2m \). Hence a \textit{ruled surface} has the property that \textit{all} its \( m \)th order osculating spaces have dimension at most \( 2m \). There are, however, non-ruled surfaces with this property (Togliatti [5], Dye [1]).

Suppose the surface \( X \) is a \textit{balanced rational normal scroll of degree} \( 2n \), i.e., \( X \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) embedded in \( \mathbb{P}^{2n+1} \) via \( \text{pr}^{1}_{*}\mathcal{O}_{\mathbb{P}^1}(1) \otimes \text{pr}^{2}_{*}\mathcal{O}_{\mathbb{P}^1}(n) \). It is shown in [2, p. 1060] that in this case \textit{all} \( m \)th order osculating spaces to \( X \) have dimension \( 2m \), for \( m \leq n \). The theorem we shall prove shows that this property characterizes the balanced rational normal scrolls.

**Theorem.** Let \( X \subset \mathbb{P}^{2n+1}, n \geq 2 \), be a smooth, projective surface, not contained in a hyperplane, such that \( \dim \text{Osc}_{x}^{m}(x) = 2m \) for all \( x \in X \) and all \( m \leq n \). Then \( X \) is a balanced rational normal scroll of degree \( 2n \).

**Proof.** In [3, Theorem, p. 221] the second and third author proved that any surface satisfying the hypotheses of the theorem is birationally ruled, but not isomorphic to \( \mathbb{P}^2 \). Moreover, the theorem was shown to hold if \( X \) is geometrically ruled (i.e., has no exceptional curves of the first kind) or if \( X \) is rational. Finally, it was shown that the theorem holds provided \( n \leq 4 \).

We shall prove the theorem by showing that the assumptions \( n \geq 5 \), \( X \) is not rational, \( X \) is birationally ruled and contains at least one exceptional curve, lead to a contradiction.

Let \( H \) denote the class of a hyperplane section of \( X \) and let \( K \) denote the class of a canonical divisor on \( X \). Since \( X \) spans \( \mathbb{P}^{2n+1} \), we have

\[
\dim H^{0}(X, \mathcal{O}_{X}(1)) \geq 2n + 2 \geq 12
\]

and \( \deg X = H^{2} \geq 2n \geq 10 \). By a theorem of Van de Ven [6, Theorem III, p. 406] it follows that \( K + H \) is very ample unless there exists an exceptional curve \( E \) on \( X \) with \( E \cdot H = 1 \). Let us show that such an \( E \) cannot occur. Recall from [3, p. 220] that \( X \) possesses a line bundle \( \mathcal{L} \), given by

\[
\mathcal{L} := \text{Coker}(a^{2} : \mathcal{O}_{X}^{N+1} \to \mathcal{P}_{X}(1)),
\]

with first Chern class

\[
c_{1}(\mathcal{L}) = \frac{1}{n-1}(2nK + (n + 3)H).
\]

Since \( E \cdot K = -1, E \cdot H = 1 \) implies

\[
c_{1}(\mathcal{L}) \cdot E = -\frac{n-3}{n-1}.
\]

But this is not an integer since we have assumed \( n \geq 5 \). So \( K + H \) must be very ample.
We can therefore argue as in [3, p. 221], using $K + H$ instead of $H$. The theorem of Sommese [4, Theorem (1.5), p. 377] and Van de Ven [6, Theorem II, p. 403] allows us to conclude that $2K + H$ is generated by its global sections, in particular that $(2K + H)^2 \geq 0$ holds.

Let $\tau = \frac{1}{3}(K^2 - 2c_2(X))$ denote the Hirzebruch index of $X$. Since $X$ is birationally ruled and not isomorphic to $\mathbb{P}^2$, we have $K^2 = 8(1 - q) - t$ and $c_2(X) = 4(1 - q) + t$, where $t$ denotes the number of exceptional curves on $X$ with respect to a relatively minimal model, and $q$ denotes the genus of the base curve of this minimal model. Therefore $\tau = -t$ holds, hence $\tau \leq 0$.

Recall [3, (2'), p. 220; p. 221] that the numerical characters of $X$ satisfy the formulas

$$n(2n + 1)K^2 + 2n(n + 5)K \cdot H + 2(n^2 + 2n + 3)H^2 = 0$$

and

$$3n(2n + 1)\tau = -2(n + 3)(nK \cdot H + (n + 1)H^2).$$

From these two formulas we deduce the following equality

$$(2K + H)^2 = \frac{54}{n + 3}\tau - 3\frac{n - 4}{n}H^2.$$ 

Since $\tau \leq 0$, $n \geq 5$, and $H^2 > 0$, we get $(2K + H)^2 < 0$, which is the desired contradiction.

REFERENCES