PRIMITIVE IDEMPOTENTS OF THE GROUP
ALGEBRA CSL(3, q)

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Introduction.

Let GL(3, q) denote the group of all nonsingular 3 × 3 matrices over the Galois field GF(q) of q elements (q = p^s, p is a prime, s ≥ 1 is an integer) and C be the field of all complex numbers. Drobotenko [3] obtained the primitive idempotents of the group algebra CGL(3, q). For this, he used the following properties:

Property 1. Let G be a finite group of order n and let K be an algebraically closed field with characteristic not dividing n. If ξ is an irreducible KG-character affording the orthogonal central idempotent e_ξ of KG, then

\[ e_ξ = ξ(1)n^{-1} \sum_{g \in G} ξ(g^{-1})g, \]

where ξ(1) is the degree of ξ.

Property 2. Let H be a subgroup of G and let ψ be a KH-character of degree 1, η an irreducible KG-character. Assume that the multiplicity of η in the induced KG-character ψ^H is 1. If η and ψ afford the orthogonal central idempotents e_η and e_ψ respectively, then e_ηe_ψ is a primitive idempotent of KG which corresponds to η.

Proof. Let M be a KH-module and N be an irreducible KG-module which afford the characters ψ and η respectively. From Wedderburn theorem [2, Theorem 2.9], it is easy to see that KHe_ψ = M and e_ηM^G = N. By definition of the induced KG-module M^G, we obtain M^G = KG e_ψ. Then we have N = KG e_ψ e_η. Moreover, it is well known that e is a primitive idempotent of KG if and only if KG e is an irreducible KG-module.

Remark. Using these properties, all irreducible K-representations of G are obtained.

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We have used these properties in [4] to obtain the primitive idempotents of CSL$(3, p)$. In this paper we will use again these properties and the some results of [3] to obtain the primitive idempotents of CSL$(3, q)$ which correspond to the irreducible CSL$(3, q)$-characters, where SL$(3, q)$ is the group of $3 \times 3$ matrices over GF$(q)$ with determinant unity.

**Method and Results.**

If $\rho$ is a generator of the multiplicative group GF$(q)^\times$ and if $d$ is the greatest common divisor of $q - 1$ and 3, we write $\langle \rho \rangle = \text{GF}(q)^\times$ and $d = (q - 1, 3)$ respectively. Let $K = \{ x \in \text{GF}(q)^\times \mid x \equiv p^u; \, u \equiv 0 \text{ (mod 3)} \}$. If $d = 1$, we have $\text{GF}(q)^\times = K$ and if $d = 3$, then

$$\text{GF}(q)^\times = K \cup \rho K \cup \rho^2 K \quad \text{(disjoint)}.$$

Let $\theta$ be a primitive element of the field GF$(q)$. Consider the above elements of $S = \text{SL}(3, q)$:

$$a_i = \begin{bmatrix} 1 & 0 & \theta^{e_i - 1} \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 & \theta^{e_i - 1} & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \theta^{e_i - 1} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$(a_i^p = b_i^p = c_i^p = I; \, i = 1, \ldots, s)$

It follows directly that $S$ has the following subgroups:

$$A = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle; \quad B = \langle b_1 \rangle \times \ldots \times \langle b_s \rangle; \quad C = \langle c_1 \rangle \times \ldots \times \langle c_s \rangle;$$

$$D = A \times B; \quad H = DC;$$

$$|A| = |B| = |C| = q; \quad |D| = q^2; \quad |H| = q^3.$$  

The irreducible CH-characters are given by table 1 [3]. It is easy to see that there exist the following relations between the conjugacy classes of $S$ which have already been determined by Dickson [1] and the conjugacy classes of $H$:

The conjugacy classes of types $\Delta_3, \Delta_4, \Delta_5$ of $H$, for $d = 1$ or 3, are included in the conjugacy class of canonical representative

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

of $S$.

The conjugacy classes of type $\Delta_5$ of $H$, for $d = 1$, are included in the conjugacy class of canonical representative

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

of $S$ and for $d = 3$, we have the following proposition:

**Proposition 1.** For $d = 3$, $(q - 1)^2/3$ conjugacy classes of type $\Delta_5$ of $H$ are included in the conjugacy classes of canonical representatives.
### TABLE 1
Irreducible CH-characters

\[ k = k_1 + k_2 \theta + \ldots + k_r \theta^{r-1}, \quad t = t_1 + t_2 \theta + \ldots + t_r \theta^{r-1}; \quad k_i, t_i, m_i, n_i = 0, \ldots, p - 1; \quad i = 1, \ldots, s; \]

\((k_1, \ldots, k_s) \neq (0, \ldots, 0), (t_1, \ldots, t_s) \neq (0, \ldots, 0), (h_1, \ldots, h_s) \neq (0, \ldots, 0), e \neq 1; \quad \varepsilon^e = 1\)

<table>
<thead>
<tr>
<th>Conjugacy class</th>
<th>canonical representative</th>
<th>number of conjugacy classes</th>
<th>Number of elements in the conj. class</th>
<th>(q_{h_1, \ldots, h_s}^k)</th>
<th>(\psi_{m_1, \ldots, m_s, n_1, \ldots, n_s}^k)</th>
</tr>
</thead>
</table>
| \(A_1\)        | \[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 \\
1
\end{bmatrix}
\] | 1                           | 1                                  | \(q\)                           | 1                      |
| \(A_2\)        | \[
\begin{bmatrix}
1 & 0 & k \\
1 & 0 \\
1
\end{bmatrix}
\] | \(q - 1\)                     | 1                                  | \(q \varepsilon_{h_1}^{k_1} \cdots \varepsilon_{h_s}^{k_s}\) | 1                      |
| \(A_3\)        | \[
\begin{bmatrix}
1 & k & 0 \\
1 & 0 \\
1
\end{bmatrix}
\] | \(q - 1\)                     | \(q\)                              | 0                  | \(\varepsilon_{m_1}^{k_1} \cdots \varepsilon_{m_s}^{k_s}\) |
| \(A_4\)        | \[
\begin{bmatrix}
1 & 0 & 0 \\
1 & t \\
1
\end{bmatrix}
\] | \(q - 1\)                     | \(q\)                              | 0                  | \(\varepsilon_{n_1}^{k_1} \cdots \varepsilon_{n_s}^{k_s}\) |
| \(A_5\)        | \[
\begin{bmatrix}
1 & k & 0 \\
1 & t \\
1
\end{bmatrix}
\] | \((q - 1)^2\)                 | \(q\)                              | 0                  | \(\varepsilon_{m_1}^{k_1} \cdots \varepsilon_{m_s}^{k_s} \varepsilon_{n_1}^{k_1} \cdots \varepsilon_{n_s}^{k_s}\) |

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 \\
1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 0 \\
1 & \rho \\
1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 0 \\
1 & \rho^2 \\
1 & 1
\end{bmatrix}
\]

of \(S\) if and only if the following conditions hold respectively:

**X)** \(k \equiv t \pmod{K}\)

**Y)** \((k, t) \in K \times \rho K \text{ or } (k, t) \in \rho K \times \rho^2 K \text{ or } (k, t) \in \rho^2 K \times K\)

**Z)** \((k, t) \in K \times \rho^2 K \text{ or } (k, t) \in \rho K \times K \text{ or } (k, t) \in \rho^2 K \times \rho K\)

Let \(G = \text{GL}(3, q)\). The irreducible CG-characters are as follows [6]:

\[\xi(z^{(j)}); \quad j = 1, \ldots, 8; \quad z = 1, \ldots, h; \quad h_1 = h_2 = h_4 = q - 1,\]

\[(1) \quad h_3 = h_5 = (q - 1)(q - 2), \quad h_6 = (q - 1)(q - 2)(q - 3)/6,\]

\[h_7 = q(q - 1)^2/2, \quad h_8 = q(q^2 - 1)/3;\]
where \( j \) denotes the type of the character and \( z \) denotes the number of the characters of the type \( j \).

The irreducible CS-characters are as follows [5]:

\[
\eta^{(j)}_r; j = 1, \ldots, 11; r = 1, \ldots, f_j; f_1 = f_2 = f_4 = 1, f_3 = f_5 = q - 2,
\]

\[
f_6 = [(q - 1)(q - 4) + 3 - d]/6, f_7 = q(q - 1)/2,
\]

\[
f_8 = (q^2 + q + 1 - d)/3, f_9 = f_{10} = f_{11} = (d - 1)/3/2;
\]

where \( d = (q - 1, 3) = 1 \) or 3 and \( j \) denotes the type of the character, \( r \) denotes the number of the characters of the type \( j \).

Let \( \xi^{(j)}_z \) be the restriction of \( \xi^{(j)}_z \) to \( S \). The relations between (1) and (2) are as follows:

For \( d = 1 \), \( \xi^{(j)}_z = \xi^{(j)}_z \), \( j = 1, \ldots, 8; z = 1, \ldots, h_j, r = 1, \ldots, f_j \)

For \( d = 3 \), \( \xi^{(j)}_z = \eta^{(j)}_r \), \( j = 1, \ldots, 8; z = 1, \ldots, h_j, r = 1, \ldots, f_j \)

\[
(3) \sum_{r=0}^{2} \eta^{(9)}_r = \xi^{(6)}_z, \sum_{r=0}^{2} \eta^{(10)}_r = \xi^{(8)}_z, \sum_{r=0}^{2} \eta^{(11)}_r = \xi^{(8)}_z,
\]

where \( \xi^{(j)}_z \) are conveniently chosen CG-characters. Using the character tables which are given in [5] and [6] we can choose easily the character \( \xi^{(j)}_z \).

Moreover, from the table of the irreducible CG-characters which is given in [6] and the table 1, it is easy to obtain the multiplicity of \( \xi^{(j)}_z \) in \( \psi^G_{m_1, \ldots, m_z; n_1, \ldots, n_z} \).

Then we have [3]:

\[
\psi^G_{m_1, \ldots, m_z; 0, \ldots, 0} = \psi^G_{0, \ldots, 0; n_1, \ldots, n_z} = \sum_{j=2}^{h_j} \sum_{z=1}^{h_z} \xi^{(j)}_z + 2 \xi^{(5)}_z + 3 \xi^{(6)}_z + \sum_{z=1}^{h_z} \xi^{(7)}_z
\]

\[
(4) \psi^G_{m_1, \ldots, m_z; n_1, \ldots, n_z} = \sum_{j=4}^{h_j} \sum_{z=1}^{h_z} \xi^{(j)}_z
\]

\((m_i, n_i = 0, \ldots, p - 1; i = 1, \ldots, s; (m_1, \ldots, m_z) \neq (0, \ldots, 0); (n_1, \ldots, n_z) \neq (0, \ldots, 0)) \)

Let \( \eta^{(j)}_{r|H} = \eta^{(j)}_r \). The values \( \eta^{(j)}_r \) are given by table 2. Let

\( (\psi^S_{m_1, \ldots, m_z; n_1, \ldots, n_z}; \eta^{(j)}_{r|H}) \)

denote the multiplicity of \( \eta^{(j)}_r \) in \( \psi^S_{m_1, \ldots, m_z; n_1, \ldots, n_z} \). By the tables 1 and 2, the proposition 1 and the Frobenius theorem, it is easy to see that, for every \( r \in \{0, 1, 2\}, (\psi^S_{m_1, \ldots, m_z; 0, \ldots, 0}; \eta^{(j)}_r) \) (and \( (\psi^S_{0, \ldots, 0; n_1, \ldots, n_z}; \eta^{(j)}_r) \)) is constant. Moreover, from (3) and (4), we have:
<table>
<thead>
<tr>
<th>Con. class</th>
<th>Canon. repres.</th>
<th>$\eta^{(1)}$</th>
<th>$\eta^{(2)}$</th>
<th>$\eta^{(3)}$</th>
<th>$\eta^{(4)}$</th>
<th>$\eta^{(5)}$</th>
<th>$\eta^{(6)}$</th>
<th>$\eta^{(7)}$</th>
<th>$\eta^{(8)}$</th>
<th>$\eta^{(9)}$</th>
<th>$\eta^{(10)}$</th>
<th>$\eta^{(11)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_1$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>$q(q + 1)$</td>
<td>$q^2 + q + 1$</td>
<td>$q^3$</td>
<td>$q(q^2 + q + 1)$</td>
<td>$(q + 1) \times (q^2 + q + 1)$</td>
<td>$(q - 1) \times (q^2 + q + 1)$</td>
<td>$(q - 1) \times (q + 1)$</td>
<td>$(q + 1) \times (q^2 + q + 1)/3$</td>
<td>$(q - 1)^2 \times (q + 1)/3$</td>
<td>$(q - 1)^2 \times (q + 1/3$</td>
</tr>
<tr>
<td>$\Delta_2$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; k \ 1 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>$q$</td>
<td>$q + 1$</td>
<td>0</td>
<td>$q$</td>
<td>$2q + 1$</td>
<td>$-1$</td>
<td>$-(q - 1)$</td>
<td>$(2q + 1)/3$</td>
<td>$-(q - 1)/3$</td>
<td>$-(q - 1)/3$</td>
</tr>
<tr>
<td>$\Delta_3$</td>
<td>$\begin{bmatrix} 1 &amp; k &amp; 0 \ 1 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>$q$</td>
<td>$q + 1$</td>
<td>0</td>
<td>$q$</td>
<td>$2q + 1$</td>
<td>$-1$</td>
<td>$-(q - 1)$</td>
<td>$(2q + 1)/3$</td>
<td>$-(q - 1)/3$</td>
<td>$-(q - 1)/3$</td>
</tr>
<tr>
<td>$\Delta_4$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 \ 1 &amp; t &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>$q$</td>
<td>$q + 1$</td>
<td>0</td>
<td>$q$</td>
<td>$2q + 1$</td>
<td>$-1$</td>
<td>$-(q - 1)$</td>
<td>$(2q + 1)/3$</td>
<td>$-(q - 1)/3$</td>
<td>$-(q - 1)/3$</td>
</tr>
<tr>
<td>$\Delta_5$</td>
<td>$\begin{bmatrix} 1 &amp; k &amp; 0 \ 1 &amp; t &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>$q\delta_{rh} -$</td>
<td>$-(q - 1)/3$</td>
<td>$-(q - 1)/3$</td>
</tr>
</tbody>
</table>

$r, h = 0, 1, 2; \delta_{rh} = \begin{cases} 1, & r = h \\ 0, & r \neq h \end{cases}$
\[(\psi_{m_1,\ldots, m_s; 0,\ldots, 0}^S; \sum_{r=0}^2 \eta_r^{(9)}) = (\psi_{0,\ldots, 0; n_1,\ldots, n_s}^S; \sum_{r=0}^2 \eta_r^{(9)}) = 3.\]

It follows directly that:
\[(\psi_{m_1,\ldots, m_s; 0,\ldots, 0}^S; \eta_r^{(9)}) = (\psi_{0,\ldots, 0; n_1,\ldots, n_s}^S; \eta_r^{(9)}) = 1,\quad (r = 0, 1, 2) \tag{5}\]

Finally, by the tables 1 and 2, (5) and the Frobenius theorem, we obtain the following relation:
\[
\psi_{m_1,\ldots, m_s; 0,\ldots, 0}^S = \psi_{0,\ldots, 0; n_1,\ldots, n_s}^S = \sum_{j=2}^4 \sum_{r=1}^{f_j} \eta_r^{(j)} + 2 \sum_{r=1}^{f_s} \eta_r^{(5)} + 3 \sum_{r=1}^{f_j} \eta_r^{(6)} + \sum_{r=1}^{f_s} \eta_r^{(7)} + \left(\frac{d-1}{2}\right) \sum_{r=0}^{2} \eta_r^{(9)},\]

\[(m_i, n_i = 0,\ldots, p - 1; i = 1,\ldots, s; (m_1,\ldots, m_s) \neq (0,\ldots, 0); (n_1,\ldots, n_s) \neq (0,\ldots, 0);\quad d = 1, 3)\]

If \((k, t)\) satisfies the condition X (or Y or Z), we write \((k, t) \in X\) (or \((k, t) \in Y\) or \((k, t) \in Z\)). It follows from the tables 1 and 2, (4) and the Frobenius theorem that:
\[
(\psi_{m_1,\ldots, m_s; n_1,\ldots, n_s}^S; \eta_r^{(j)}) = q^{-3} \left[ (q^3 - q)/3 + (q \delta_{r0} - (q - 1)/3)q \sum_{(k, t) \in X} \varepsilon^{\alpha} + (q \delta_{r1} - (q - 1)/3)q \sum_{(k, t) \in Y} \varepsilon^{\alpha} + (qd_{r2} - (q - 1)/3)q \sum_{(k, t) \in Z} \varepsilon^{\alpha} \right] = 1 \text{ or } 0, \tag{7}
\]

\[(j = 9, 10, 11; r = 0, 1, 2; \delta_{rh} = \begin{cases} 1; r = h \\ 0; r \neq h \end{cases}; \varepsilon^{\alpha} = 1, k = k_1 + k_2 \theta + \ldots + k_s \theta^{s-1},
\]
\[t = t_1 + t_2 \theta + \ldots + t_s \theta^{s-1}; \alpha = k_1 m_1 + \ldots + k_s m_s + t_1 n_1 + \ldots + t_s n_s;\]
\[n_i = 0,\ldots, p - 1; i = 1,\ldots, s; (m_1,\ldots, m_s) \neq (0,\ldots, 0); (n_1,\ldots, n_s) \neq (0,\ldots, 0),\]
\[(k_1,\ldots, k_s) \neq (0,\ldots, 0), (t_1,\ldots, t_s) \neq (0,\ldots, 0))\]

Using (7), we then obtain the following result, which follows from the solution of a simple system of linear equations:

**Proposition 2.** Let \(x = \sum_{(k, t) \in X} \varepsilon^{\alpha}, y = \sum_{(k, t) \in Y} \varepsilon^{\alpha}, z = \sum_{(k, t) \in Z} \varepsilon^{\alpha}\) in (7). One of the \(x, y, z\) is equal to \((2q + 1)/3\) and each of the others is equal to \((1 - q)/3\).

We now prove the following basic result:
PROPOSITION 3. For $j = 9, 10, 11$,

$$(\psi^{S}_{m,n_{1},...,m_{s};n_{1},...,n_{s}}; \eta^{(j)}_{r}) = 1; \ (m_{i}, n_{i}) \in Z, \ r = 1;$$

$$(m_{i}, n_{i}) \in Y, \ r = 2.$$

$$(m_{i}, n_{i}) = 0, ..., p - 1; m_{i} = 0 \iff n_{i} = 0; i = 1, ..., s; (m_{1}, ..., m_{s}) \neq (0, ..., 0),$$

$$(n_{1}, ..., n_{s}) \neq (0, ..., 0)).$$

PROOF. Set $n_{i} = m_{i} (i = 1, ..., s)$ in (7). Then we have:

$$(\psi^{S}_{m_{1},...,m_{s};m_{1},...,m_{s}}; \eta^{(j)}_{0}) = q^{-3}[(q^{3} - q)/3 + ax + by + bz] = 1 \ or \ 0;$$

$$a = (q - (q - 1)/3)q, \ b = -(q - 1)q/3, \ j = 9, 10, 11.$$

Since the elements $k_{i}, t_{i} (i = 1, ..., s)$ obtained for $(k, t) \in Y$ are the same for $(t, k) \in Z$, proposition 2 implies that $x = (2q + 1)/3$ and $y = z = (1 - q)/3$. Hence, we have:

$$(8) \ (\psi^{S}_{m_{1},...,m_{s};m_{1},...,m_{s}}; \eta^{(j)}_{0}) = q^{-3}[(q^{3} - q)/3 + ax + by + bz] = 1; \ j = 9, 10, 11.$$

For all elements $m$ of $\{0, ..., p - 1\} \cap \rho K$, we have the following relations:

$$(k, t) \in X \Rightarrow (k, mt) \in Y,$$

$$(k, t) \in Y \Rightarrow (k, mt) \in Z,$$

$$(k, t) \in Z \Rightarrow (k, mt) \in X.$$

Putting $\beta = m_{1}k_{1} + ... + m_{s}k_{s} + mm_{1}t_{1} + ... + mm_{s}t_{s}$ and

$$\beta' = m_{1}k_{1} + ... + m_{s}k_{s} + m_{1}t'_{1} + ... + m_{s}t'_{s}; \ t_{i}' = mt_{i}; \ i = 1, ..., s,$$

$$t' = t'_{1} + t'_{2} + ... + t'_{s} \theta^{s-1}.$$

We obtain:

$$\sum_{(k, t) \in X} \varepsilon^{\beta} = \sum_{(k, t) \in Y} \varepsilon^{eta'} = y, \ \sum_{(k,t') \in Y} \varepsilon^{\beta'} = \sum_{(k,t') \in Z} \varepsilon^\beta = z, \ \sum_{(k,t') \in Z} \varepsilon^\beta = \sum_{(k,t') \in X} \varepsilon^\beta = x.$$

After putting $mm_{i} = n_{i} (i = 1, ..., s)$, we obtain:

$$(9) \ (\psi^{S}_{m_{1},...,m_{s};n_{1},...,n_{s}}; \eta^{(j)}_{2}) = q^{-3}[(q^{3} - q)/3 + by + bz + ax] = 1$$

$$(m_{i} = 0 \iff n_{i} = 0; (m_{i}, n_{i}) \in Y; \ i = 1, ..., s, j = 9, 10, 11).$$

We note that we pass from (8) to (9) with the permutation $(xzy)$ afforded by $\sum \varepsilon^\beta$. Moreover, for all elements $m$ of $\{0, ..., p - 1\} \cap p K$, we have:

$$(k, t) \in X \Rightarrow (mk, t) \in Z, \ (k, t) \in Y \Rightarrow (mk, t) \in X, \ (k, t) \in Z \Rightarrow (mk, t) \in Y.$$
It follows that $\sum \varepsilon_i^r; \gamma = mm_1k_1 + \ldots + mm_2k_s + m_1t_1 + \ldots + m_st_s$ affords the permutation $(xyz)$. Hence, we obtain:

\begin{equation}
(\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S; \eta_1^{(j)}) = 1
\end{equation}

\begin{equation}
(m_i = 0 \iff n_i = 0; (m_i, n_i) \in \mathbb{Z}; i = 1, \ldots, s; j = 9, 10, 11)
\end{equation}

For all elements $m$ of $\{0, \ldots, p - 1\} \cap \rho^2 \mathbb{K}$, we have:

\begin{equation}
(k, t) \in X \Rightarrow (mk, t) \in Y \text{ and } (k, mt) \in Z
\end{equation}

\begin{equation}
(k, t) \in Y \Rightarrow (mk, t) \in Z \text{ and } (k, mt) \in X
\end{equation}

\begin{equation}
(k, t) \in Z \Rightarrow (mk, t) \in X \text{ and } (k, mt) \in Y
\end{equation}

It follows directly that $\sum \varepsilon_i^r$ and $\sum \varepsilon_i^s$ afford the permutation $(xyz)$ and $(xyz)$ respectively. Hence, we obtain:

\begin{equation}
(\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S; \eta_1^{(j)}) = 1; (m_i, n_i) \in Y,
\end{equation}

\begin{equation}
(\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S; \eta_1^{(j)}) = 1; (m_i, n_i) \in Z,
\end{equation}

\begin{equation}
(m_i = 0 \iff n_i = 0; i = 1, \ldots, s; j = 9, 10, 11)
\end{equation}

In the same way, for all elements $m$ of $\{0, \ldots, p^{-1}\} \cap \mathbb{K}$, $\sum \varepsilon_i^s$ afford the identity permutation. Hence, we obtain:

\begin{equation}
(\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S; \eta_0^{(j)}) = 1
\end{equation}

\begin{equation}
(m_i = 0 \iff n_i = 0; (m_i, n_i) \in \mathbb{Z}; i = 1, \ldots, s; j = 9, 10, 11)
\end{equation}

This completes the proof of proposition.

Finally, by the tables 1 and 2, (9), (10), (11), (12) and the Frobenius theorem, we obtain the following relations:

\begin{equation}
\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S = \sum_{j=4}^{8} \sum_{r=1}^{k_j} \eta_r^{(j)} + \left(\frac{d - 1}{2}\right) \sum_{j=9}^{11} \eta_0^{(j)}; (m_i, n_i) \in \mathbb{X},
\end{equation}

\begin{equation}
\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S = \sum_{j=4}^{8} \sum_{r=1}^{k_j} \eta_r^{(j)} + \left(\frac{d - 1}{2}\right) \sum_{j=9}^{11} \eta_0^{(j)}; (m_i, n_i) \in \mathbb{Z},
\end{equation}

\begin{equation}
\psi_{m_1, \ldots, m_s; n_1, \ldots, n_s}^S = \sum_{j=4}^{8} \sum_{r=1}^{k_j} \eta_r^{(j)} + \left(\frac{d - 1}{2}\right) \sum_{j=9}^{11} \eta_0^{(j)}; (m_i, n_i) \in \mathbb{Y},
\end{equation}

\begin{equation}
(m_i, n_i = 0, \ldots, p - 1; m_i = 0 \iff n_i = 0; i = 1, \ldots, s; (m_1, \ldots, m_s) \neq (0, \ldots, 0),
\end{equation}

\begin{equation}
(n_1, \ldots, n_s) \neq (0, \ldots, 0)
\end{equation}

To summarize, using (3), (4), (6), (13) and property 2, we have established the following theorem:
THEOREM. If $e_r^{(j)}$ and $e_{m_1,\ldots,m_z;\ldots;0}$ are the orthogonal central idempotents afforded by the irreducible CSL$(3, q)$-character $\eta_r^{(j)}$ and the irreducible CH-character $\psi_{m_1,\ldots,m_z;\ldots;0}$ respectively, then the primitive idempotents of the group algebra CSL$(3, q)$ which correspond to $\eta_r^{(j)}$ are as follows:

For $j = 2, 3, 4, 9; e_r^{(j)} e_{m_1,\ldots,m_z;\ldots;0}, e_r^{(j)} e_{0,\ldots,0;\ldots;0}$

For $j = 4, 5, 6, 7, 8; e_r^{(j)} e_{m_1,\ldots,m_z;\ldots;0}$

$(m_i, n_i = 0, \ldots, p - 1; i = 1, \ldots, s; (m_1, \ldots, m_z) \neq (0, \ldots, 0), (n_1, \ldots, n_z) \neq (0, \ldots, 0))$

For $j = 9, 10, 11; e_0^{(j)} e_{m_1,\ldots,m_z;\ldots;0}, (m_i, n_i) \in X,$

$e_1^{(j)} e_{m_1,\ldots,m_z;\ldots;0}, (m_i, n_i) \in Z,$

$e_2^{(j)} e_{m_1,\ldots,m_z;\ldots;0}, (m_i, n_i) \in Y.$

$(m_i, n_i = 0, \ldots, p - 1; m_i = 0 \Leftrightarrow n_i = 0; i = 1, \ldots, s; (m_1, \ldots, m_z) \neq (0, \ldots, 0), (n_1, \ldots, n_z) \neq (0, \ldots, 0))$

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