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ON CONNECTED TRANSVERSALS IN INFINITE GROUPS

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1. Introduction.

The multiplication group and the inner mapping group of a loop are important tools when studying the structure of loops. These tools were introduced by Bruck [2] and he used them in order to investigate centrally nilpotent loops. A purely group theoretical characterization of multiplication groups was given in [6] and by using this characterization it was shown that if a finite loop Q has a cyclic inner mapping group then Q is an abelian group. In this paper we show that this result is also true in the case that Q is infinite and the multiplication group of Q is either locally finite or subgroup separable. The concept of connected transversals has an important role in the theory of multiplication groups. Therefore we show here how H-connected transversals influence the derived length of a group in the case that H is cyclic. We cover here the locally finite and the residually finite case. For the definitions and results in loop theory the reader should have a look at [2], [5] and [6]. In group theory the standard results can be found in [4]. Finally, we remark that $L_G(H)$ denotes the core of a subgroup H in G.

2. Preliminaries.

If G is a group, $H \leqq G$ and A and B are left transversals to H in G such that $[A, B] \leqq H$, then we say that A and B are H-connected in G. The importance of connected transversals is shown by the following

THEOREM 1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $L_G(H) = 1$ and H-connected transversals A and B satisfying $G = \langle A, B\rangle$.*

The following result is of purely group theoretical character.

THEOREM 2. *Let H be a cyclic subgroup of a finite group G. Then $G' \leqq H$ if and*

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only if there exists a pair A, B of H-connected transversals in G such that $G = \langle A, B \rangle$.

We still need the following two lemmas.

LEMMA 1. *Let $H \leqq G$ and A and B be H-connected transversals in G. Let $C \subseteq A \cup B$ and $K = \langle H, C \rangle$, then $C \subseteq L_G(K)$.*

LEMMA 2. *Let $H \leqq G$, $L_G(H) = 1$ and $[A, B] = 1$. If H is cyclic and $G = \langle A, B \rangle$, then $G = A = B$ is abelian.*

For the proofs the reader is advised to consult [6] (theorems 3.5 and 4.1, lemmas 2.5 and 3.4).

3. Infinite cases.

We first recall that a group is locally finite if every finitely generated subgroup is finite.

THEOREM 3. *Let G be locally finite, $H < G$, H cyclic, $L_G(H) = 1$ and $G = \langle A, B \rangle$, where A and B are H-connected transversals. Then G is abelian and $H = 1$.*

PROOF. We write $H = \langle h \rangle$ and since $G = \langle A, B \rangle$, it follows that $h = x_1^{n_1} \dots x_k^{n_k}$, where $x_i \in A \cup B$, $n_i = \pm 1$ and k is an integer. Let $a \in A$ and $b \in B$ be arbitrary and consider the subgroup

$$T = \langle a, b, x_1, \dots, x_k, H \rangle$$

Clearly, T is finite and we can write $T = A_1 H = B_1 H$, where $A_1 \subset A$ and $B_1 \subset B$ are H-connected transversals in T. Now $a \in A_1$, $b \in B_1$ and $x_i \in \langle A_1, B_1 \rangle$ ($i = 1, \dots, k$). Thus $T = \langle A_1, B_1 \rangle$ and by theorem 2, $T' \leqq H$. By Lemma 1, a and b are elements of $L_G(T)$. Now $L_G(T)' \leqq T' \leqq H$ and since $L_G(T)'$ is a characteristic subgroup of $L_G(T)$, we conclude that $L_G(T)'$ is normal in G, hence $L_G(T)' = 1$. But then $[a, b] = 1$ and since a and b were arbitrary, it follows that $[A, B] = 1$. Now it is clear by lemma 2, that G is abelian and $H = 1$. The proof is complete.

As is very often the case with locally finite groups the proof of theorem 3 relies heavily on the finite case. Another alternative is to use finite factor groups of G. For this purpose we need the following

DEFINITION. (a) Let K be a subgroup of G and let $x \in G - K$. Then x is *separable* from K if and only if there exists a normal subgroup N of finite index in G such that $xN \notin KN/N$ in G/N.

(b) Let K be a subgroup of G. Then G is called *K-separable* if and only if x is separable from K whenever $x \in G - K$.

(c) We say that G is *subgroup separable* if and only if G is K-separable for every finitely generated subgroup K of G.

THEOREM 4. *Let G be a group. $H < G$ cyclic and assume that A and B are H-connected transversals in G. If G is H-separable, $L_G(H) = 1$ and $G = \langle A, B\rangle$, then G is abelian and $H = 1$.*

PROOF. Assume that G is not abelian. Then there exists $x, y \in G$ such that $z = [x, y] \notin H$, since $L_G(H) = 1$. Now we have a normal subgroup N of G such that $\bar{G} = G/N$ is finite and $\bar{z} = [\bar{x}, \bar{y}] \notin \bar{H}$. But this clearly contradicts theorem 2 and our proof is complete.

As a direct consequence of theorems 1, 3 and 4 we get

COROLLARY. *If Q is a loop such that the multiplication group of Q is either locally finite or subgroup separable and the inner mapping group is cyclic, then Q is an abelian group.*

REMARK 1. For an interested reader we mention that free groups, polycyclic groups, free products of two free groups with a cyclic amalgamation and free products of two finitely generated nilpotent groups with a cyclic amalgamation are subgroup separable (these and some other examples can be found in [3] and [9]).

REMARK 2. In our corollary it would have been enough to consider groups which are π_c (if G is K-separable for all cyclic subgroups K of G then G is said to be π_c; for the details, see [7]).

REMARK 3. In [6] it was shown that if Q is any loop and the inner mapping group of Q is cyclic of prime power order, then Q is an abelian group.

4. Derived length.

If $H \leqq G$ and A and B are H-connected transversals in G, then it is clear that the structure of H has certain influence on the structure of G. In this section we study the case where H is cyclic. We start by introducing an old but interesting result by Ito ([4], Satz 4.4, p. 674–675).

LEMMA 3. *If $G = KL$, where K and L are abelian subgroups of G, then $G'' = 1$.*

THEOREM 5. *Let G be a locally finite group, $H < G$ and let A and B be H-connected transversals. If H is cyclic, then $G^{(3)} = 1$.*

PROOF. Assume first that $L_G(H) = 1$. If $G = \langle A, B\rangle$, then G is abelian by theorem 3. Let $G > T = \langle A, B\rangle$ and denote $E = T \cap H$. If $1 < K \leqq E$ and K is normal in T, then K is normal in G, hence $K = 1$. Thus $L_T(E) = 1$ and it is also clear that A and B are E-connected transversals in T. By theorem 3, T is abelian and $E = 1$. It follows that $T = A = B$ is an abelian group. Now $G = AH$ and by lemma 3, $G'' = 1$. If $L_G(H) > 1$, then $G'' \leqq L_G(H)$ and thus $G^{(3)} = 1$.

We say that a group G is residually finite if for every $1 \neq x \in G$ there exists a normal subgroup N_x with G/N_x finite and $x \notin N_x$. Now we show that theorem 5 holds for residually finite groups, too.

THEOREM 6. *Let G be a residually finite group, $H < G$ and let A and B be H-connected transversals. If H is cyclic, then $G^{(3)} = 1$.*

PROOF. Assume that G satisfies our conditions but $G^{(3)} \neq 1$. Thus there exists $g \in G^{(3)}$ such that $g \neq 1$. Now we have a normal subgroup N of G such that $\bar{G} = G/N$ is finite and $\bar{g} \neq \bar{1}$ but this clearly contradicts the result of theorem 5.

REMARK 4. Naturally all subgroup separable groups are residually finite. Also free products of residually finite groups are residually finite and the same is true for finitely generated cyclic extensions of free groups. It is also interesting that finitely generated subgroups of $\mathrm{GL}(n, F)$ (here $n \geqq 2$ and F is a field) are residually finite (for the details, see [1] and [8]).

Finally we state two problems.

PROBLEM 1. Is theorem 3 true if G is residually finite? (It is not difficult to see that the answer is positive if H is finite.)

PROBLEM 2. Is it true that if G is infinite and $H < G$ is cyclic with H-connected transversals A and B, then G can not have a free subgroup of rank two?

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