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SUMMATION OF FORMAL POWER SERIES THROUGH ITERATED LAPLACE INTEGRALS

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Introduction.

Very recently, the question of *summation* of formal solutions of meromorphic differential equations has attracted increasing attention: Based upon results of G. N. Watson and F. Nevanlinna, J. P. Ramis has defined and studied *k-summability* of formal power series (for the definition, see Section 2). In essence, this concept is equivalent to a representation of the *sum* f of such a (divergent) series $\hat{f}$ as a Laplace integral over a function having a locally convergent expansion which is formally obtained from $\hat{f}$ by termwise inversion of the Laplace integral. According to (earlier) results of J. Horn, W. J. Trjitzinsky, and H. L. Turrittin, this summation process applies to formal solutions of meromorphic differential equations obeying some restrictive assumption. In general, J. P. Ramis and the author have independently shown that formal solutions are a *(matrix) product* of terms which individually are k-summable (with k depending upon the factor). Since this factorization cannot be so easily achieved, this result, although theoretically of great importance, is not easily applied to explicit examples. This led J. Ecalle to defining a more powerful summation method which he called *multi-summability*. Essentially, his method differs from the previous one by factoring the Laplace transform L_k (of index $k > 0$) into a product

$$L_{k_1} \circ A_{k_1,k_2} \circ \ldots \circ A_{k_{p-1},k_p} \quad (k_1 > k_2 > \ldots > k_p = k)$$

of a Laplace operator L_{k_1} and so-called *acceleration operators* A_{k_{j-1},k_j}, and then studying the (larger) class of functions to which this iteration of operators can be applied. From general properties of multi-summability and the factorization of formal solutions into (individually) k-summable factor, it follows immediately that *all such formal solutions are multi-summable.*

A (slight) disadvantage of multi-summability is the fact that a class of integral operators (the acceleration operators) is used, which involve a kernel more

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complicated than the Laplace operator. The main goal of this paper is to present a variant of multi-summability which avoids using acceleration operators and instead involves an iteration of Laplace operators only. While Ecalle's method is more flexible in proving certain theoretical properties (compare Martinet-Ramis [5]), this new one can be more easily applied, or even defined, since it is inductive (in a certain sense explained in Section 4). Since we succeed in proving that both methods are equivalent, we may think of the new method to give a *better representation formula* for the sum of multi-summable series.

Recently, W. B. Jurkat [4] has discovered a much more direct way of summing a multi-summable formal power series. However, while our representation formula gives the sum in a relatively large sector, his may be seen to work only in a much smaller sector. Moreover, the theory of multi-summability (either in Ecalle's or our new way) shows that the *Stokes' phenomenon* is closed related to the singular behavior of finitely many analytic functions at (finitely many) singular points, and no such result appears to exist for Jurkat's summation method.

1. Domain and image of Laplace operators.

For convenience of the reader, we give a few definitions which (partially in slightly different form) are standard and/or have been introduced in [1]:

For arbitrary real d and positive reals α, r, let

$$S_{d,\alpha} = \{z \mid |z| > 0, d - \alpha/2 < \arg z < d + \alpha/2\},$$
$$S_{d,\alpha,r} = S_{d,\alpha} \cap \{z \mid |z| < r\};$$

both sectors on the Riemann surface of $\log z$, so that z^λ, for arbitrary complex λ, is a well-defined analytic function in every such sector. With d as above, $\beta \geqq 0$ and $0 < k \leqq \infty$, we write

$$f \in \mathscr{A}(d, \beta; k)$$

iff f is a function having the following two properties:

(a) There exist $r > 0$ and $\alpha > \beta$ such that f is analytic in $S_{d,\alpha,r}$.

(b) There exists a formal power series

$$\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$$

and constants $C, K > 0$ such that for every $N \geqq 0$ and every $z \in S_{d,\alpha,r}$

$$\left| f(z) - \sum_{n=0}^{N-1} f_n z^n \right| \leqq C K^N |z|^N \Gamma(1 + N/k). \tag{1.1}$$

It is well-known that $\hat{f}$ as in (b) corresponds uniquely to f, and we frequently write $\hat{f} = J(f)$.

Suppose that, for some $\tilde{k} > 0$, a function $f \in \mathscr{A}(d, \beta, k)$ additionally satisfies the following condition:

(c) The function f is analytic in $S_{d,\alpha}$ (for some $\alpha > \beta$), and for suitably large c_1, $c_2 > 0$ and every $z \in S_{d,\alpha}$

$$|f(z)| \leqq c_1 \exp\{c_2 |z|^{\tilde{k}}\}. \tag{1.2}$$

If this is so, we write

$$f \in \mathscr{A}(d, \beta; k, \tilde{k}),$$

and for convenience, we set

$$\mathscr{A}(d, \beta; k, \infty) = \mathscr{A}(d, \beta; k).$$

For arbitrary $f \in \mathscr{A}(d, \beta; k, \tilde{k})$ (and α as in (a)), let τ be such that $|\tau - d| < \alpha$ and define in accordance with [1] and Martinet-Ramis [5]

$$L_{\tilde{k}}(f)(z) = \int_0^{\infty(\tau)} f(u) \exp\{-(u/z)^{\tilde{k}}\}\, d(u^{\tilde{k}}) \tag{1.3}$$

(integrating along $\arg u = \tau$). The integral obviously converges absolutely for every z satisfying

$$\operatorname{Re}\{(e^{i\tau}/z)^{\tilde{k}}\} > c_2$$

(with c_2 as in (1.2)), defining an analytic function in z. A change of τ results in an analytic continuation of this function, and therefore $L_{\tilde{k}}(f)$ is (at least) analytic for $z \in S_{d,\gamma,\rho}$, with sufficiently small $\rho > 0$ and $\gamma = \gamma(\rho) > \beta + \pi/\tilde{k}$. Defining

$$\mathscr{L}_{\tilde{k}}(f) = z^{-\tilde{k}} L_{\tilde{k}}(f), \tag{1.4}$$

it follows from standard results upon Laplace transforms and their inverses that *the transformation* (1.4) *is a bijective map from* $\mathscr{A}(d, \beta; k, \tilde{k})$ *onto* $\mathscr{A}(d, \beta + \pi/\tilde{k}; k\tilde{k}/(k + \tilde{k}))$, *and if we set* $g = \mathscr{L}_{\tilde{k}}(f)$, *then*

$$J(g) = \sum_{n=0}^{\infty} f_n \Gamma(1 + n/\tilde{k}) z^n.$$

The inverse map to $\mathscr{L}_{\tilde{k}}$ will be denoted by $\mathscr{B}_{\tilde{k}}$, and $\hat{\mathscr{B}}_{\tilde{k}}$ will denote an operator applied to formal power series $\hat{f}$, namely if $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$, then

$$\hat{\mathscr{B}}_{\tilde{k}}(\hat{f}) = \sum_{n=0}^{\infty} f_n z^n / \Gamma(1 + n/\tilde{k}). \tag{1.5}$$

2. Summable power series.

Let a formal power series $\hat{f}$, a real number d, and some k with $0 < k \leqq \infty$ be arbitrarily given, and assume existence of $f \in \mathscr{A}(d, \alpha; k)$ (for some $\alpha > 0$) such that $\hat{f} = J(f)$. In case $k = \infty$, condition (b) of Section 1 is equivalent to f being single-valued and analytic at the origin, and $\hat{f}$ being its (convergent) power series expansion; hence f is uniquely determined by $\hat{f}$. Generally, for $\beta < \pi/k$, one always finds infinitely many $f \in \mathscr{A}(d, \alpha; k)$ with $\hat{f} = J(f)$, but for $\beta \geqq \pi/k$, Watson's Lemma again proves uniqueness of f (provided that any such f exists!). This leads to the following characterization of k-summability:

A formal power series $\hat{f}$ is said to be k-summable in direction d iff $f \in \mathscr{A}(d, \pi/k; k)$ exists with $\hat{f} = J(f)$. The function f then is uniquely associated with $\hat{f}$, and we call it the k-sum of $\hat{f}$ (or: the sum of $\hat{f}$) in direction d, and write

$$f = \textstyle\sum_{d;k}(\hat{f}).$$

For $k = \infty$, this simply means that $\hat{f}(z)$ converges to f for sufficiently small $|z|$, in which case we write $f = \sum(\hat{f})$.

The notion of k-summability was originally introduced by J. P. Ramis. His definition and the one above are equivalent, as one can see by taking $\tilde{k} = \infty$ in the following

PROPOSITION 1. *For arbitrary d, and $0 < k < \tilde{k} \leqq \infty$, let $\hat{f}$ be k-summable in direction d, and define*

$$\hat{g} = \hat{\mathscr{B}}_\kappa(\hat{f}), \tag{2.1}$$

with

$$1/\kappa = 1/k - 1/\tilde{k}. \tag{2.2}$$

Then $\hat{g}$ is $\tilde{k}$-summable in direction d, $\sum_{d;\tilde{k}}(\hat{g}) \in \mathscr{A}(d, \pi/\tilde{k}; \tilde{k}, \kappa)$, and

$$\textstyle\sum_{d;k}(\hat{f}) = \mathscr{L}_\kappa(\sum_{d;\tilde{k}}(\hat{g})). \tag{2.3}$$

PROOF. Let $f = \sum_{d;k}(\hat{f})$ $(\in \mathscr{A}(d, \pi/k; k))$, then according to Section 1, there is precisely one $g \in \mathscr{A}(d, \pi/\tilde{k}; \tilde{k}, \kappa)$ such that $f = \mathscr{L}_\kappa(g)$, and $J(g) = \hat{g}$. Obviously, $\mathscr{A}(d, \pi/\tilde{k}; \tilde{k}, \kappa) \subset \mathscr{A}(d, \pi/\tilde{k}; \tilde{k})$; hence $\hat{g}$ is $\tilde{k}$-summable in direction d, and $g = \sum_{d;\tilde{k}}(\hat{g})$.

In the following result, we restrict to $1/2 \leqq k < \tilde{k} \leqq \infty$, because in general our proof would give a decomposition (2.4) with $\hat{f}_1$, $\hat{f}_2$ being *series in roots of z*. The proof may be seen to work under slightly weaker assumptions, but we will not need this in our applications.

PROPOSITION 2. *For arbitrary d, and $1/2 \leqq k < \hat{k} \leqq \infty$, let $\hat{f}$ be so that $\hat{g}$ as in (2.1) is $\tilde{k}$-summable in direction d. Then*

$$\hat{f} = \hat{f}_1 + \hat{f}_2, \tag{2.4}$$

where $\hat{f}_2$ is k-summable in direction d, and $\hat{g}_1 = \hat{B}_\kappa(\hat{f}_1)$ converges for sufficiently small $|z|$.

PROOF. Let $g = \sum_{d;\tilde{k}}(\hat{g})$ $(\in \mathscr{A}(d, \pi/\tilde{k}; \tilde{k}))$. Then for some $r > 0$ and $\alpha > \pi/\hat{k}$, $g(z)$ is analytic in $S_{d,\alpha,r}$. Taking α and r a little smaller, we may assume that $g(z)$ even is analytic on the (positively oriented) boundary of $S_{d,\alpha,r}$ (except for the origin, where it has an asymptotic expansion). Let γ_1 resp. γ_2 be the circular section resp. the two straight-line segments of the boundary of $S_{d,\alpha,r}$. Then, according to Cauchy's Formula,

$$g(z) = g_1(z) + g_2(z),$$

$$g_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{g(u)}{u - z}\, du, \quad z \in S_{d,\alpha,r}.$$

Obviously, $g_1(z)$ is analytic for $|z| < r$, and we denote its power series expansion by $\hat{g}_1$. The difference $\hat{g}_2 = \hat{g} - \hat{g}_1$ then is $\tilde{k}$-summable in direction d, and

$$\begin{aligned}\textstyle\sum_{d;\tilde{k}}(\hat{g}_2) &= \textstyle\sum_{d;\tilde{k}}(\hat{g}) - \sum(\hat{g}_1) \\ &= g - g_1 = g_2.\end{aligned}$$

Obviously, $g_2 \in \mathscr{A}(d, \pi/\tilde{k}; \tilde{k}, \kappa)$, hence we conclude that $f_2 = \mathscr{L}_\kappa(g_2) \in \mathscr{A}(d, \pi/k; k)$. Therefore, $\hat{f}_2 = J(f_2)$ is k-summable in direction d. Defining $\hat{f}_1 = \hat{f} - \hat{f}_2$, we have

$$\hat{\mathscr{B}}_\kappa(\hat{f}_1) = \hat{g} - J(g_2) = \hat{g} - \hat{g}_2 = \hat{g}_1,$$

which completes the proof.

3. Multi-summability of formal power series.

For the definition of multi-summability, due to Ecalle, we refer the reader to [1], or *Martinet-Ramis* [5]. Here we will do with the following characterization of multi-summability (which is the main result of [1]):

Given an arbitrary real d and real numbers

$$k_1 > k_2 > \ldots > k_p \geqq 1/2 \quad (p \geqq 1),$$

a formal power series $\hat{f}$ is $(k_1, \ldots, k_p)$-summable in direction d iff

(3.1) $$\hat{f} = \hat{f}_1 + \ldots + \hat{f}_p,$$

where $\hat{f}_j$ is a formal power series being k_j-summable in direction d ($j = 1, \ldots, p$).

The restriction to $k_p \geqq 1/2$ is made in order to avoid $\hat{f}_j$ to become a power series in a root of z, but this is no loss in generality: For an arbitrary natural number q we see immediately from the above characterization of multi-summability and the definition of k_j-summability that $\hat{f}(z)$ *is* $(k_1, \ldots, k_p)$*-summable in direction d iff $\hat{f}(z^q)$ is $(qk_1, \ldots, qk_p)$-summable in direction d/q*, and this may serve as characterization of multi-summability in case of arbitrary reals $k_1 > k_2 > \ldots > k_p > 0$.

In case of $k_p \geqq 1/2$ and $\hat{f}$ being $(k_1, \ldots, k_p)$-summable in direction d, we define (in view of (3.1))

(3.2) $$\textstyle\sum_{d;k_1,\ldots,k_p}(\hat{f}) = \sum_{d;k_1}(\hat{f}_1) + \ldots + \sum_{d;k_p}(\hat{f}_p).$$

Despite of the fact that in (3.1) the series $\hat{f}_j$ are never uniquely associated with $\hat{f}$, one can show, using results on k_j-summability from *Martinet-Ramis* [5], that the right-hand side of (3.2) *is independent of the decomposition of $\hat{f}$* into a sum as in (3.1). Moreover, for every natural q

(3.3) $$\textstyle\sum_{d;k_1,\ldots,k_p}(\hat{f}(z)) = \sum_{d/q;qk_1,\ldots,qk_p}(\hat{f}(z^q)),$$

provided that $\hat{f}$ is $(k_1, \ldots, k_p)$-summable in direction d, and this extends the definition of $\sum_{d;k_1,\ldots,k_p}$ to arbitrary $k_1 > k_2 > \ldots > k_p > 0$.

4. Summability through iterated Laplace integrals.

Let $p \geqq 1$ and $\kappa_j > 0$, $1 \leqq j \leqq p$, be arbitrarily given. By *induction with respect to p*, we are going to define a summability method of formal power series through iterated Laplace integrals, which we will denote as $(\kappa_1, \ldots, \kappa_p)$-*iL*-summability. If $\hat{f}$ *is* $(\kappa_1, \ldots, \kappa_p)$-*iL*-summable in direction d, we write

$$f = \textstyle\sum^{d;\kappa_1,\ldots,\kappa_p}(\hat{f})$$

to denote its sum (in direction d). The definition can be phrased as follows:

(a) *For $p = 1$, a power series $\hat{f}$ is (κ_1)-iL-summable in direction d iff it is κ_1-summable in direction d, and*

$$\textstyle\sum^{d;\kappa_1}(\hat{f}) = \sum_{d;\kappa_1}(\hat{f}).$$

(b) *For $p \geqq 2$, a power series $\hat{f}$ is $(\kappa_1, \ldots, \kappa_p)$-iL-summable in direction d iff*

$$\hat{g} = \hat{\mathscr{B}}_{\kappa_p}(\hat{f})$$

is $(\kappa_1, \ldots, \kappa_{p-1})$-iL-summable in direction d, and in addition (for suitable $k > 0$)

$$\sum^{d;\kappa_1,\dots,\kappa_{p-1}}(\hat{g}) \in \mathscr{A}(d, 0; k, \kappa_p).$$

If this is so, we define

$$\sum^{d;\kappa_1,\dots,\kappa_p}\hat{f} = \mathscr{L}_{\kappa_p}\left(\sum^{d;\kappa_1,\dots,\kappa_{p-1}}(\hat{g})\right).$$

Suppose that $\hat{f}$ is $(\kappa_1, \dots, \kappa_p)$-$iL$-summable in direction d. Then the following statements are direct consequences of the definition and the properties of Laplace transforms listed in Section 1:

(i) Let $\hat{f}_p = \hat{f}, \hat{f}_{j-1} = \hat{\mathscr{B}}_{\kappa_j}(\hat{f}_j)$, $1 \leqq j \leqq p$. Then $\hat{f}_0$ converges for $|z|$ sufficiently small, while for $j = 1, \dots, p$, $\hat{f}_j$ is $(\kappa_1, \dots, \kappa_j)$-$iL$-summable in direction d.

(ii) With

$$f_0 = \sum(\hat{f}_0),$$
$$f_j = \sum^{d;\kappa_1,\dots,\kappa_j}(\hat{f}_j), \quad j = 1, \dots, p,$$
$$1/\tilde{k}_j = \sum_{\nu=1}^{j} 1/\kappa_\nu, \quad j = 1, \dots, p,$$
$$\tilde{k}_0 = \infty$$

we have

$$f_j \in \mathscr{A}(d, 0; \tilde{k}_j, \kappa_{j+1}), \quad j = 0, \dots, p-1,$$
$$f_j = \mathscr{L}_{\kappa_j}(f_{j-1}) \in \mathscr{A}(d, \pi/\kappa_j; \tilde{k}_j), \quad j = 1, \dots, p.$$

(iii) For any natural number q, the series $\hat{f}(z^q)$ is $(q\kappa_1, \dots, q\kappa_p)$-$iL$-summable in direction d/q, and

$$\sum^{d/q;q\kappa_1,\dots,q\kappa_p}(\hat{f}(z^q)) = \sum^{d;\kappa_1,\dots,\kappa_p}(\hat{f}).$$

From (i), (ii) we see why we call this summation method *summation through iterated Laplace integrals*: Obviously, $f = \sum^{d;\kappa_1,\dots,\kappa_p}(\hat{f}) = \mathscr{L}_{\kappa_p} \circ \dots \circ \mathscr{L}_{\kappa_1}(f_0)$ is an iterated Laplace integral over a function f_0 locally represented by a convergent power series $\hat{f}_0$, which is obtained from $\hat{f}$ as $\hat{f}_0 = \hat{\mathscr{B}}_{\kappa_1} \circ \dots \circ \hat{\mathscr{B}}_{\kappa_p}(\hat{f})$.

THEOREM 1. *Let $p \geqq 1$, d, and $\kappa_j > 0$, $1 \leqq j \leqq p$, be arbitrarily given, and define $k_1 > k_2 > \dots > k_p > 0$ by*

$$1/k_j = \sum_{\nu=p+1-j}^{p} 1/\kappa_\nu, \quad 1 \leqq j \leqq p. \tag{4.1}$$

Moreover, assume $k_p \geqq 1/2$ (i.e. $\sum_1^p 1/\kappa_\nu \leqq 2$). *Then a formal power series $\hat{f}$ is $(\kappa_1, \dots, \kappa_p)$-$iL$-summable in direction d, iff*

$$\hat{f} = \hat{f}_1 + \ldots + \hat{f}_p, \tag{4.2}$$

where each $\hat{f}_j$ is k_j-summable in direction d, $1 \leqq j \leqq p$.
Moreover,

$$\textstyle\sum^{d;\kappa_1,\ldots,\kappa_p}(\hat{f}) = \sum_{d;k_1}(\hat{f}_1) + \ldots + \sum_{d;k_p}(\hat{f}_p). \tag{4.3}$$

PROOF. We proceed by induction with respect to p: Obviously, Theorem 1 holds (trivially) for $p = 1$, hence from now on, let $p \geqq 2$.

(a) Suppose $\hat{f}$ is $(\kappa_1, \ldots, \kappa_p)$-$iL$-summable in direction d. Then by definition, $\hat{g} = \hat{\mathscr{B}}_{\kappa_p}(\hat{f})$ is $(\kappa_1, \ldots, \kappa_{p-1})$-$iL$-summable in direction d, hence by induction assumption

$$\hat{g} = \hat{g}'_1 + \ldots + \hat{g}'_{p-1},$$

where each $\hat{g}'_j$ is k'_j-summable in direction d, with

$$1/k'_j = \sum_{\nu=p-j}^{p-1} 1/\kappa_\nu = 1/k_{j+1} - 1/\kappa_p, \; 1 \leqq j \leqq p-1,$$

and

$$g = \textstyle\sum^{d;\kappa_1,\ldots,\kappa_{p-1}}(\hat{g}) = g'_1 + \ldots + g'_{p-1},$$
$$g'_j = \textstyle\sum_{d;k'_j}(\hat{g}'_j), \; 1 \leqq j \leqq p-1.$$

With $\hat{f}'_j$ so that $\hat{\mathscr{B}}_{\kappa_p}(\hat{f}'_j) = \hat{g}'_j$, we apply Proposition 2 (with $k = k_{j+1}$, $\tilde{k} = k'_j, \kappa = \kappa_p$) and find

$$\hat{f}'_j = \hat{f}_{j+1} + \hat{h}_j,$$

with $\hat{f}_{j+1}$ being k_{j+1}-summable in direction d, and $\hat{\mathscr{B}}_{\kappa_p}(\hat{h}_j)$ convergent for sufficiently small $|z|$. This implies

$$\hat{f} = \hat{f}'_1 + \ldots + \hat{f}'_{p-1}$$
$$= \hat{f}_2 + \ldots + \hat{f}_p + \hat{h}_1 + \ldots + \hat{h}_{p-1}.$$

Since $\hat{g}_0 := \hat{\mathscr{B}}_{\kappa_p}(\hat{h}_1 + \ldots + \hat{h}_{p-1})$ converges for sufficiently small $|z|$, and, according to Proposition 1, $\hat{g}_j = \hat{\mathscr{B}}_{\kappa_p}(\hat{f}_{j+1})$ is k'_j-summable in direction d $(1 \leqq j \leqq p)$, we find

$$\begin{aligned} \textstyle\sum(\hat{g}_0) = & \textstyle\sum^{d;\kappa_1,\ldots,\kappa_{p-1}}(\hat{g}) \\ & - \textstyle\sum_{d;k'_1}(\hat{g}_1) - \ldots - \sum_{d;k'_{p-1}}(\hat{g}_{p-1}) \\ & \in \mathscr{A}(d, 0; \infty, \kappa_p). \end{aligned} \tag{4.4}$$

This implies $\hat{f}_1 = \hat{h}_1 + \ldots + \hat{h}_{p-1}$ being k_1-summable in direction d, hence (4.2), and (4.3) follows from (4.4) and (2.3).

(b) Conversely, assume (4.2). Define $\hat{g}_{j-1} = \hat{\mathscr{B}}_{\kappa_p}(\hat{f}_j)$, $1 \leqq j \leqq p$, we conclude from Proposition 1 that $\hat{g}_j$ is k'_j-summable in direction d, with $1/k'_j = 1/k_{j+1} - 1/\kappa_p$, $0 \leqq j \leqq p-1$. Therefore,

$$\hat{g} = \hat{g}_0 + \ldots + \hat{g}_{p-1} = \hat{\mathscr{B}}_{\kappa_p}(\hat{f})$$

is, by induction assumption, $(\kappa_1, \ldots, \kappa_{p-1})$-$iL$-summable in direction d (also note that $\hat{g}_0$ converges for sufficiently small $|z|$, hence is k-summable in direction d for arbitrary $k > 0$). Moreover, from (2.3) we see

$$\sum\nolimits^{d;\kappa_1,\ldots,\kappa_{p-1}}(\hat{g}) = \sum_{j=0}^{p-1} \sum\nolimits_{d;k'_j}(\hat{g}_j)$$

$$\in \mathscr{A}(d, 0; k, \kappa_p)$$

(for sufficiently large $k > 0$), hence by definition $\hat{f}$ is $(\kappa_1, \ldots, \kappa_p)$-$iL$-summable in direction d.

THEOREM 2. *Let $p \geqq 1$, and $\kappa_j > 0$, $1 \leqq j \leqq p$, be arbitrarily given, and define $k_1 > k_2 > \ldots > k_p > 0$ by* (4.1). *Then a formal power series $\hat{f}$ is $(\kappa_1, \ldots, \kappa_p)$-$iL$-summable in direction d iff it is $(k_1, \ldots, k_p)$-summable in direction d, and*

$$\sum\nolimits^{d;\kappa_1,\ldots,\kappa_p}(\hat{f}) = \sum\nolimits_{d;k_1,\ldots,k_p}(\hat{f}). \tag{4.5}$$

PROOF. In view of (iii) (following from the definition of iL-summability) we may restrict to cases where in addition $k_p \geqq 1/2$ is satisfied. In such a situation, however, Theorem 1 and the above characterization of multi-summability complete the proof.

The following result shows that multi-summability also may be characterized in an inductive manner:

THEOREM 3. *Let $p \geqq 2$, d, and $k_1, \ldots, k_p > 0$ be given, and define*

$$1/k'_j = 1/k_{j+1} - 1/k_1, \quad 1 \leqq j \leqq p-1. \tag{4.6}$$

Then a formal power series $\hat{f}$ is $(k_1, \ldots, k_p)$-summable in direction d iff $\hat{g} = \hat{\mathscr{B}}_{k_1}(\hat{f})$ is $(k'_1, \ldots, k'_{p-1})$-summable in direction d and, in addition, for some $k > 0$

$$\sum\nolimits_{d;k'_1,\ldots,k'_{p-1}}(\hat{g}) \in \mathscr{A}(d, 0; k, k_1).$$

If this is so, then

$$\sum\nolimits_{d;k_1,\ldots,k_p}(\hat{f}) = \mathscr{L}_{k_1}\left(\sum\nolimits_{d;k'_1,\ldots,k'_{p-1}}(\hat{g})\right).$$

PROOF. Define $\kappa_1, \ldots, \kappa_p$ so that (4.1) holds, and use Theorem 2 together with the definition of iL-summability.

5. An example.

One can relatively easily find examples of k-summable series, and using Theorem 1, we will construct a series $\hat{f}(z)$ which is (1,1)-iL-summable, but not k-summable in any direction d, for arbitrary $k > 0$.

Using standard notations from the theory of special functions, let

$$\hat{f}_1(z) = \sum_0^\infty (1/2)_n z^n,$$

then

$$\hat{g}_1 = \hat{\mathscr{B}}_1 \hat{f} = F(1/2, 1, 1; z),$$

and from the definition of k-summability, we conclude that $\hat{f}_1$ is 1-summable. Similarly,

$$\hat{f}_2(z) = \sum_0^\infty \Gamma^2(1 + n) z^n$$

can be seen to be 1/2-summable, and consequently,

$$\hat{f}_1(z) + \hat{f}_2(z) = \hat{f}(z)$$

is $(1, 1)$-iL-summable. Its sum $f(z)$ can be represented as

$$f(z) = z^{-1} \int_0^{\infty(d)} h(u) \exp\{-u/z\}\, du,$$

$$h(u) = u^{-1} \int_0^{\infty(d)} g(w) \exp\{-w/u\}\, dw,$$

with

$$g(w) = {}_1F_1(1/2, 1; w) + (1 - w)^{-1}$$

(omitting the question of where the integrals represent the functions). Due to the fact that

$$\hat{\mathscr{B}}_{1/2} \hat{f} = e^{z/4} + F(1, 1, 1/2; z/4),$$

one can see easily that $\hat{f}$ is 1/2-summable along rays in the left halfplane, but not along such in the right. Taking

$$\hat{g}(z) = \hat{f}_1(z) + \hat{f}_1(-z) + \hat{f}_2(z),$$

one easily finds that $\hat{g}(z)$ is again $(1, 1)$-iL-summable, but not $1/2$-summable in any direction d (not even for $d = \pm \pi/2$, since summability along d always implies the same type of summability along neighboring rays). For $k > 1/2$, the coefficients of $\hat{g}$ grow too rapidly for k-summability. For $0 < k < 1/2$, if $\hat{f}$ were k-summable in direction d, it would also be $(1/2, k)$-summable in direction d, and the fact that $\hat{\mathscr{B}}_{1/2}\hat{g}$ converges can be seen to imply $1/2$-summability in direction d, contradicting the above argument.

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