MATH. SCAND. 70 (1992), 91–111

WEAK COMPACTNESS OF MULTIPLICATION OPERATORS ON SPACES OF BOUNDED LINEAR OPERATORS*

EERO SAKSMAN and HANS-OLAV TYLLI

Dedicated to Klaus Vala

0. Introduction.

Let E_i, $i = 1,2,3,4$, be Banach spaces and fix the bounded linear operators $A \in L(E_1, E_2)$ and $B \in L(E_3, E_4)$. The wedge product of A and B is the bounded linear composition operator $A \wedge B : L(E_2, E_3) \to L(E_1, E_4)$ defined through

$$X \mapsto BXA, \quad X \in L(E_2, E_3).$$

The compactness of the wedge operator was considered by K. Vala [V]. He showed, given non-zero operators A and B, that the wedge product $A \wedge B$ is a compact operator on $L(E_2, E_3)$ if and only if A and B are compact operators. The above simple multiplication operator has been studied in several contexts, in particular with a view to determining how the properties of A and B are reflected in the properties of $A \wedge B$, see [BMSW, 0.6, 0.7 and C*.1] for some aspects. In addition, various spectral properties of the so called elementary operators (i.e. finite sums of wedge operators) have been considered, initially by Lumer and Rosenblum in the 50's and continued in particular by Fialkow and his collaborators in a series of papers, cf. [F] and [CF] for additional references.

This paper addresses a very natural problem: when is the operator $A \wedge B$ weakly compact? That is, we consider the relative compactness of the image $\{BXA \mid X \in L(E_2, E_3), \|X\| \leqq 1\}$ of the unit ball in the weak topology of $L(E_1, E_4)$ as a Banach space. The only known explicit result in this direction is contained in [AW], where the Hilbert space case is settled: $A \wedge B$ is weakly compact on $L(l^2)$ if and only if A or B are compact operators on l^2. There is also an extension in [M] to the multiplication operator $x \mapsto bxa$ on C^*-algebras. Moreover, Diestel and Faires [DF] considered the closely related problem of the stability of weak

* This research was supported by the Academy of Finland.
Received January 28, 1991.

compactness under the ε-tensor product on spaces with the Dunford-Pettis property. Here we study the Banach algebra case $L(E)$, where E is an arbitrary Banach space over the real or complex field, although our methods are also applicable to the general case. It appears that the results strongly depend on the underlying Banach space E.

In section 2 we develop the required tools and under some restrictions on E general characterizations of the weak compactness of the wedge operator are given. Sufficient conditions for weak compactness are proved on arbitrary Banach spaces. In section 3 the criteria are applied and precise conditions are obtained for several concrete Banach spaces. These examples testify to the diversity of the requirements on A and B in order that the wedge product $A \wedge B$ be weakly compact. These conditions are influenced by several properties of the underlying Banach space E, such as reflexivity, the Radon-Nikodym property, the approximation property, the Dunford-Pettis property, as well as the structure of the closed ideals of $L(E)$.

We are grateful to G. Racher for drawing our attention to [DF].

1. Preliminaries.

We will use standard Banach space notation as of [LT2]. Recall here that $B_E = \{x \in E : \|x\| \leqq 1\}$ denotes the closed unit ball of the Banach space E. We identify E with its canonical image in E'' whenever there is no possibility of confusion. If E and F are Banach spaces, and $x' \in E'$, $y \in F$, then $x' \otimes y$ stands for the operator $E \to F$ given by $x \to \langle x', x \rangle y$.

Apart from $L(E, F)$, the bounded linear operators, and $K(E, F)$, the compact operators $E \to F$, we will use the components of some other ideals of operators between the Banach spaces E and F. The operator $S : E \to F$ is said to be weakly compact, and denoted $S \in W(E, F)$, if SB_E is relatively compact in the weak topology of F. The ideal of the strictly singular operators consists of the operators $R : E \to F$ such that the restriction $R|_M$ fails to be an isomorphism $M \to RM$ for any closed infinite-dimensional subspace $M \subset E$.

We briefly recollect some relevant Banach space properties. A Banach space E is said to possess the Radon-Nikodym property (abbreviated RNP), if for any probability space (Ω, Σ, μ) and for any μ-continuous vector measure $m : \Sigma \to \mathrm{E}$ having finite variation there is a Bochner integrable function $f : \Omega \to E$ such that

$$m(A) = \int_A f d\mu, \quad A \in \Sigma.$$

Separable dual spaces and reflexive spaces are among the spaces with the RNP while $L^1(0, 1)$ or spaces containing a copy of c_0 fail to have this property. The monograph [DU] is a convenient reference. A Banach space E has the approximation property (the AP), if for all compact subsets $D \subset E$ and all $\varepsilon > 0$ there is a finite-dimensional operator $R: E \to E$ satisfying

$$\sup_{x \in D} \|x - Rx\| < \varepsilon.$$

We refer to [LT2, 1.e] for a comprehensive discussion of this property. A Banach space E has the Dunford-Pettis property if any weakly compact operator $S: E \to F$ maps weakly compact sets $B \subset E$ to norm compact sets SB.

The projective tensor product $E \hat{\otimes}_\pi F$ of the Banach spaces E and F is the completion of $E \otimes F$ with respect to the π-norm, by definition $\pi(u) = \inf\{\sum_{k=1}^{n} \|x_k\| \|y_k\| : u = \sum_{k=1}^{n} x_k \otimes y_k\}$. The trace $\operatorname{tr}(v)$ of an element $v \in E' \hat{\otimes}_\pi E$ is introduced by $\operatorname{tr}(v) = \sum_{k=1}^{\infty} x_k'(x_k)$, where $v = \sum_{k=1}^{\infty} x_k' \otimes x_k$ is any tensor representation of v satisfying $\sum_{k=1}^{\infty} \|x_k'\| \|x_k\| < \infty$.

2. General results.

We commence by recording an elementary general observation that delimits our task. The proof is included for completeness.

PROPOSITION 2.1. *Let E be a Banach space and let $A, B \in L(E)$ be non-zero operators. Then $A \in W(E)$ and $B \in W(E)$ whenever $A \wedge B \in W(L(E))$.*

PROOF. If $x' \in E'$ and $z \in E$, then

$$(A \wedge B)x' \otimes z = A'x' \otimes Bz.$$

Observe that each pair of elements $u' \in E'$ and $w'' \in E''$ induces an evaluation functional $\delta_{(u', w'')}$ on $L(E)$ through

$$\langle \delta_{(u', w'')}, R \rangle = \langle R''w'', u' \rangle, \quad R \in L(E).$$

The set $\{A'x' \otimes Bz \mid x' \in B_{E'}, z \in B_E\}$ is relatively weakly compact in $L(E)$ by assumption. It is then easily seen from the Eberlein-Smulian theorem that $\{Bz \mid z \in B_E\}$ and $A'B_{E'}$, are relatively weakly compact in E and E', respectively, by evaluating with the functionals $\delta_{(w', u)}$ for $w' \in E'$ and $\delta_{(w', u'')}$ for $u'' \in E''$. In the first case pick $x' \in B_{E'}$ and $u \in E$ with $\langle A'x', u \rangle \neq 0$ and in the second case ensure that $\langle w', Bz \rangle \neq 0$. Hence A is a weakly compact operator according to Gantmacher's theorem [DS, VI.4.8].

The intractable nature of $L(E)'$ constitutes an obvious obstacle in connection with the weak compactness of composition operators on $L(E)$. However, it is possible to approach the problem in an important class of spaces via explicit representations of $K(E)'$ and $K(E)''$. We first state an efficient and well-known fundamental criterion for the weak compactness of operators.

LEMMA 2.2. [DS, VI.4.2] *Let E and F be Banach spaces. A linear operator $S \in L(E, F)$ is weakly compact if and only if $S''E'' \subset F$.*

We require some details of the duality theory of $K(E)$ given in [FS]. The first general results in this direction are due to Grothendieck. Assume E is a Banach space such that E' or E'' has the RNP. Consider the linear operator $V: E'\hat{\otimes}_\pi E'' \to K(E)'$,

$$\langle Vu, R\rangle = \mathrm{tr}(R''u),$$

whenever $u \in E'\hat{\otimes}_\pi E''$ and $R \in K(E)$. Then V is a metric surjection, that is $VB_{E'\hat{\otimes}_\pi E''} = B_{K(E)'}$, and $K(E)' \simeq E'\hat{\otimes}_\pi E''/\mathrm{Ker}\, V$ isometrically. Moreover, if E' has the AP then V is injective. For a proof of these claims see [FS, Theorem 1]. In particular, if E' or E'' has the RNP, then $K(E)''$ is identified with the w^*-closure of $\{R'' \mid R \in K(E)\}$ in $L(E'')$. Here the w^*-topology is taken with respect to the well-known duality $(E'\hat{\otimes}_\pi E'')' = L(E'')$, where

$$\langle R, u\rangle = \mathrm{tr}(Ru)$$

for $u \in E'\hat{\otimes}_\pi E''$ and $R \in L(E'')$. Under this identification the canonical embedding $K(E) \to K(E)''$ is the map $S \mapsto S''$. Note also that $K(E)'' = L(E'')$ whenever E' has in addition the AP and that in general this equality fails to hold if E' fails the AP. For these claims see [FS, 1.1 and Remark 3] and [GS, 1.1]. Observe that the arguments in [GS, 1.1 and 1.2] are independent of the scalar field despite the fact that the paper in question only considers real Banach spaces. Similar representations are valid for $K(E, F)'$, when either E'' or F' has the RNP.

The following theorem establishes a basic general characterization of the weak compactness of the wedge product (or of more general composition operators) on the above class of Banach spaces for which the duality theory of $K(E)$ is satisfactory. The basic idea to use bi-duality in this problem is found for instance in [AW], where it is applied in a slightly disguised manner to the C^*-algebra $L(l^2)$. The theorem reduces the question to the particular range-inclusion problems (2.1) and (2.4) below. In section 3 we determine the exact conditions on A and B that result out of this characterization on several concrete Banach spaces. Range-inclusion questions of analogous type were studied in [FoS] and [AF] for elementary operators on $L(l^2)$.

THEOREM 2.3. *Assume E is a Banach space such that E' or E'' has the Radon-Nikodym property and E' has the* AP. *Let $A, B \in L(E)$ be non-zero operators.*

Then the following conditions are equivalent

(i) $A \wedge B : L(E) \to L(E)$ *is weakly compact,*

(ii) $A, B \in W(E)$ *and*

$$\{B''XA'' \mid X \in L(E'')\} \subset K(E''), \tag{2.1}$$

(iii) $\{B''XA'' \mid X \in L(E'')\} \subset \{R'' \mid R \in K(E)\}$.

PROOF. Let $A \wedge B|_{K(E)} \in L(K(E))$ be the restriction of $A \wedge B$ to $K(E)$. We start by verifying that its second adjoint satisfies the equality

$$(A \wedge B|_{K(E)})'' = A'' \wedge B'' : L(E'') \to L(E'') \tag{2.2}$$

under these conditons on E. Let $\phi : L(E) \to L(E'')$ be the isometric embedding $\phi(S) = S''$, $S \in L(E)$. It is enough for (2.2) to establish that $A'' \wedge B''$ is w^*-w^* continuous on $L(E'')$, since $A'' \wedge B''|_{\phi K(E)} = \phi(A \wedge B|_{K(E)})$ and since $\phi K(E)$ is invariant under $A'' \wedge B''$. Indeed, this allows one to identify $(A \wedge B|_{K(E)})'' = A'' \wedge B''$ on account of the w^*-density of $\phi K(E)$ in $L(E'')$. On the other hand, the continuity of $A'' \wedge B'' : (L(E''), w^*) \to (L(E''), w^*)$ is a consequence of the duality

$$A'' \wedge B'' = (B' \hat{\otimes}_\pi A'')'. \tag{2.3}$$

Here $B' \hat{\otimes}_\pi A'' \in L(E' \hat{\otimes}_\pi E'')$ stands for the bounded linear extension of $(B' \otimes A'')x' \otimes y'' = B'x' \otimes A''y''$, $x' \otimes y'' \in E' \otimes E''$. In order to verify (2.3), let $x' \otimes y'' \in E' \hat{\otimes}_\pi E''$ be a simple tensor. One obtains for all $X \in L(E'')$ that

$$\begin{aligned}\langle (A'' \wedge B'')X, x' \otimes y'' \rangle &= \langle B''XA'', x' \otimes y'' \rangle = \langle B''XA''y'', x' \rangle \\ &= \langle XA''y'', B'x' \rangle,\end{aligned}$$

while also

$$\langle (B' \hat{\otimes}_\pi A'')'X, x' \otimes y'' \rangle = \langle X, B'x' \otimes A''y'' \rangle = \langle XA''y'', B'x' \rangle.$$

Next we proceed to establish the equivalence of the conditions. If $A \wedge B \in W(L(E))$, then $A, B \in W(E)$ according to Proposition 2.1. Clearly also the restriction $A \wedge B|_{K(E)} \in L(K(E))$ is weakly compact. From Lemma 2.2 and (2.2) one deduces that

$$(A'' \wedge B'')L(E'') = (A \wedge B|_{K(E)})''L(E'') \subset \{R'' \mid R \in K(E)\} \subset K(E'').$$

Assume that condition ii) is satisfied. If $X \in L(E'')$, then $B''XA'' \in K(E'')$ by assumption. Observe that $B''XA'' \in \{R'' \mid R \in K(E)\}$ if $B''XA''E \subset E$ and if $B''XA''$ is w^* continuous on E''. Indeed, this enables us to identify $B''XA'' = (V|_E)''$, where $V : E'' \to E$ is the astriction of the compact operator $B''XA''$. Clearly $B''XA''E \subset E$ according to Lemma 2.2 and the weak compactness of B. Let $W = \{x'' \in E'' : |\langle x'', x'_k \rangle| < \varepsilon, k = 1, \ldots, n\}$ be a basic w^*-neighborhood of 0 in E''

for given $x'_k \in E'$ and $\varepsilon > 0$. Then

$$(B''XA'')^{-1}W = \{x'' \in E'' : |\langle x'', A'''X'B'''x'_k\rangle| < \varepsilon, k = 1, \ldots, n\}$$

is a w^*-neighborhood of 0 since $A'''E''' \subset E'$ on account of the weak compactness of A'. This entails condition iii).

If condition iii) is satisfied, then Lemma 2.2 and (2.2) imply that the restriction $A \wedge B|_{K(E)}$ is weakly compact $K(E) \to K(E)$. Gantmacher's theorem [DS, VI.4.8] yields that its second adjoint $A'' \wedge B'' \in L(L(E''))$ is also weakly compact. Hence $(A \wedge B)B_{L(E)}$ is a relatively weakly compact subset of $L(E)$, since $\phi((A \wedge B)B_{L(E)}) \subset (A'' \wedge B'')B_{L(E'')}$ and since $(A'' \wedge B'')B_{L(E'')} \subset \phi K(E)$.

The conditions of the previous theorem are particularly simple on reflexive Banach spaces.

COROLLARY 2.4. *Let E be a reflexive Banach space with the* AP *and let $A, B \in L(E)$ be non-zero operators. Then the product $A \wedge B$ is weakly compact on $L(E)$ if and only if*

$$\{BXA \mid X \in L(E)\} \subset K(E). \tag{2.4}$$

PROOF. Reflexive spaces have the RNP [DU, p. 76] and a reflexive Banach space E has the AP if and only if E' also has the AP, cf. [LT2, 1.e.7].

REMARKS 2.5. i) In a similar manner one may prove a characterization for the weak compactness of $A \wedge B : L(E_2, E_3) \to L(E_1, E_4)$, if $A \in L(E_1, E_2)$, $B \in L(E_3, E_4)$ are non-zero operators. In order to utilize the duality theory of [FS], one requires the following assumptions on the spaces E_i, $i = 1, 2, 3, 4$:

- E_2'' or E_3' has the RNP, and E_1'' or E_4' has the RNP,
- E_2'' or E_3' has the AP.

Then $A \wedge B \in W(L(E_2, E_3), L(E_1, E_4))$ if and only if $A \in W(E_1, E_2)$, $B \in W(E_3, E_4)$ and

$$(A'' \wedge B'')L(E_2'', E_3'') \subset K(E_1'', E_4'').$$

The general case reveals phenomena that are not clear in the algebra setting. For instance, fix $1 < p < r < \infty$. Then $A \wedge B \in W(L(l^r, l^p), L(l^r, l^p))$ whenever $A \in L(l^r)$ and $B \in L(l^p)$. This is immediate from the above criteria and the fact that any operator $X : l^r \to l^p$ is compact [LT2, 2.c.3]. Alternatively, one may appeal to the fact that $L(l^r, l^p)$ is reflexive when $1 < p < r < \infty$, cf. [FS, Corollary 1.4].

ii) The proof of 2.3 yields the following criterion for the weak compactness of $A \wedge B|_{K(E)}$. Let E' or E'' have the RNP and let $A, B \in L(E)$ be non-zero operators. Then $A \wedge B \in W(K(E))$ if and only if $A, B \in W(E)$ and $(A'' \wedge B'')K(E)'' \subset K(E'')$. Analogous criteria may also be stated for the weak compactness of certain induced operators (for instance, $B'\hat{\otimes}_\pi A''$) between π- or ε-tensor products of Banach spaces, see also [DF].

iii) Let $I(E)$ be the components of a normed operator ideal I in the sense of Pietsch [P]. It is a general problem, which will not be pursued here, to determine when the induced operator $A \wedge B : I(E) \to I(E)$ is weakly compact.

The condition of RNP in Theorem 2.3 excludes concrete spaces such as $C(0,1)$ and $L^1(0,1)$, see [DU, p. 219]. Moreover, there exist subspaces $M \subset l^p$, $2 < p < \infty$, failing the AP, such that $K(M)'' \subsetneq L(M)$, see [FS, Remark 3]. Our next aim is to derive sufficient conditions for the weak compactness of the wedge product on arbitrary Banach spaces.

A useful representation of the compact operators was considered by Kalton [K]. Let E be a Banach space and set $K = B_{E'} \times B_{E''}$, which is a compact Hausdorff space when $B_{E'}$ and $B_{E''}$ are equipped with their respective w^*-topologies. Let $\chi : L(E) \to l^\infty(K)$ be the linear isometric embedding defined by $\chi(R)(y', x'') = \langle y', R''x'' \rangle$ for $(y', x'') \in K$ and $R \in L(E)$. The "if" part of the following useful observation is already contained in [K, Lemma 1], whereas the converse is crucial for our purpose.

LEMMA 2.6. *Let E be a Banach space and let $R \in L(E)$. Then $\chi(R) \in C((K, w^* \times w^*))$ if and only if $R \in K(E)$.*

PROOF. There remains to prove the "only if" part. Assume that $\chi(R)$ is $w^* \times w^*$-continuous on K and let L be the compact Hausdorff space $(B_{E''}, w^*)$. Consider for any $y' \in B_{E'}$ the continuous functions $f_{y'} : L \to \mathbb{K}$, $f_{y'}(x'') = \langle y', R''x'' \rangle$, $x'' \in B_{E''}$. Clearly $\{f_{y'} \mid y' \in B_{E'}\}$ is a bounded subset of $C(L)$; indeed $\|f_{y'}\|_\infty \leqq \|R\|$ for all $y' \in B_{E'}$. We claim that $\{f_{y'} \mid y' \in B_{E'}\}$ is an equicontinuous family in $C(L)$. Assume this is not the case. According to the linearity of R it is easily seen that the equicontinuity fails at 0. Let $\mathscr{U}$ be some neighborhood basis for the relative w^*-topology at 0 in $B_{E''}$. Consequently there is $\varepsilon > 0$ such that for any neighborhood $U \in \mathscr{U}$ there are elements $x_U'' \in U$ and $y_U' \in B_{E'}$ satisfying

$$|f_{y_U'}(x_U'')| = |\langle y_U', R''x_U'' \rangle| > \varepsilon.$$

Observe that $(x_U'')_{U \in \mathscr{U}}$ is a net (under reverse inclusion on $\mathscr{U}$) which converges w^* to 0. Moreover, the net $(y_U')_{U \in \mathscr{U}}$ has a w^*-convergent subnet $(y_\alpha')_{\alpha \in \Lambda}$ due to the compactness of $(B_{E'}, w^*)$. Let $y_\alpha' \xrightarrow{w^*} y'$ as $\alpha \in \Lambda$. It follows that $\lim_{\alpha \in \Lambda} \langle y_\alpha', R''x_\alpha'' \rangle = \langle y', 0 \rangle = 0$ because of the joint $w^* \times w^*$-continuity of $\chi(R)$. Clearly this contradicts the fact that $|\langle y_\alpha', R''x_\alpha'' \rangle| > \varepsilon$ for all $\alpha \in \Lambda$.

Consequently $\{f_{y'} \mid y' \in B_{E'}\}$ is a precompact subset of $C(L)$ according to the Ascoli-Arzela theorem. Let $\mu > 0$ be arbitrary. Then there is a finite set

$\{y'_1,\dots,y'_n\} \subset B_{E'}$ such that for any $y' \in B_{E'}$ one may pick $k \in \{1,\dots,n\}$ with the property that

$$\begin{aligned}\|R'y' - R'y'_k\| &= \sup_{x''\in B_{E''}} |\langle R'y' - R'y'_k, x''\rangle| = \sup_{x''\in B_{E''}} |\langle y' - y'_k, R''x''\rangle| \\ &= \|f_{y'} - f_{y'_k}\|_\infty \leqq \mu.\end{aligned}$$

This states that $R' : E' \to E'$ is a compact operator and hence so is R.

The previous representation yields a fundamental criterion for weak convergence in $K(E)$.

LEMMA 2.7. (See [K, Theorem 1].) *Let E be a Banach space and let $(K_n) \subset K(E)$ be a bounded sequence. Then $K_n \xrightarrow{w} K$ as $n \to \infty$, for some $K \in K(E)$, if and only if $\lim_{n\to\infty} \langle y', K''_n x''\rangle = \langle y', K''x''\rangle$ pointwise for all $(y', x'') \in B_{E'} \times B_{E''}$.*

The global strategy of our argument is to reduce the general case to the case of reflexive spaces via suitable factorizations. Hence we commence by establishing the requisite reflexive version, where the assumptions are tailored to the requirements of the reduction.

PROPOSITION 2.8. *Assume that E, G and H are Banach spaces with G and H reflexive. Let $A \in L(G,E)$ and $B \in L(E,H)$. Then $A \wedge B \in W(L(E), L(G,H))$ whenever $A \in K(G,E)$ or $B \in K(E,H)$.*

PROOF. The strategy is based on the criterion of Lemma 2.7. On account of the Eberlein-Smulian theorem and the fact that $K(G,H)$ is a closed subspace of $L(G,H)$ it is enough to find for each sequence $(X_n) \subset B_{L(E)}$ some subsequence (still denoted by (X_n)) such that the sequence of compact operators (BX_nA) converges weakly in $K(G,H)$ to a compact operator $R : G \to H$ as $n \to \infty$. General results are available which allow one to deduce the existence of w-Cauchy subsequences of $(BX_nA)_{n\in\mathbb{N}}$, see [FS, 1.3]. However, we are required to make this construction more explicit in order to see where the properties of A or B enter. We treat the two possible cases separately, since the respective choices are different.

Case 1. $A \in K(G,E)$. The compactness of BX_nA, $n \in \mathbb{N}$, implies that $M = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{Im}(BX_nA)}$ is a separable subspace of H. However, in case H is already separable we put $M = H$ and then the following construction is simpler. Let $U_n : G \to M$ be the astriction of BX_nA, $n \in \mathbb{N}$, and $I : M \to H$ the inclusion. Observe that according to elementary duality $M' \simeq H'/M^\perp$ is separable and reflexive, $I' = Q$ is the quotient map $H' \to H'/M^\perp$ while $A'X'_nB' = U'_nQ$, $n \in \mathbb{N}$. The separability of $H'/M^\perp$ provides us with a linearly independent sequence $(z'_k + M^\perp)_{k\in\mathbb{N}}$ such that $\|z'_k\| \le 2$, $k \in \mathbb{N}$, and with the corresponding "rational" linear hull L of

$\{z'_k + M^\perp \mid k \in \mathsf{N})$ norm-dense in $H'/M^\perp$. Clearly $\{U'_n(z'_k + M^\perp) \mid n \in \mathsf{N}, k \in \mathsf{N}\} \subset 2\|B\| A' B_{E'}$ is a relatively compact subset of G' due to the compactness of A'. Diagonalization furnishes us with a subsequence, still denoted by (U'_n), with the property that

$$\lim_{n\to\infty} U'_n(z'_k + M^\perp) = w'_k, \text{ for each } k \in \mathsf{N},$$

in the norm of G'. Define the linear mapping $\tilde{R}: L \to G'$ through $\tilde{R}(\sum_j r_j(z'_j + M^\perp)) = \sum_j r_j w'_j$. Observe that there is according to norm-density a bounded linear extension $\tilde{R}: H'/M^\perp \to G'$ such that

$$\lim_{n\to\infty} U'_n(x' + M^\perp) = \tilde{R}(x' + M^\perp) \text{ for all } x' \in H',$$

since $\|\tilde{R}u\| \leqq \|B\| \, \|u\|$ for any $u \in L$. Let $R = \tilde{R}Q \in L(H', G')$. Then the norm-limit $\lim_{n\to\infty} A'X'_n B'x' = Rx'$ for all $x' \in H'$ and in particular

$$(2.5) \qquad \lim_{n\to\infty} \langle x', BX_n Ay\rangle = \lim_{n\to\infty} \langle A'X'_n B'x', y\rangle = \langle Rx', y\rangle = \langle x', R'y\rangle$$

pointwise for $(x', y) \in B_{H'} \times B_G$.

We claim that the mapping $(x', y) \mapsto \langle Rx', y\rangle$ is $w \times w$-continuous on $B_{H'} \times B_G$. From the decomposition $\langle R(x'_\alpha - x'), y_\alpha - y\rangle = \langle Rx'_\alpha, y_\alpha\rangle - \langle Rx', y_\alpha\rangle - \langle Rx'_\alpha, y\rangle + \langle Rx', y\rangle$ it is easily seen that it suffices to verify continuity at $(0, 0)$. Let $((x'_\alpha, y_\alpha))_{\alpha \in \Lambda}$ be a net in $B_{H'} \times B_G$ such that $x'_\alpha \xrightarrow{w} 0$, $y_\alpha \xrightarrow{w} 0$ as $\alpha \in \Lambda$ and let $\varepsilon > 0$. According to the compactness of A' there is a finite set $K \subset G'$ satisfying

$$\{A'X'_n B'x'_\alpha \mid n \in \mathsf{N}, \alpha \in \Lambda\} \subset K + \frac{\varepsilon}{2} B_{G'}.$$

Observe that the norm-limit $Rx'_\alpha \in K + \frac{\varepsilon}{2} B_{G'}$ so that for each α we may choose $k_\alpha \in K$ with $\|Rx'_\alpha - k_\alpha\| \leqq \frac{\varepsilon}{2}$. Since $y_\alpha \xrightarrow{w} 0$ as $\alpha \in \Lambda$ there is $\alpha_0 \in \Lambda$ with the property that

$$\sup_{k \in K} |\langle k, y_\alpha\rangle| < \frac{\varepsilon}{2}$$

for all $\alpha \geqq \alpha_0$. This yields the estimate

$$|\langle Rx'_\alpha, y_\alpha\rangle| \leqq \|Rx'_\alpha - k_\alpha\| \, \|y_\alpha\| + |\langle k_\alpha, y_\alpha\rangle| < \varepsilon$$

whenever $\alpha \geqq \alpha_0$.

Consequently Lemmas 2.6 and 2.7 together with (2.5) entail that $BX_nA \xrightarrow{w} R' \in K(G,H)$ in $L(G,H)$ as $n \to \infty$. We use implicitly the elementary fact that the w- and the w^*-topologies coincide on the unit balls of reflexive spaces.

Case 2. $B \in K(E,G)$. We only outline the construction of the limit operator, since the rest of the argument is analogous to that of Case 1. From the compactness of B' one obtains that $N = \overline{\bigcup_{n \in \mathbb{N}} \operatorname{Im}(A'X_n'B')}$ is a separable subspace of G'. However, if G is itself separable we put $N = G'$. Elementary duality gives $N' \simeq G/{}^{\perp}N$, which is separable and reflexive. Let Q be the quotient map $G \to G/{}^{\perp}N$. There is for every $n \in \mathbb{N}$ a compact operator $U_n : G/{}^{\perp}N \to H$ such that $U_nQ = BX_nA$ while $Q'U_n' = A'X_n'B'$, where Q' is the inclusion $N \to G'$ and U_n' is the astriction $H' \to N$ of $A'X_n'B'$, $n \in \mathbb{N}$. Pick some linearly independent sequence $(x_k + {}^{\perp}N) \subset B_{G/{}^{\perp}N}$, $\|x_k\| \leqq 2$ for $k \in \mathbb{N}$, with the corresponding "rational" linear hull norm-dense in the separable space $G/{}^{\perp}N$. According to the compactness of B one obtains that $\{U_n(x_k + {}^{\perp}N) \mid k \in \mathbb{N}, n \in \mathbb{N}\} \subset 2\|A\|B(B_E)$ is a relatively compact set. Diagonalization and density give rise to a subsequence, still denoted by (U_n), and a bounded linear operator $\tilde{S} : G/{}^{\perp}N \to H$ satisfying $\lim_{n \to \infty} U_n(x + {}^{\perp}N) = \tilde{S}(x + {}^{\perp}N)$, $x \in G$. Upon setting $S = \tilde{S}Q$ one secures that $\lim_{n \to \infty} BX_nAx = Sx$ in the norm for all $x \in G$. One establishes the $w \times w$-continuity of the mapping $(x', y) \mapsto \langle x', Sy \rangle$ on $B_{H'} \times B_G$ from the compactness of B just as in Case 1. As before this means that $BX_nA \xrightarrow{w} S$, as $n \to \infty$, in $L(G,H)$ and we are done.

These preparations pave the way for announced result.

THEOREM 2.9. *Let E be an arbitrary Banach space and let $A, B \in L(E)$. Then $A \wedge B \in W(L(E))$ whenever ($A \in K(E)$ and $B \in W(E)$) or ($A \in W(E)$ and $B \in K(E)$).*

PROOF. The Davis-Figiel-Johnson-Pelczynski factorization theorem [DU, VIII.4.8] states that there exists relative to the weakly compact operator A a reflexive space G as well as operators $A_0 \in L(E,G)$, $I \in L(G,E)$ satisfying $A = IA_0$. Moreover, for any $\varepsilon > 0$ there is $n(\varepsilon) > 0$ such that $IB_G \subset n(\varepsilon)\overline{AB_E} + \varepsilon B_E$. In particular, this means that I is a compact operator whenever A is compact. We refer to [N, 2.1] for a discussion of the DFJP-factorization from this point of view. We factorize B by applying a "predual" factorization scheme due to Gonzales and Onieva. According to [GO, Proposition 1.1.] there is for each $B \in W(E)$ a reflexive space H as well as $Q \in L(E,H)$, $B_0 \in L(H,E)$ with the property that $B = B_0Q$. Moreover, we may ensure that $B' = Q'B_0'$ is the DFJP-factorization of B' through H'. In particular, if $B \in K(E)$ then Q' and also Q are compact operators by the above observation. Now the claim follows from the factorization

$$A \wedge B = (A_0 \wedge B_0)(I \wedge Q)$$

and Proposition 2.8.

Note that a similar sufficiency result can be established in the case of several spaces by making the required notational changes in the arguments of 2.8 and 2.9. The condition of the previous theorem is exact on the Hilbert space l^2 [AW, 2.3] as well on the non-reflexive James space J (see Proposition 3.8 below). We proceed to establish that the maximal condition of Proposition 2.1 is the correct one on several classical Banach spaces with the Dunford-Pettis property. There is thus a complete analogue of the compactness result of Vala in these cases. This was stated as a problem in the first version of the paper. It was solved by G. Racher who applied [DF, Theorem 4] and duality theory in his proof. However, we prefer to indicate a direct proof based on the criterion of Remark 2.5.i.

THEOREM 2.10. *Suppose that E is a Banach space such that E'' has the Dunford-Pettis property and E' has the approximation property. Then $A \wedge B \in W(L(E))$ whenever $A, B \in W(E)$.*

PROOF. Factorize the weakly compact operators A and B through reflexive spaces G and H as $A = A_1 A_0$, $B = B_1 B_0$ with $A_0 \in L(E, G), A_1 \in L(G, E)$, $B_0 \in L(E, H)$ and $B_1 \in L(H, E)$ [DU, VIII.4.8]. It suffices to verify that

$$\{B_0'' S A_0'' : S \in L(G, E'')\} \subset K(E'', H) \tag{2.6}$$

in order to see that $A_0 \wedge B_0 \in W(L(G, E), L(E, H))$, in view of Remark 2.5.i. One requires the fact that the reflexive spaces G and H' have the Radon-Nikodym property [DU, III.2.13] and that E' has the approximation property by assumption. Since $B_0'' \in W(E'', H)$ and since any $S \in L(G, E'')$ is weakly compact it follows from the Dunford-Pettis property of E'' that $B_0'' S$ is compact and hence (2.6) holds. Moreover, $\{S A_1 : S \in B_{L(E)}\} \subset c \cdot B_{L(G,E)}$ with $c = \|A_1\|$ and thus

$$\begin{aligned}\{BSA : S \in B_{L(E)}\} &= (\mathrm{id} \wedge B_1)\{(A_0 \wedge B_0) S A_1 : S \in B_{L(E)}\} \\ &\subset c \cdot (\mathrm{id} \wedge B_1)(A_0 \wedge B_0) B_{L(G,E)}\end{aligned}$$

is a relatively weakly compact subset of $L(E)$.

We refer to [LT1, II.5.b] for the definitions of the $\mathscr{L}^1$- and the $\mathscr{L}^\infty$-spaces. Recall that $L^1(0, 1)$ and $C(0, 1)'$ are among the $\mathscr{L}^1$-spaces while $C(0, 1)$, l^∞ and $L^\infty(0, 1)$ are examples of $\mathscr{L}^\infty$-spaces.

COROLLARY 2.11. *Suppose that E is a $\mathscr{L}^1$- or a $\mathscr{L}^\infty$-space and that $A, B \in L(E)$ are non-zero operators. Then*

$$A \wedge B \in W(L(E)) \text{ if and only if } A, B \in W(E).$$

PROOF. It is known that E' is a $\mathscr{L}^\infty$-space if E is a $\mathscr{L}^1$-space and that F' is a $\mathscr{L}^1$-space if F is a $\mathscr{L}^\infty$-space [LT1, II.5.7]. Moreover, any $\mathscr{L}^1$- or $\mathscr{L}^\infty$-space has the approximation property [LT1, II.5.9] as well as the Dunford-Pettis property [LT1, II.4.30 and II.5.7]. The claim follows from Theorem 2.10 and 2.1.

3. Concrete examples.

In this section we determine the precise conditions on $A, B \in L(E)$ for some concrete Banach spaces E in order that the wedge product $A \wedge B$ be a weakly compact operator. The main purpose of these examples is to point out how these conditions may vary within the limits allowed by Proposition 2.1 and Theorem 2.9. Most of the examples are drawn from the class of spaces where Theorem 2.3 provides an exact abstract characterization of weak compactness.

The subspaces of l^p are among the simplest Banach spaces from a structural point of view. In the reflexive cases the result is similar to that of Akemann and Wright [AW, 2.3] for $L(l^2)$. We commence by stating an operator theoretic version of block basis techniques from Banach space theory in order to be able to handle non-compact operators.

LEMMA 3.1. *Let E be a Banach space with a normalized Schauder basis (e_k). If $R \notin K(E)$, then there is a constant $\delta > 0$ such that for all $\varepsilon > 0$ there are block bases (x_k) and (y_k) with respect to (e_k) satisfying the following properties:*
(i) $\|x_k\| = 1$ *and* $\operatorname{dist}(Rx_k, [Rx_l : 1 \leqq l \leqq k-1]) \geqq \delta$, $k \in \mathbf{N}$,
(ii) $\sum_{k=1}^{\infty} \|Rx_k - y_k\| < \varepsilon$.

PROOF. See [T, 1.2.]

PROPOSITION 3.2. *Assume that $M \subset l^p$, $1 < p < \infty$, is a closed infinite-dimensional subspace and that A, $B \in L(M)$. Then $A \wedge B \in W(L(M))$ if and only if $A \in K(M)$ or $B \in K(M)$.*

PROOF. In view of Proposition 2.8 one has $A \wedge B \in W(L(M))$ whenever $A \in K(M)$ or $B \in K(M)$. Towards the converse, according to Remark 2.5.ii the restriction $A \wedge B|_{K(M)}$ is weakly compact on $K(M)$ if and only if

$$BXA \in K(M) \text{ for all } X \in K(M)''. \tag{3.1}$$

Here $K(M)'' = \overline{K(M)}^\tau$, the closure in $L(M)$ with respect to the topology τ of uniform convergence on compact sets in M, [GS, 1.2]. Hence it suffices on account of (3.1) to exhibit, for any given $A \notin K(M)$ and $B \notin K(M)$, an operator $X \in K(M)''$ such that $BXA \notin K(M)$. It is well known that an operator $R \in L(M)$ is compact if and only if

$$\inf\{\|Q_N R\| : N \subset M \text{ subspace, } \dim N < \infty\} = 0. \tag{3.2}$$

Here $Q_N : M \to M/N$ is the quotient map.

Consequently, if $A \notin K(M)$, then one finds inductively with the help of (3.2) a sequence (x_k) in B_M with the properties $\|x_k\| = 1$ and

$$\operatorname{dist}(Ax_k, [Ax_l : 1 \leqq l \leqq k-1]) > \delta/2 \tag{3.3}$$

for some $\delta > 0$ and for all $k \in \mathbf{N}$. We may assume due to the reflexivity of M that (x_k) is weakly convergent in M. Passing to $w_k = x_{2k+1} - x_{2k}$, $k \in \mathbf{N}$, one ensures that $w_k \xrightarrow{w} 0$ and $Aw_k \xrightarrow{w} 0$, as $k \to \infty$, while $\|Aw_k\| > \frac{1}{2}\delta$ from (3.3). Hence standard applications of the Bessaga-Pelczynski selection principle [LT2, 1.a.12], here considered for l^p, yield blocks (z_k) and (u_k) with respect to the standard coordinate basis in l^p as well as a subsequence of (w_k) (still denoted by (w_k) for simplicity) such that $\|z_k\| = 1$ and $\frac{1}{2}\delta \leqq \|u_k\| \leqq 2\|A\|$ for all $k \in \mathbf{N}$, while

$$\sum_{k=1}^{\infty} \|w_k - z_k\| < \tfrac{1}{8} \text{ and } \sum_{k=1}^{\infty} \|Aw_k - u_k\| < \tfrac{1}{8}.$$

Well-known perturbation results for Schauder bases (see [LT2, 1.a.9]) guarantee that the sequences (w_k) and (Aw_k) are equivalent to the unit vector basis of l^p and moreover that their closed linear spans $[w_k]$ and $[Aw_k]$ are complemented in M. This means that there are constants $c_i > 0$, $i = 1, 2, 3, 4$, such that

$$c_1\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} \leqq \left\|\sum_{k=1}^{\infty} \lambda_k w_k\right\| \leqq c_2\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p}, \tag{3.4}$$

and

$$c_3\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} \leqq \left\|\sum_{k=1}^{\infty} \lambda_k Aw_k\right\| \leqq c_4\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} \tag{3.5}$$

for all $(\lambda_k) \in l^p$. Clearly (3.4) together with (3.5) express the fact that the restriction $A|_N$, $N = [w_k]$, is an isomorphism $N \to AN$ and that $AN \sim l^p$. The assumption that $B \notin K(M)$ provides us similarly with a (complemented) copy R of l^p in M such that the restriction $B|_R$ is an isomorphism $R \to BR$. Let X_0 be an isomorphism $AN \to R$ and let $P : M \to AN$ be a projection. Denote $X = X_0 P \in L(M)$. Clearly $BXA \notin K(M)$ since the restriction $BXA|_N$ is an isomorphism.

It remains to verify that $X \in K(M)''$. Observe first the elementary fact that $K(M)'' = \overline{K(M)}^{\tau}$ is a closed 2-sided ideal of $L(M)$. Here τ denotes the topology of uniform convergence on compact sets. Since $AN \sim l^p$ has the approximation property there is a bounded net $(S_\alpha)_{\alpha\in\Lambda}$ of finite-dimensional operators on AN satisfying $S_\alpha \xrightarrow{\tau} \mathrm{id}_{AN}$ on AN as $\alpha \in \Lambda$. Clearly the net $(S_\alpha P)_{\alpha\in\Lambda}$ converges in τ to P on M and consequently both P and $X = X_0 P$ belong to $K(M)''$.

The following simple proposition exhibits cases where actually all weakly compact wedge operators $A \wedge B$ are compact. Recall that the Banach space E has the Schur property if all weakly convergent sequences (x_n) in E are norm-convergent. The standard example is l^1 [DU, p. 105].

PROPOSITION 3.3. *Let E be a Banach space such that $W(E) = K(E)$. If A, $B \in L(E)\setminus\{0\}$, then $A \wedge B \in W(L(E))$ if and only if $A \in K(E)$ and $B \in K(E)$. The equality $W(E) = K(E)$ holds for instance if E or E' has the Schur property or if $E \subset c_0$ is a closed subspace.*

PROOF. The first statement is obvious from Proposition 2.1 and the known compact case [V]. The equality $W(M) = K(M)$ lies somewhat deeper for subspaces $M \subset c_0$. Assume that $R \notin K(M)$. Lemma 3.1 provides us with a normalized basic sequence (x_n) in M such that (Rx_n) is also a basic sequence, and with the property that (x_n) and (Rx_n) are equivalent to blocks of the unit basis in c_0. Consequently $R|_{[x_k]}$ is an isomorphism onto $[Rx_k] \sim c_0$, and one has $R \notin W(M)$.

We mention in passing that it follows from the proof of Proposition 3.2 that $K(M)$ coincides with the ideals consisting of the strictly singular and the strictly cosingular operators whenever $M \subset l^p$. It is of interest to compare this to the fact that $K(l^p)$ is the unique proper closed ideal of $L(l^p)$ [P, 5.1], since there exists a subspace $M \subset l^p, 2 < p < \infty$, failing the approximation property and satisfying the strict inclusions $\overline{\mathscr{F}(M)} \subsetneq K(M) \subsetneq K(M)'' \subsetneq L(M)$ (see [FS, Remark 3]). Here $\mathscr{F}(M)$ denotes the finite-dimensional operators on M.

More complicated conditions are already seen on the direct sums $l^p \oplus l^r$, $1 < p < r < \infty$. Prior to this we record an elementary observation on the ideal property of the preimages of the weakly compact operators under the wedge product.

LEMMA 3.4. *Let E be a Banach space and fix the operators $A, B \in L(E)$. Then the sets*

$$\Lambda_{r,A} = \{U \in L(E) \mid A \wedge U \in W(L(E))\}$$

and

$$\Lambda_{l,B} = \{U \in L(E) \mid U \wedge B \in W(L(E))\}$$

are closed 2-sided ideals of $L(E)$.

PROOF. $\Lambda_{r,A}$ and $\Lambda_{l,B}$ are the preimages of the closed subspace $W(L(E))$ under the bounded linear operators $U \mapsto A \wedge U$ and $U \mapsto U \wedge B$, respectively, and hence norm-closed subspaces of $L(E)$. The ideal property is seen from straight-forward composition formulae such as

$$(AX) \wedge B = (X \wedge \mathrm{id})(A \wedge B), \text{ for } X \in L(E).$$

It will be convenient to represent bounded operators S on direct sums $E \oplus F$ as 2×2 operator matrices $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. Here the operators $S_{11} \in L(E)$, $S_{12} \in L(F, E)$ etc. are introduced by $S_{ij} = P_i S I_j$, for $i,j = 1,2$, where $P_1 : E \oplus F \to E$; $P_2 : \mathrm{E} \oplus F \to F$ are the natural projections and $I_1 : E \to E \oplus F$, $I_2 : F \to E \oplus F$ the corresponding embeddings. Consequently any closed ideal $I \subset L(E \oplus F)$ is uniquely determined by its norm-closed components

$$I_{ij} = \{S_{ij} : \exists S \in I, S_{ij} = P_i S I_j\}$$

for $i,j = 1,2$, which possess obvious properties.

The spaces $l^p \oplus l^r$, $1 < p < r < \infty$, admit the somewhat surprising property that the actual conditions for the weak compactness of $A \wedge B$ are non-symmetric in A and B.

PROPOSITION 3.5. *Let $1 < p < r < \infty$ and let $A, B \in L(l^p \oplus l^r)$. Then $A \wedge B \in W(L(l^p \oplus l^r))$ if and only if*

$$A \in K(l^p \oplus l^r) \text{ or } B \in K(l^p \oplus l^r) \text{ or} \tag{3.6}$$

$$\left(A \in \begin{pmatrix} K(l^p) & L(l^r, l^p) \\ L(l^p, l^r) & L(l^r) \end{pmatrix} \text{ and } B \in \begin{pmatrix} L(l^p) & L(l^r, l^p) \\ L(l^p, l^r) & K(l^r) \end{pmatrix}\right).$$

PROOF. Observe first that $I_1 = \begin{pmatrix} K(l^p) & L(l^r, l^p) \\ L(l^p, l^r) & L(l^r) \end{pmatrix}$ and $I_2 = \begin{pmatrix} L(l^p) & L(l^p, l^r) \\ L(l^p, l^r) & K(l^r) \end{pmatrix}$ are maximal proper closed ideals of $L(l^p \oplus l^r)$, since $K(l^s)$ is the unique proper closed ideal of $L(l^s)$ for $1 < s < \infty$ [P, 5.1].

The criterion of Corollary 2.4 states in this case that $A \wedge B \in W(L(l^p \oplus l^r))$ if and only if

$$BXA \in K(l^p \oplus l^r) \text{ for any } X \in L(l^p \oplus l^r). \tag{3.7}$$

Hence it remains to verify towards the "if" part that condition (3.7) is satisfied whenever $A \in I_1$ and $B \in I_2$. Recall that all operators $S : l^r \to l^p$ are compact since $1 < p < r < \infty$ [LT2, 2.c.3]. Consequently there is no loss of generality in assuming, by disregarding compact perturbations in (3.7), that A and B are of the respective forms

$$A = \begin{pmatrix} 0 & 0 \\ U & V \end{pmatrix} \in I_1 \text{ and } B = \begin{pmatrix} Z & 0 \\ Y & 0 \end{pmatrix} \in I_2,$$

and that $X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix} \in L(l^p \oplus l^r)$. A simple computation shows then that $BXA = 0$. This means that $A \wedge B \in W(L(l^p \oplus l^r))$ whenever (3.6) holds.

In order to establish the converse we claim that $A \wedge B \notin W(L(l^p \oplus l^r))$ whenever the pair (A, B) of operators fails to satisfy (3.6). A simple consideration shows that the possibilities for $A, B \in L(l^p \oplus l^r)$ are among the following.

i) $A \notin I_1$ and $B \in I_2 \backslash K(l^p \oplus l^r)$,

ii) $B \notin I_2$ and $A \in I_1 \backslash K(l^p \oplus l^r)$,

iii) $A \notin I_1$ and $B \notin I_2$.

The cases i) and ii) are handled with the help of Lemma 3.4 and a maximality argument.

Case i). According to the first part of the proof one has for each $B \in I_2 \backslash K(l^p \oplus l^r)$ that

$$I_1 \subset \mathcal{A}_{l,B} = \{A \in L(l^p \oplus l^r) \mid A \wedge B \in W(L(l^p \oplus l^r))\}.$$

Here $\mathcal{A}_{l,B}$ is a closed 2-sided ideal of $L(l^p \oplus l^r)$ (Lemma 3.4). In order to conclude that $\mathcal{A}_{l,B} = I_1$ for $B \in I_2 \backslash K(l^p \oplus l^r)$ it suffices to verify that $\mathcal{A}_{l,B} \neq L(l^p \oplus l^r)$, since I_1 is a maximal proper closed ideal of $L(l^p \oplus l^r)$. It is enough to note that $\mathrm{id}_{l^p \oplus l^r} \wedge B$ fails to be weakly compact on $L(l^p \oplus l^r)$. Indeed, $B = B\,\mathrm{id}_{l^p \oplus l^r} \notin K(l^p \oplus l^r)$ since $B \in I_2 \backslash K(l^p \oplus l^r)$. Then (3.7) implies that $A \wedge B$ is not weakly compact in this case.

Case ii) is handled similarly while the remaining case is verified directly.

Case iii). If $A \notin I_1$ and if $B \notin I_2$, then it is again assumed by neglecting compact perturbations in (3.7) that

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix},$$

where $A_{11} \notin K(l^p)$ and $B_{22} \notin K(l^r)$. Let $X = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \in L(l^p \oplus l^r)$ and observe that $BXA = \begin{pmatrix} 0 & 0 \\ B_{22}ZA_{11} & 0 \end{pmatrix}$.

In view of (3.7) it remains to exhibit some operator $Z \in L(l^p, l^r)$ such that $B_{22}ZA_{11} \notin K(l^p, l^r)$. As in the proof of Proposition 3.2 one finds from $A_{11} \notin K(l^p)$ a block basic sequence (x_k) with respect to the unit basis in l^p such that the restriction $A|_{[x_k]}$ is an isomorphism $[x_k] \to [Ax_k]$ and with $[Ax_k]$ complemented in l^p by a projection P. The assumption $B_{22} \notin K(l^r)$ yields similarly a copy N of l^r such that $B|_N : N \to BN$ is an isomorphism. Let $U : [Ax_k] \to l^p$, $V : N \to l^r$ be isomorphisms and denote by $I : l^p \to l^r$ the non-compact natural inclusion. Then the choice $Z = V^{-1}IU\,P \in L(l^p, l^r)$ is easily seen to provide us with non-compact products $B_{22}ZA_{11}$ and BXA. This ends the proof of the proposition.

PROBLEM 3.6. Determine the contents of the characterization in Corollary 2.4 in the case of the reflexive spaces $L^p(0,1)$, $1 < p < \infty$, $p \neq 2$. It is well-known that $l^p \oplus l^2$ is isomorphic to a complemented subspace of $L^p(0,1)$ and consequently similar behavior as in the previous proposition is already seen on this copy. Moreover, the structure of closed ideals in $L(L^p(0,1))$ is very complicated [P, 5.3.9].

We proceed to solve as examples the range-inclusion problem in Theorem 2.3 for some simple non-reflexive Banach spaces.

PROPOSITION 3.7. *Let p satisfy $1 < p < \infty$ and let $A, B \in L(l^p \oplus c_0)$ be non-zero operators. Then $A \wedge B \in W(L(l^p \oplus c_0))$ if and only if*

$$(3.8) \qquad (A \in W(l^p \oplus c_0) \text{ and } B \in K(l^p \oplus c_0)) \text{ or}$$

$$\left(A \in \begin{pmatrix} K(l^p) & L(c_0, l^p) \\ L(l^p, c_0) & K(c_0) \end{pmatrix} \text{ and } B \in W(l^p \oplus c_0)\right).$$

PROOF. We may assume, due to Proposition 2.1, that $A, B \in W(l^p \oplus c_0)$. Observe that $W(l^p \oplus c_0) = \begin{pmatrix} L(l^p) & L(c_0, l^p) \\ L(l^p, c_0) & K(c_0) \end{pmatrix}$ since $W(c_0) = K(c_0)$. Clearly $(l^p \oplus c_0)' = l^{p'} \oplus l^1$ has the RNP as well as the approximation property. Theorem 2.3 states that $A \wedge B \in W(L(l^p \oplus c_0))$ if and only if

$$(3.9) \qquad B''XA'' \in K(l^p \oplus l^\infty) \text{ for all } X \in L(l^p \oplus l^\infty).$$

Again there is no loss in neglecting compact perturbations and hence in writing

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix} \in L(l^p \oplus c_0)$$

since $L(c_0, l^p) = K(c_0, l^p)$ [LT2, 2.c.3]. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in L(l^p \oplus l^\infty)$. A computation yields the equality

$$(3.10) \qquad B''XA'' = \begin{pmatrix} B_1(X_{11}A_1 + X_{12}A_2'') & 0 \\ B_2''(X_{11}A_1 + X_{12}A_2'') & 0 \end{pmatrix}.$$

Here $X_{12}A_2''$ admits the factorization $l^p \xrightarrow{A_2''} l^\infty \xrightarrow{X_{12}} l^p$ through the space l^∞ having the Dunford-Pettis property. It is then obvious that the product $X_{12}A_2'' \in K(l^p)$ (see [LT1, II.4.k]), since both A_2'' and X_{12} are weakly compact operators. Consequently (3.10) implies that $B''XA'' \in K(l^p \oplus l^\infty)$ for all $x \in L(l^p \oplus {}^{\nu\infty})$ precisely when

(3.11) $\quad B_1X_{11}A_1 \in K(l^p)$ and $B_2''X_{11}A_1 \in K(l^p, l^\infty)$ for all $X_{11} \in L(l^p)$.

Clearly (3.11) holds for all $X_{11} \in L(l^p)$ whenever

(3.12) $\quad (B_1 \in K(l^p)$ and $B_2 \in K(l^p, c_0))$ or $A_1 \in K(l^p)$.

It remains to show the converse, that is, to exhibit $Z \in L(l^p)$ such that (3.11) and equivalently also (3.9) fail to hold whenever the weakly compact operators A and B on $l^p \oplus c_0$ do not satisfy the condition (3.12). Consequently we suppose that the pair $A = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix} \in L(l^p \oplus c_0)$ has the property that $(B_1 \notin K(l^p)$ or $B_2 \notin K(l^p, c_0))$ and $A_1 \notin K(l^p)$.

Case 1. If $A_1, B_1 \notin K(l^p)$, then one finds $Z \in L(l^p)$ such that the product B_1ZA_1 is non-compact as in the proof of Proposition 3.2.

Case 2. Suppose that $A_1 \notin K(l^p)$ and that $B_2 \notin K(l^p, c_0)$. As before one obtains a normalized block basis (x_n) of the unit basis in l^p such that the restriction $A_1|_{[x_k]}$ is an isomorphism $[x_k] \to [A_1x_k]$ and such that there is a projection $P: l^p \to [A_1x_k]$. Moreover, since $B_2 \notin K(l^p, c_0)$ an application of Lemma 3.1 yields a normalized block basic sequence (y_k) with respect to the unit basis of l^p and $\delta > 0$ such that $\|B_2y_k\| \geqq \delta$, $k \in \mathbf{N}$. Let $Z = X_0P$, where $X_0: [A_1x_k] \to [y_k]$ is the isomorphism mapping A_1x_k to y_k, $k \in \mathbf{N}$. Observe that $x_k \xrightarrow{w} 0$, $k \to \infty$, while $\|B_2''ZA_1x_k\| \geqq \delta$ for all $k \in \mathbf{N}$. It is then well known that $B_2''ZA_1$ cannot be compact.

As our final non-reflexive example we show that the minimal conditions of Theorem 2.9 for the weak compactness of the multiplication operator $A \wedge B$ are exactly realized on the standard quasi-reflexive James space J. Recall that the James space J consists of all sequences $x = (x_n)$ of scalars converging to 0 for which the norm

$$\|x\| = \sup \frac{1}{\sqrt{2}} \left(\sum_{k=1}^{m-1} |x_{p_k} - x_{p_{k+1}}|^2 + |x_{p_m} - x_{p_1}|^2 \right)^{1/2} < \infty.$$

The supremum is taken over all choices of $m \in \mathbf{N}$ and $p_1 < p_2 < \ldots < p_m$. It is known that J is of codimension 1 in J'', see [LT2, 1.d.2]. We will identify $J'' = J \oplus \mathbf{K}e$, where e is the constant sequence $(1, 1, 1, \ldots)$.

PROPOSITION 3.8. *Let $A, B \in L(J)$ be non-zero operators. Then $A \wedge B \in W(L(J))$ if and only if*

$$(A \in K(J) \text{ and } B \in W(J)) \text{ or } (A \in W(J) \text{ and } B \in K(J)).$$

PROOF. Operators $S \in L(J'')$ are written in the form $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ with $S_{11} \in L(J)$ and where S_{12}, S_{21} and S_{22} are suitable, at most 1-dimensional operators. We may restrict attention to operators A, $B \in W(J)$ because of Proposition 2.1. Hence $A'' = \begin{pmatrix} A & A_0 \\ 0 & 0 \end{pmatrix}$, $B'' = \begin{pmatrix} B & B_0 \\ 0 & 0 \end{pmatrix}$, since $A''J'' \subset J$ (Lemma 2.2). Observe that J' has the RNP as a separable dual [DU, III.3.1]. Theorem 2.3 states, after discarding finite-dimensional perturbations, that $A \wedge B$ is weakly compact on $L(J)$ if and only if

$$BS_{11}A \in K(J) \text{ for all } S_{11} \in L(J). \tag{3.13}$$

Clearly we are left with the task of verifying that (3.13) fails to hold whenever $A \notin K(J)$ and $B \notin K(J)$.

Let (e_k) be the standard unit vector basis of J. Assume first that $A \notin K(J)$. Lemma 3.1 provides us with a constant $\delta > 0$ and for all preassigned $\mu > 0$ with block basic sequences (x_k) and (z_k) with respect to (e_k) such that the following properties are satisfied:

(i) $\|x_k\| = 1$ and $\|Ax_k\| \geqq \delta$, $k \in \mathbf{N}$,

(ii) $\sum_{k=1}^{\infty} \|Ax_k - z_k\| < \mu$.

In the inductive constructions of (x_k) and (z_k) one may easily ensure, by starting the next block further out if necessary, that the supports of (x_k) and (z_k) are separated in the sense that

$$x_k = \sum_{j=p_k}^{q_k} \alpha_j e_j,$$

where $p_{k+1} - q_k \geqq 2$ for all $k \in \mathbf{N}$ (and similarly for (z_k)). This implies, according to [HW, Lemma 1] that the basic sequences (x_k) and (z_k) are equivalent to the unit vector basis of l^2. A standard perturbation argument for Schauder bases [LT2, 1.a.9] ensures that (Ax_k) is also a basic sequence equivalent to the unit basis in l^2 once μ is chosen small enough. Deduce as in the proof of Proposition 3.2 that the restriction $U = A|_{[x_k]}$ determines an isomorphism $[x_k] \to [Ax_k]$. The application of [CLL, Corollary 11] yields an infinite-dimensional subspace $M \subset [Ax_k]$ which is complemented in J by a projection $P: J \to M$. In a similar manner, from $B \notin K(J)$ one obtains a copy N of l^2 in J where the restriction $B|_N$ is an isomorphism onto BN. Finally, let V be some isomorphism $M \to N$ and let $S_{11} = VP$. Clearly the restriction $BS_{11}A|_{V^{-1}M}$ is bounded below. Consequently (3.13) fails to hold with this choice. This completes the proof.

REMARK 3.9. Observe that $K(J) \subsetneq W(J)$, since J contains complemented copies of l^2, see [CLL]. The above argument actually shows that if R is a non-compact operator on J, then R is an isomorphism between suitable (complemented) copies of l^2. Hence $K(J)$ coincides with the strictly singular and the strictly cosingular operators on J.

As our final topic we briefly review the relevance of our results on the Banach algebra $L(E)$ for some attempts to define "weakly compact" elements of arbitrary Banach algebras. Since this problem is not of central interest from the point of view of this paper we refer to [BMSW, R.5 and C^*.1] for a comprehensive discussion and for further references.

Let A be a Banach algebra. An element $a \in A$ is said to be weakly completely continuous [O], if both the left and the right multiplications $x \mapsto ax$ and $x \mapsto xa$ are weakly compact operators $A \to A$. This concept is a trivial one in the Banach algebra $L(E)$ for non-reflexive Banach spaces E because of Proposition 2.1. The closed ideal consisting of the weakly completely continuous elements of $L(E)$ coincides with the set of compact operators on the closed subspaces $E \subset l^p$ and on $E = l^p \oplus l^r$, $1 < p < r < \infty$, according to Propositions 3.2 and 3.5. It is unclear to us whether this always holds.

An element $a \in A$ is weakly semi-completely continuous if the multiplication operator $x \mapsto axa$ determines a weakly compact operator on A. This concept was considered in [TW] and it is the weakly compact analogue of the compact elements previously introduced by Vala and J. C. Alexander. If E is a Banach space we set

$$\mathrm{Wsc}(E) = \{A \in L(E) \mid A \text{ is a weakly semi-completely continuous element of } L(E)\}.$$

PROPOSITION 3.10. i) *If E is a Banach space, then*

$$K(E) \subset \mathrm{Wsc}(E) \subset W(E).$$

ii) $\mathrm{Wsc}(l^p) = K(l^p)$, $1 < p < \infty$, $\mathrm{Wsc}(J) = K(J)$.

iii) $\mathrm{Wsc}(l^p \oplus l^r) = \begin{pmatrix} K(l^p) & L(l^r, l^p) \\ L(l^p, l^r) & K(l^r) \end{pmatrix}$, $1 < p < r < \infty$.

iv) $\mathrm{Wsc}(E) = W(E)$ *if E is a $\mathscr{L}^1$- or a $\mathscr{L}^\infty$-space.*

PROOF. Part i) follows from [V, Theorem 3] and Proposition 2.1. Parts ii)–iv) are restatements of 3.2, 3.5, 3.8 and 2.11 for the special case A $\wedge$ A.

Ylinen [Y1, 3.1], [Y2, 3.1] (cf. also [AW, p. 146]) established the surprising fact that the set of weakly completely continuous elements of any C^*-algebra A coincides with the set of weakly semi-completely continuous elements of A. This equality fails in the natural class of Banach algebras of the form $L(E)$ on reflexive spaces E, since in part iii) of the above proposition $K(l^p \oplus l^r) \subsetneq \mathrm{Wsc}(l^p \oplus l^r)$.

REFERENCES

[AW] C. Akemann and S. Wright, *Compact actions on C^*-algebras*, Glasgow Math. J. 21 (1980), 143–149.

[AF] C. Apostol and L. A. Fialkow, *Structural properties of elementary operators*, Canad. J. Math. 38 (1986), 1485–1524.

[BMSW] B. A. Barnes, G. J. Murphy, M. R. F. Smyth and T. T. West, *Riesz and Fredholm theory in Banach algebras*, Pitman Research Notes in Mathematics, vol. 67, Pitman, 1982.

[CLL] P. G. Casazza, B.-L. Lin and R. H. Lohman, *On James' quasireflexive Banach space*, Proc. Amer. Math. Soc. 67 (1977), 265–271.

[CF] R. Curto and L. A. Fialkow, *The spectral picture of* $[L_A, R_B]$, J. Funct. Anal. 71 (1987), 371–392.

[DF] J. Diestel and B. Faires, *Remarks on the classical Banach operator ideals*, Proc. Amer. Math. Soc. 58 (1976), 189–196.

[DU] J. Diestel and J. J. Uhl, jr., *Vector Measures*, AMS Mathematical Surveys vol. 15, AMS, 1977.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators, vol.* I, Interscience, 1958.

[FS] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. 21 (1975), 38–49.

[F] L. A. Fialkow, *Essential spectra of elementary operators*, Trans. Amer. Math. Soc. 267 (1981), 157–174.

[FoS] C. K. Fong and A. R. Sourour, *On the operator identity* $\sum A_k X B_k \equiv 0$, Canad. J. Math. 31 (1979), 845–857.

[GS] G. Godefroy and P. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. 32 (1988). 672–695.

[GO] M. Gonzales and V. Onieva, *Factorization of linear operators and duality*, preprint, 1988.

[HW] R. Herman and R. Whitley, *An example concerning reflexivity*, Studia Math. 28 (1967), 289–294.

[K] N. J. Kalton, *Spaces of compact operators*, Math. Ann. 208 (1974). 267–278.

[LT1] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Lecture Notes in Mathematics vol. 338, Springer, 1973.

[LT2] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces* I. *Sequence Spaces*, Ergebnisse der Mathematik vol. 92, Springer, 1977.

[M] M. Mathieu, *Compact and weakly compact multiplications on* C^**-algebras*, Ann. Acad. Sci. Fenn. A I Math. 14 (1989), 57–62.

[N] R. D. Neidinger, *Factoring operators through hereditarily* l^p*-spaces*, Lecture Notes in Mathematics vol. 1166, Springer, 1985, 116–128.

[O] T. Ogasawara, *Finite dimensionality of certain Banach algebras*, J. Sci. Hiroshima Univ. 17(A) (1954), 359–364.

[P] A. Pietsch, *Operator Ideals*, North-Holland, 1980.

[TW] B. J. Tomiuk and P. K. Wong, *Weakly semi-completely continuous* A^**-algebras*, Illinois Math. J. 16 (1972), 653–662.

[T] H.-O. Tylli, *Lifting non-topological divisors of zero modulo the compact operators* (in preparation).

[V] K. Vala, *On compact sets of compact operators*, Ann. Acad. Sci. Fenn. A I Math. 351 (1964).

[Y1] K. Ylinen, *Weakly completely continuous elements of* C^**-algebras*, Proc. Amer. Math. Soc. 52 (1975), 323–326.

[Y2] K. Ylinen, *Dual* C^**-algebras, weakly semi-completely continuous elements, and the extreme rays of the positive cone*, Ann. Acad. Sci. Fenn. A I Math. 599 (1975).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
HALLITUSKATU 15
SF-00100 HELSINKI
FINLAND