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ANALYTICITY THEOREMS FOR PARAMETER-DEPENDENT CURRENTS

WANG XIAOQIN*

Abstract.

Plurisubharmonic functions of two groups of complex variables $(x_1,\ldots,x_n)$ and $(a_1,\ldots,a_m)$ are considered; their partial functions are defined by $f_a(x) = f(x,a)$. We discuss analyticity theorems for the level sets associated to Lelong numbers of the parameter-dependent currents $dd^c f_a$.

1. Introduction.

Let $D \subset \mathbf{C}^n$ be an open set and let PSH(D) denote the set of all plurisubharmonic functions in D. Given $f \in \mathrm{PSH}(D)$, the Lelong numbers $\nu(f,x)$, $x \in D$, are well defined. A fundamental result of Siu states that all superlevel sets

$$E_c(f) = \{x \in D; \nu(f,x) \geqq c\}, \quad c > 0,$$

are analytic subsets of D.

Now let $A \subset \mathbf{C}^m$ be open and $f \in \mathrm{PSH}(D \times A)$. Fix $a \in A$ and let us consider the partial function $f_a \in \mathrm{PSH}(D)$ defined by $f_a(x) = f(x,a)$, $x \in D$. In this case the superlevel sets

$$X_c(f) = \{(x,a) \in D \times A; \nu(f_a,x) \geqq c\}, \quad c > 0,$$

can be much more complicated; it may happen that they are not analytic. The main purpose of this paper is to study analyticity theorems for the superlevel sets $X_c(f)$.

Section 2 is a survey of known results. We shall describe the Lelong number, the directional (or refined) Lelong number, the generalized Lelong number, and the analyticity theorems that hold for them.

In section 3 we present the main tool used for establishing analyticity of superlevel sets: the Hörmander-Bombieri theorem.

In section 4 we shall state sufficient conditions under which all the level sets $X_c(f)$ are analytic subsets of $D \times A$ and also present examples that show that it may happen that these sets are not analytic.

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In section 5, finally, we compare various Lelong numbers. In particular the partial Lelong numbers appear as limiting cases of the directional Lelong numbers for functions which satisfy a condition which we call upper Hölder regularity.

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2. A survey of known results.

In this section we shall describe the Lelong number, the directional (or refined) Lelong number, the generalized Lelong number, and the analyticity theorems that hold for them.

Let $D \subset \mathbf{C}^n$ be open and $f \in \mathrm{PSH}(D)$. The Lelong number of f at a point $x \in D$, denoted by $\nu(f, x)$, is defined as the $(2n-2)$-dimensional density of the mass $\mu = \frac{1}{2\pi}\Delta f$ at the point x:

$$\nu(f, x) = \lim_{r \to 0} \mu(x + rB)/\lambda_{2n-2}(rB \cap \mathbf{C}^{n-1}), \tag{2.1}$$

where $x + rB$ is the ball of center x and radius r, and λ_{2n-2} is Lebesgue measure in $\mathbf{R}^{2n-2}$.

We shall use the following notation:

$$u(x, t) = \int_{|z|=1} f(x + ze^t);$$

$$U(x, t) = \sup_{|z|=1} f(x + ze^t),$$

where $x \in D$, $t \in \mathbf{R}$, assuming that $x + z \in D$ for all z such that $|z| \leqq e^t$;

$$\begin{aligned} v(x, y) &= \int_{|z_j|=e^{y_j}} f(x + z) \\ &= (2\pi)^{-n} \int_0^{2\pi} ds_1 \ldots \int_0^{2\pi} ds_n f(x_1 + e^{y_1 + is_1}, \ldots, x_n + e^{y_n + is_n}); \end{aligned}$$

$$V(x, y) = \sup_{|z_j|=e^{y_j}} f(x + z),$$

where $x \in D$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$ are such that $x + z \in D$ for all z with $|z_j| \leqq e^{y_j}$. The barred integral sign indicates mean value.

LEMMA 2.1. *Assume h is a negative harmonic function in the ball $|x| < r$ in $\mathbf{R}^m$. Then the following inequalities hold there*

$$h(0)\frac{1 - |x|^2/r^2}{(1 - |x|/r)^m} \leqq h(x) \leqq h(0)\frac{1 - |x|^2/r^2}{(1 + |x|/r)^m}. \tag{2.2}$$

By this lemma we can prove the following chain of equalities for the Lelong number.

PROPOSITION 2.2 (Kiselman [1987]). *Let $x \in D$, $f \in \mathrm{PSH}(D)$. Then we have*

$$\nu(f, x) = \lim_{t \to -\infty} \frac{u(x,t)}{t} = \lim_{t \to -\infty} \frac{U(x,t)}{t} = \lim_{t \to -\infty} \frac{v(x,ta)}{t} = \lim_{t \to -\infty} \frac{V(x,ta)}{t}, \tag{2.3}$$

where $a = (1, \ldots, 1) \in \mathbf{R}^n$.

PROOF. As a consequence of the maximum principle, $u(x,t)$, $U(x,t)$, $v(x,t)$, and $V(x,t)$ are increasing in t; by Hadamard's three-circle theorem, they are all convex functions of t. Therefore their slopes at $-\infty$ exist. Let $x \in D$ and $r < d(x, \partial D)$. Assume that f is sufficiently smooth; then we have, writing $r = e^t$,

$$\mu(x + rB) = \frac{1}{2\pi}\int_{x+rB} \Delta f = \frac{1}{2\pi}\int_{x+rS} \frac{\partial f}{\partial r}\, ds = \frac{1}{2\pi}\frac{\partial u}{\partial t}\frac{dt}{dr}\int_{rS} ds.$$

Using the fact that $dt/dr = 1/r$ and that

$$\int_{rS^{2n-1}} ds = r^{2n-1}\int_{S^{2n-1}} ds = 2\pi r^{2n-1}\int_{B^{2n-2}} d\lambda_{2n-2} = 2\pi r\int_{rB^{2n-2}} d\lambda_{2n-2},$$

we obtain the equality

$$\frac{\mu(x + rB)}{\lambda_{2n-2}(rB^{2n-2})} = \frac{\partial u}{\partial t}(x,t).$$

If f is not smooth, replacing $\partial u/\partial t$ by

$$\partial_t^+ u(x,t) = \lim_{\varepsilon \to 0+} (u(x, t + \varepsilon) - u(x,t))/\varepsilon,$$

we still have

$$\frac{\mu(x + r\bar{B})}{\lambda_{2n-2}(rB^{2n-2})} = \partial_t^+ u(x,t),$$

where again $r = e^t$, $t \in \mathbb{R}$. When $t \to -\infty$, we get

$$v(f, x) = \lim_{t \to -\infty} \frac{u(x, t)}{t}, \; x \in D.$$

This is the first equality of (2.3).

If f is subharmonic in a neighborhood of the ball $e^s \bar{B}$ in $\mathbb{C}^n$ we can consider its harmonic majorant h there, which satisfies $f(x) \leqq h(x)$ and

$$h(0) = \int_{z \in S} h(e^s z) = \int_{z \in S} f(e^s z) = u(0, s).$$

Therefore Lemma 2.1 shows that

$$U(0, t) = \sup_{e^t S} f \leqq \sup_{e^t S} h \leqq \frac{1 - e^{t-s}}{(1 + e^{t-s})^{2n-1}} u(0, s), \quad t < s,$$

provided only $f \leqq 0$ in $e^s B$. If we apply this inequality to the function $f - U(0, s)$ which is $\leqq 0$ in $e^s B$, we get, writing $U(t)$ instead of $U(0, t)$ for simplicity:

$$U(t) - U(s) \leqq \frac{1 - e^{t-s}}{(1 + e^{t-s})^{2n-1}} (u(s) - U(s)),$$

i.e.,

$$U(t) \leqq (1 - \lambda_{s-t}) U(s) + \lambda_{s-t} u(s), \quad t < s,$$

where λ_s is defined for $s > 0$ as

$$\lambda_s = \frac{1 - e^{-s}}{(1 + e^{-s})^{2n-1}}.$$

Taking $s = t + 1$, we get that

$$U(t) \leqq (1 - \lambda_1) U(t + 1) + \lambda_1 u(t + 1),$$

whence

$$\frac{U(t)}{t} \geqq (1 - \lambda_1) \frac{U(t + 1)}{t} + \lambda_1 \frac{u(t + 1)}{t}, \quad t < 0.$$

Letting t tend to $-\infty$ we see that

$$\lim_{t \to -\infty} \frac{u(x, t)}{t} \leqq \lim_{t \to -\infty} \frac{U(x, t)}{t}.$$

The other direction follows from $u \leqq U$.

Similarly we can prove

$$\lim_{t\to-\infty}\frac{v(x,ta)}{t}=\lim_{t\to-\infty}\frac{V(x,ta)}{t}.$$

We finally compare U and V. The maximum principle for plurisubharmonic functions implies that, if $a=(1,\ldots,1)$,

$$U(x,t)\leqq V(x,ta)\leqq U(x,t+\frac{1}{2}\log n).$$

This is because the ball of radius e^t is contained in the polydisk of radii e^t, which in turn is contained in the ball of radius $\sqrt{n}e^t$. If $t \ll 0$ we have

$$\frac{U(x,t)}{t}\geqq\frac{V(x,ta)}{t}\geqq\frac{U(x,t+\frac{1}{2}\log n)}{t+\frac{1}{2}\log n}\left(1+\frac{1}{2t}\log n\right),$$

hence

$$\lim_{t\to-\infty}\frac{U(x,t)}{t}=\lim_{t\to-\infty}\frac{V(x,ta)}{t}.$$

We also note the following results.

PROPOSITION 2.3. (a) *We have*

$$\{x\in D;\, v(f,x)>0\}\subset\{x\in D;\, f(x)=-\infty\}$$

and $v(f,x)\geqq 0$ *for all* $x\in D$.

(b) *We have*

$$\limsup_{w\to x} v(f,w)=v(f,x)$$

for all $x\in D$.

(c) *For* $L\in P_{n-1}(\mathbf{C})=\mathsf{P}_{n-1}$ *we also denote by* L *the corresponding line through the origin in* $\mathbf{C}^n$. *If* $x\in D$, *we let* $f|_{x+L}$ *be the restriction of* f *to the affine line* $x+L$. *Then for all* $x\in D$ *we have:*

$$\inf_{L\in\mathsf{P}_{n-1}} v(f|_{x+L},x)=v(f,x).$$

Furthermore, $v(f|_{x+L,x})=v(f,x)$ *for all* $L\in\mathsf{P}_{n-1}$, *except possibly for a locally pluripolar set of lines in* P_{n-1}. (Siu 1974)

(d) *If* $G:D\to D'$ *is a biholomorphism between two open sets* D *and* D' *in* $\mathbf{C}^n$ *and if* $f\in\mathrm{PSH}(D')$, *then* $v(G^*f,x)=v(f,G(x))$ *for all* $x\in D$. (Siu 1974)

PROOF. (a) is obvious from the definition of the Lelong number as the slope at minus infinity, and (b) follows easily from its interpretation as a density: the mean density in $x + rB$ is an increasing function of r.

We now prove (c). From the expression for ν in terms of U it follows that

$$\nu(f|_{x+L}, x) \geqq \nu(f, x). \tag{2.4}$$

On the other hand the expression for ν in terms of u gives

$$\int_{L \in \mathsf{P}_{n-1}} \nu(f|_{x+L}, x)\omega = \nu(f, x), \tag{2.5}$$

where ω is the volume element of the Fubini-Study metric of P_{n-1}. Property (c) follows from (2.4) and (2.5).

Using the mean value property of $f \in \mathrm{PSH}(D')$ and the fact that G is a holomorphic mapping we see that

$$\begin{aligned}\sup_{|z|=1} f(G(x + ze^t)) &= \sup_{|z|=1} f(G(x) + G'(x)ze^t + o(e^t)) \\ &\leqq \sup_{|z|=1} f(G(x) + ze^{t+M})\end{aligned}$$

for some constant M. So we have:

$$\begin{aligned}\nu(G^*f, x) &= \lim_{t\to-\infty} \frac{1}{t} \sup_{|z|=1} f(G(x + ze^t)) \geqq \lim_{t\to-\infty} \frac{1}{t} \sup_{|z|=1} f(G(x) + ze^{t+M}) \\ &= \lim_{s\to-\infty} \frac{s}{s-M}\frac{1}{s} \sup_{|z|=1} f(G(x) + ze^s) = \nu(f, G(x)),\end{aligned}$$

i.e., $\nu(G^*f, x) \geqq \nu(f, G(x))$. Applying this to the inverse of G, we see that

$$\nu(f, G(x)) \geqq \nu(G^*f, x).$$

So we have proved (d).

By using the Legendre transform of the convex function $t \mapsto u(x, t)$, Kiselman [1979] gave a very simple proof of the following fundamental analyticity theorem due to Siu [1974]. This proof is also in Hörmander [1990: Theorem 4.4.12].

THEOREM 2.4 (Siu [1974]). *For every constant $c > 0$, the set*

$$E_c(f) = \{x \in D; \nu(f, x) \geqq c\}$$

is an analytic subvariety of D of codimension $\geqq 1$.

Even though the Lelong number $\nu(f, x)$ gives information on the local structure of a plurisubharmonic function f, it is not enough to determine the Lelong number of a composition $f \circ h$ of f and a holomorphic mapping h. This was one of the reasons behind the introduction of the "refined Lelong number" in Kiselman [1987]. We shall use the following terminology, where the adjective "directional" refers to the vector y which determines the shape of the polydisks:

DEFINITION 2.5. *Assume $f \in \mathrm{PSH}(D)$, $x \in D$ and $y \in \mathbb{R}^n_+$. We define the directional Lelong number of a function f at the point x in the direction y, denoted by $\nu(f, x, y)$, as*

$$\nu(f, x, y) = \lim_{t \to -\infty} \frac{1}{t} \sup_{|z_j| = e^{y_j t}} f(x + z) = \lim_{t \to -\infty} \frac{V(x, ty)}{t}.$$

Using the proof of Proposition 2.2 we can deduce:

PROPOSITION 2.6. *We have the following equalities*

$$\nu(f, x, y) = \lim_{t \to -\infty} \frac{1}{t} \int_{|z_j| = e^{y_j t}} f(x + z) = \lim_{t \to -\infty} \frac{v(x, ty)}{t};$$

and

$$\nu(f, x, by) = b\nu(f, x, y).$$

where b is any positive number. Furthermore for $a = (1, 1, \ldots, 1) \in \mathbb{R}^n_+$, the directional Lelong number is the usual Lelong number:

$$\nu(f, x, a) = \nu(f, x).$$

For the directional Lelong number we have the following analogue of Siu's theorem.

THEOREM 2.7 (Kiselman [MS]). *Assume $f \in \mathrm{PSH}(D)$ and fix $y \in \mathbb{R}^n_+$. Then for every $c > 0$, the set*

$$E_c(f, y) = \{x \in D; \nu(f, y) \geqq c\}$$

is an analytic subset of D.

For the directional Lelong number Proposition 2.3 (d) is not true. The following example shows this.

EXAMPLE 2.8. Take $f(z_1, z_2) = \log(|z_1|^\alpha + |z_2|^\beta)$, $(z_1, z_2) \in \mathbb{C}^2$. We can calculate the directional Lelong number as

$$\nu(f, (0, 0), (y_1, y_2)) = \min(\alpha y_1, \beta y_2).$$

Let $F : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \mapsto (z_2, z_1)$. Then F is a biholomorphism of $\mathbb{C}^2$, but we have

$$\nu(F^* f, (0, 0), (y_1, y_2)) = \min(\beta y_1, \alpha y_2).$$

Demailly generalized the definition of the Lelong number and gave a very general and beautiful definition of the generalized Lelong number.

Let T be a positive, closed (p, p) current on D, and let φ be a plurisubharmonic function which is semi-exhaustive on D, i.e., there exists $R < 0$ such that the set $\{x \in D; \varphi(x) < R\}$ is relatively compact in D. We also assume that e^{φ} is continuous. For every real number $r < R$ let

$$\nu(T, \varphi, r) = \frac{1}{(2\pi)^{n-p}} \int_{\varphi < r} T \wedge (dd^c \max(\varphi, s))^{n-p},$$

where s is a constant $< r$. Using Stokes' formula and the fact that $dT = 0$ we deduce that $\nu(T, \varphi, r)$ is independent of the choice of s and that $r \mapsto \nu(T, \varphi, r)$ is increasing on $]-\infty, R[$. So $\lim_{r \to -\infty} \nu(T, \varphi, r)$ exists.

Definition 2.9. *The limit described above will be called the generalized Lelong number with respect to φ. We shall denote it by $\nu(T, \varphi)$. If T is of the special form $T = dd^c f$, we shall write $\nu(f, \varphi)$ for $\nu(dd^c f, \varphi)$.*

Proposition 2.10. *With the notation and assumptions above we have the following equalities:*

$$\nu(f(z), \log|z - x|) = \nu(f(z), x)$$

$$= \lim_{t \to -\infty} \frac{1}{(\pi e^{2t})^{n-1}} \int_{|z-x| < e^t} \frac{1}{2\pi} dd^c f(z) \wedge \left(\frac{1}{4} dd^c |z - x|^2\right)^{n-1};$$

and

$$\nu\left(f(z), \max_j \frac{1}{y_j} \log|z_j - x_j|\right) = \frac{1}{y_1 \cdots y_n} \nu(f(z), x, (y_1, \ldots, y_n))$$

for every $y = (y_1, \ldots, y_n) \in \mathbf{R}^n_+$.

We need the following definitions in order to describe the analogue of Siu's theorem for the generalized Lelong number proved by Demailly.

Let $D \subset \mathbf{C}^n$, $A \subset \mathbf{C}^m$ be two open sets and $\varphi: D \times A \to [-\infty, +\infty[$ a plurisubharmonic function.

Definition 2.11. *$\varphi(x, y)$ is called semi-exhaustive on D if for every compact subset K of A there exists a real number $R = R(K) < 0$ such that the set $\{(x, y) \in D \times K; \varphi(x, y) < R\}$ is relatively compact in $D \times A$.*

Definition 2.12. *A function g is called locally Hölder continuous with respect to A if for every compact subset $K \subset D \times A$ there exist constants M and $r > 0$ such that*

$$|g(x, y_1) - g(x, y_2)| \leqq M|y_1 - y_2|^r$$

holds for all $(x, y_1) \in K$, $(x, y_2) \in K$.

Now we are ready to state Demailly's theorem.

THEOREM 2.13 (Demailly [1987]). *Assume T is a positive, closed (p, p) current and $\varphi : D \times A \to [-\infty, +\infty[$ a plurisubharmonic function which is semi-exhaustive on D and such that* $\exp \varphi(x, y)$ *is continuous and locally Hölder continuous with respect to A. Write $\varphi_y(x) = \varphi(x, y)$. Then for every $c > 0$ the set*

$$E_c(T, \varphi_y) = \{y \in A; \nu(T, \varphi_y) \geqq c\}$$

is an analytic subset of A.

By Proposition 2.10 we know that Theorem 2.13 implies Theorems 2.4 and 2.7.

3. The Hörmander-Bombieri theorem.

The fundamental tool which will provide us with analytic sets is the Hörmander-Bombieri Theorem.

THEOREM 3.1 (Hörmander 1990: Theorem 4.4.4). *Let Ω be a pseudoconvex open set in $\mathbf{C}^n$, and let $\varphi \in \mathrm{PSH}(\Omega)$. For every $a \in \Omega$ such that $e^{-\varphi} \in L^2_{\mathrm{loc}}(a)$ there exists a holomorphic function $h \in \mathcal{O}(\Omega)$ such that $h(a) = 1$ and*

$$\int_\Omega |h|^2 e^{-2\varphi}(1 + |z|^2)^{-3n} d\lambda(z) < +\infty. \tag{3.1}$$

When applying this theorem it is useful to have the following sufficient condition for integrability of $e^{-\varphi}$.

THEOREM 3.2 (Hörmander 1990: Theorem 4.4.5). *If $\varphi \in \mathrm{PSH}(\Omega)$ has a finite value at a point $a \in \Omega$, then $e^{-\varphi} \in L^2_{\mathrm{loc}}(a)$.*

We write $\mathcal{O}(\Omega, \varphi)$ for the holomorphic functions in Ω which satisfy (3.1). Then the intersection

$$Z(\Omega, \varphi) = \bigcap_{h \in \mathcal{O}(\Omega, \varphi)} h^{-1}(0)$$

is an analytic subvariety of Ω. Theorem 3.1 says that

$$\{a \in \Omega; e^{-\varphi} \notin L^2_{\mathrm{loc}}(a)\} = Z(\Omega, \varphi),$$

and Theorem 3.2 that $Z(\Omega, \varphi)$ is contained in the polar set $\varphi^{-1}(-\infty)$ of φ.

One can prove easily using Jensen's inequality that $\nu_\varphi(a) \geqq n$ implies $e^{-\varphi} \notin L^2_{\text{loc}}(a)$, and so $a \in Z(\Omega, \varphi)$. In the other direction it is known that $\nu_\varphi(a) < 1$ implies $e^{-\varphi} \in L^2_{\text{loc}}(a)$; see Skoda [1972: Proposition 7.1.] This gives

$$E_{nc}(\varphi) \subset Z(\Omega, \varphi/c) \subset E_c(\varphi) \subset Z(\Omega, n\varphi/c),$$

so that the superlevel sets $E_c(\varphi)$ can be compared with the analytic varieties $Z(\Omega, c'\varphi)$ for various choices of the constants c and c'.

4. Analyticity theorems.

Let $D \subset \mathbf{C}^n$, $A \subset \mathbf{C}^k$ be two open sets. For a function f defined in $D \times A$ we shall write as before $f_a(x) = f(x, a)$ for its partial functions. Let us make the following definitions:

$$E_c(f) = \{(x, a) \in D \times A; \nu(f, (x, a)) \geqq c\};$$

$$X_c(f) = \{(x, a) \in D \times A; \nu(f_a, x) \geqq c\} = \bigcup_{a \in A} E_c(f_a) \times \{a\}.$$

From the discussion in section 2 we know that every $E_c(f)$ is an analytic subset of $D \times A$ and that every

$$E_c(f_a) = \{x \in D; \nu(f_a, x) \geqq c\}$$

is an analytic subvariety of D. So it is natural to ask whether the level sets $X_c(f)$ are analytic subsets of $D \times A$.

REMARK 4.1. The obvious inequality $\nu(f_a, x) \geqq \nu(f, (x, a))$ shows that

$$E_c(f) \subset X_c(f).$$

In this section we shall give conditions under which all level sets $X_c(f)$ are analytic subsets of $D \times A$ and also some examples that show that it may happen that these level sets are not analytic.

EXAMPLE 4.2. Let us consider a plurisubharmonic function of two variables (x, a)

$$f(x, a) = \sum \log(|a - a_k|^{\alpha_k} + |x - x_k|^{\beta_k}) \geqq \sum \alpha_k \log|a - a_k| = g(a),$$

where $|x_k|, |a_k| < 1$ and the α_k and β_k are positive. If $\sum \alpha_k$ is finite, the function g is locally integrable, both as a function of a and as a function of (x, a). The function f gives rise to a parameter-dependent current $T_a = d_x d_x^c f_a$ in the unit disk D in $\mathbf{C}$, but of course also to a current $T = dd^c f$ in the polydisk $D \times D$ in $\mathbf{C}^2$. For all a except for those in a set of measure zero (in fact of capacity zero) we have $g(a) > -\infty$, and for these a we must have $\nu(T_a, x) = 0$ for all x. We see that $\nu(T_{a_k}, x_k) \geqq \beta_k$, for

$$f(x, a_k) = \beta_k \log|x - x_k| + f_k(x),$$

where f_k denotes the sum of the terms of index $\neq k$ and thus a plurisubharmonic function. Now the β_k are arbitrary positive numbers, and if they are all larger than $c > 0$, the superlevel set of ν_{T_a} in the product space,

$$X_c(T) = \{(x, a); \nu(T_a, x) \geqq c\}$$

contains all points (x_k, a_k). On the other hand, $X_c(T)$ cannot contain any point (x, a) with $g(a) > -\infty$ as we have seen, so if we choose for instance $\{a_k\}$ to be dense in the unit disk and $x_k = 0$, it cannot be an analytic set. In the terminology to be introduced below, see Definition 4.5, f is not a Siu function in this case. With the superlevel set of T,

$$E_c(T) = \{(x, a); \nu(T, (x, a)) \geqq c\},$$

the situation is different, for we get only $\nu(T, (x_k, a_k)) \geqq \min(\alpha_k, \beta_k) \to 0$.

EXAMPLE 4.3. Let us now look at

$$f(x, a) = \sum \log(|a - a_k - x^{m_k}|^{\alpha_k} + |x|^{\beta_k}).$$

Now the points $(0, a_k)$ appear with weights (at least) $\min(m_k\alpha_k, \beta_k)$ in the current T_a. So if $\limsup m_k\alpha_k > 0$, Siu's theorem cannot hold. Note that in all cases we have a minorant

$$g(x, a) = \sum \alpha_k \log|a - a_k - x^{m_k}| \in L^1_{\mathrm{loc}}(D \times D)$$

where the choice of the exponents m_k now plays no role as far as integrability is concerned, for $\log|a - a_k - x^{m_k}|$ is comparable to $\log|a - a_k|$ from the point of view of functions in (x, a). At $x = 0$ we have

$$g(0, a) = \sum \alpha_k \log|a - a_k|$$

which is finite for almost all a; hence $\nu(T_a)(0) = 0$ for these a. The superlevel set

$$X_c(T) = \{(x, a); \nu(T_a, x) \geqq c\}$$

contains all points $(0, a_k)$ if $m_k\alpha_k \geqq c > 0$, but no point $(0, a)$ with $f(0, a) \geqq g(0, a) > -\infty$. Therefore it cannot be an analytic set. If we take for example $\alpha_k = 1/k^2, m_k = k$, the situation is very different from $\alpha_k = 1/k^2, m_k = k^2$.

EXAMPLE 4.4. Let $F: A \to D$ be an analytic mapping and $g \in \mathrm{PSH}(D)$. Let $f(x, a) = g(x - F(a))$ and $f_a(x) = f(x, a)$. Then for any $x \in D$, $a \in A$

$$\nu(f_a, x) = \nu(g, x - F(a)) = \nu(g, \varphi_{(x,a)}),$$

where $\varphi_{(x,a)}(z) = \log|z - x + F(a)|$. The first equality follows from Proposition 2.3 (d), the second from Proposition 2.10. Therefore we have the equality

$$X_c(f) = \{(x,a) \in D \times A; \nu(g, \varphi_{(x,a)}) \geqq c\}.$$

Hence we deduce from Theorem 2.13 that all $X_c(f)$ are analytic subsets of $D \times A$.

In view of the importance of Siu's theorem we make following definition.

DEFINITION 4.5. *A function $f \in \mathrm{PSH}(D \times A)$ will be called a Siu function if all its superlevel sets $X_c(f)$ are analytic subsets of $D \times A$.*

In order to simplify the statements we will give the following definition.

DEFINITION 4.6. *Let $f \in \mathrm{PSH}(D \times A)$. We shall say that f is upper Hölder regular with respect to A if for every compact subset $K \subset D \times A$ there exist constants M and $r > 0$ such that the inequality*

$$\sup_{|z|=1} f(x_0 + ze^t, a) \leqq M|a - a_0|^r + \sup_{|z|=1} f(x_0 + z(e^t + M|a - a_0|^r), a_0)$$

holds for all $(x_0, a), (x_0, a_0) \in K, t \ll 0$.

The following are some examples of such functions.

EAMPLE 4.7. Assume $f \in \mathrm{PSH}(D \times A)$ and $\exp f$ is locally Hölder continuous with respect to A. Then we can see easily that $\exp f$ is upper Hölder regular with respect to A.

Example 4.8. Let $g \in \mathrm{PSH}(D)$ and let $F: A \to D$ be an analytic mapping. Let $f(x,a) = g(x - F(a))$ as in Example 4.4. Then both f and $\exp f$ are upper Hölder regular with respect to A.

PROOF. Take a point $(x_0, a_0) \in D \times A$. Since F is analytic we see that

$$|F(a) - F(a_0)| \leqq M|a - a_0|$$

for every a near a_0. Therefore $g \in \mathrm{PSH}(D)$ gives the following estimates

$$\sup_{|z|=1} f(x + ze^t, a) = \sup_{|z|=1} g(x + ze^t - F(a))$$

$$\leqq \sup_{|z|=1} g(x - F(a_0) + z(e^t + |F(a) - F(a_0)|))$$

$$\leqq \sup_{|z|=1} g(x - F(a_0) + z(e^t + M|a - a_0|)) = \sup_{|z|=1} f(x + z(e^t + M|a - a_0|), a_0).$$

This implies the same estimate for e^f (the first term $M|a - a_0|^r$ in the estimate in Definition 4.6 is not needed here).

REMARK 4.9. It is clear that if the function f is upper Hölder regular with respect to A, then $\exp f$ is upper Hölder regular with respect to A. But the converse does not hold.

THEOREM 4.10. *Assume* $f \in \mathrm{PSH}(D \times A)$.
(a) *If* $\exp f$ *is upper Hölder regular with respect to* A, *then* f *is a Siu function.*
(b) *If* $\exp f$ *is locally upper Hölder semicontinuous with respect to* D, *and if* f *is a Siu function, then* $\{(x, a) \in D \times A; f(x, a) = -\infty\}$ *is an analytic set.*

PROOF. (a) It suffices to consider a relatively compact open subset ω of $D \times A$ and prove that f is a Siu function in ω. Then

$$\sup_{|z|=1} f(x + ze^t, a) \in \mathrm{PSH}(\omega)$$

for every fixed $t \ll 0$. Let us consider minus the Legendre transform of this function:

$$U_\tau(x, a) = \inf_t \Big(\sup_{|z|=1} f(x + ze^t, a) - \tau t; t < \min(0, \log d(x, \partial D))\Big).$$

The minimum principle for plurisubharmonic functions (Kiselman [1978], Theorem 2.2) tells us that $U_\tau \in \mathrm{PSH}(\omega)$ for all positive numbers τ.

Assume $(x_0, a_0) \in X_c(f)$, i.e. $\nu(f_{a_0}, x_0) \geqq c$. Assume also that $f(x_0, a_0) < 0$; this is no restriction of generality. By the chain of equalities (2.3) and the convexity of U, we can find a real number $t_0 < 0$ such that

$$\frac{1}{t} \sup_{|z|=1} f(x_0 + ze^t, a_0) \geqq c \tag{4.1}$$

holds for all $t < t_0$. Let τ be any number less than c. From the assumption that $\exp f$ is upper Hölder regular with respect to A it follows that

$$\sup_{|z|=1} f(x_0 + ze^t, a) \leqq \log\Big[M|a - a_0|^r + \exp \sup_{|z|=1} f(x_0 + z(e^t + M|a - a_0|^r), a_0)\Big], \tag{4.2}$$

holds if a is sufficiently close to a_0 and $t \ll 0$. We may assume without loss of generality that $(x_0, a_0) = (0, 0)$. In view of the upper Hölder regularity we can then estimate f as follows:

$$\sup_{|z|=1} f(x + ze^t, a) \leqq \sup_{|z|=1} f(Z(|x| + e^t), a) \leqq$$

$$\leqq \log\Big[M|a|^r + \exp \sup_{|z|=1} f(Z(|x| + e^t + M|a|^r), 0)\Big] \leqq$$

$$\leqq \log\big[M|a|^r + (|x| + e^t + M|a|^r)^c\big],$$

where the last inequality follows from (4.1). The inequalies hold for all $t \ll 0$. A good choice for t is

$$t = \log(|x| + M|a|^r + (M|a|^r)^{1/c})$$

for then

$$M|a|^r \leqq e^{ct} \quad \text{and} \quad |x| + M|a|^r \leqq e^t,$$

so that

$$\sup_{|z|=1} f(x + ze^t, a) \leqq \log[e^{ct} + (2e^t)^c] = \log(1 + 2^c) + ct = C + ct.$$

Therefore the transform U_τ can be estimated by

$$U_\tau(x, a) \leqq \sup_{|z|=1} f(x + ze^t, a) - \tau t \leqq C + (c - \tau)t.$$

For any constant $N > 0$ we have

$$\exp[-NU_\tau(x, a)] \geqq e^{-NC}(|x| + M^{1/c}|a|^{r/c})^{-N(c-\tau)}.$$

This shows that we can take $N = N_\tau$ so big that $\exp[-N_\tau U_\tau(x, a)]$ is not locally integrable near the point $(x_0, a_0) = (0, 0)$. Let

$$Z_\tau = \{(x, a) \in \omega; \exp(-N_\tau U_\tau) \text{ is not locally integrable near } (x, a)\}.$$

Theorem 3.1 shows that Z_τ is an analytic subset of ω. We have proved that

$$X_c(f) \subset \bigcap_{\tau < c} Z_\tau.$$

On the other hand, if $\nu(f_{a_0}, x_0) < c$, then for any τ, satisfying $\nu(f_{a_0}, x_0) < \tau < c$ we have $U_\tau(x_0, a_0) > -\infty$. By Theorem 3.2 we have $(x_0, a_0) \notin Z_\tau$. So finally

$$X_c(f) = \bigcap_{\tau < c} Z_\tau.$$

(b) if $\exp f$ is locally upper Hölder semicontinuous with respect to D we have

$$e^{f(x,a)} - e^{f(x_0,a)} \leqq M|x - x_0|^r$$

for every (x, a) in some neighborhood of (x_0, a_0). So for every (x_0, a_0) such that $f(x_0, a_0) = -\infty$ we have

$$e^{f(x,a_0)} \leqq M|x - x_0|^r,$$

which shows that

$$\nu(f_{a_0}, x_0) \geqq r.$$

Therefore

$$X_c(f) = \{(x, a) \in D \times A; f(x, a) = -\infty\}$$

for all c, $0 < c < r$. This shows that $\{(x, a) \in D \times A; f(x, a) = -\infty\}$ is analytic if f is a Siu function, and completes the proof.

Let us write $T_1 \geqq T_2$ if T_1 and T_2 are two currents such that $T_1 - T_2$ is positive. We have the following approximation theorem.

THEOREM 4.11. *Assume $f^j, f \in \mathrm{PSH}(D \times A)$, $j = 1, 2, 3, \ldots$, and that f^j are all Siu functions with $\exp f^j$ and $\exp f$ continuous. Let $f_a(x) = f(x, a)$, $f_a^j(x) = f^j(x, a)$, and assume that, for every fixed j, the $f_a - f_a^j$ are locally uniformly bounded in $L^1(D)$. If the sequence $(dd^c f_a^j)_j$ of positive currents is increasing and $dd^c f_a^j$ tends weakly to $dd^c f_a$ for every fixed $a \in D$, then f is a Siu function.*

We will need the following lemma for the proof of Theorem 4.11.

LEMMA 4.12. *Let D be a bounded set in $\mathbf{C}^n$ and let $f \in \mathrm{PSH}(D)$ with $\|f\|_{L^1(D)} < +\infty$. Then $\max(f, b)$ tends to f as $b \to -\infty$ with a certain uniformity: for every positive ε there is a number b_0 which depends only on D, $\|f\|_{L^1(D)}$ and ε such that $\int_D(\max(f, b) - f)d\lambda < \varepsilon$ for all $b \leqq b_0$.*

PROOF. If the conclusion were not true, there would exist a positive ε and plurisubharmonic functions f_j of bounded L^1 norm such that $\int(\max(f_j, -j) - f_j) \geqq \varepsilon$. Then a subsequence f_{j_k} of (f_j) must converge weakly in the space of measures. However, since $dd^c f_j \geqq 0$, also the weak limit g must satisfy $dd^c g \geqq 0$, i.e., it is a plurisubharmonic function. It follows from Hörmander [1983: Proposition 16.1.2] that f_{j_k} tends to g also in L^1. Then of course $\max(g, b)$ is the limit of $\max(f_{j_k}, b)$, so $\int(\max(g, b) - g) \geqq \varepsilon$ for all b. But this contradicts the fact that, as b tends to $-\infty$, $\max(g, b)$ tends to g in L^1.

PROOF OF THEOREM 4.11. Because analyticity is a local property, we can assume that D and A are bounded and that all functions are defined in a fixed neighborhood of the closure of $D \times A$.

Fix a point $(x_0, a_0) \in D \times A$. From the weak convergence $dd^c f_{a_0}^j \rightharpoonup dd^c f_{a_0}$ we deduce that for any $c > 0$ and any integer k there exists an integer m_k such that

$$G(x_0, a_0, t) = \frac{1}{(\pi e^{2t})^{n-1}} \int_{|x - x_0| < e^t} \frac{1}{2\pi}(dd^c f_{a_0}(x) - dd^c f_{a_0}^{m_k}(x)) \wedge \left(\frac{1}{4} dd^c |x - x_0|^2\right)^{n-1} < \frac{c}{k}$$

holds for some $t < 0$.

By using the assumptions that $f, f^{m_k} \in \mathrm{PSH}(D \times A)$ and that $dd^c(f_a - f_a^j)$ are positive currents we see that $f - f^{m_k} \in \mathrm{PSH}(D \times A)$. To be precise one should say that the difference $f \dot{-} f^{m_k}$ is first defined only almost everywhere, but then it can be extended by a classical result. Using now Proposition 2.10 we obtain

$$\nu(f_{a_0} - f_{a_0}^{m_k}, x_0) < \frac{c}{k}. \tag{4.3}$$

Here m_k depends on x_0 and a_0. We shall prove that the same m_k can serve in a neighborhood of the original point. This will follow from the fact that the function G is continuous in (x_0, a_0), and this in turn is clear when both f and f^{m_k} are continuous real-valued functions. In fact, integration over the ball $\{x; |x - x_0| < e^t)$ can be replaced by the action of the integrand on a positive test function $\varphi(x - x_0)$, and the action of the dd^c operator moved to that test function, showing continuity in (x_0, a_0):

$$\int \varphi(x - x_0)(dd^c f_{a_0}(x) - dd^c f_{a_0}^{m_k}(x)) \wedge (dd^c|x - x_0|^2)^{n-1}$$
$$= \int dd^c \varphi(x - x_0) \wedge (f_{a_0}(x) - f_{a_0}^{m_k}(x)) \wedge (dd^c|x - x_0|^2)^{n-1}.$$

In the general case, when we assume only that $\exp f^{m_k}$ and $\exp f$ are continuous, we use Lemma 4.12 to approximate the plurisubharmonic function $f - f^{m_k}$ by a continuous real-valued function, and then the result follows.

Therefore we can find a neighborhood of (x_0, a_0) such that in this neighborhood we have the inequality

$$\nu(f_a, x) < \nu(f_a^{m_k}, x) + \frac{c}{k}.$$

Because we assume that all functions are defined in a fixed neighborhood of the closure of $D \times A$, we can find an integer j_k such that the inequality

$$\nu(f_a, x) \leqq \nu(f_a^{j_k}, x) + \frac{c}{k} \tag{4.4}$$

holds for all $(x, a) \in D \times A$.

We will prove the following equality

$$X_c(f) = \bigcap_{k=2}^{\infty} X_{c(1-1/k)}(f^{j_k}), \tag{4.5}$$

where j_k, $k = 2, 3, \ldots$ are the numbers such that (4.4) hold. If (4.5) is true, then the fact that f^{j_k} $(k = 2, 3, \ldots)$ are Siu functions shows that also f is a Siu function. If $(x, a) \in X_c(f)$, i.e., $\nu(f_a, x) \geqq c$, by using (4.4) we deduce

$$\nu(f_a^{j_k}, x) \geqq c(1 - 1/k), \quad k = 2, 3, \ldots.$$

Therefore

$$(x, a) \in \bigcap_{k=2}^{\infty} X_{c(1-1/k)}(f^{j_k}).$$

Conversely if

$$(x, a) \in \bigcap_{k=2}^{\infty} X_{c(1-1/k)}(f^{j_k})$$

we have

$$\nu(f_a, x) \geqq \nu(f_a^{j_k}, x) \geqq c(1 - 1/k), \quad k = 2, 3, \ldots.$$

So finally $\nu(f_a, x) \geqq c$.

The following example is an application of Theorem 4.11.

EXAMPLE 4.13. Consider the function discussed in Example 4.3

$$f(x, a) = \sum_{1}^{\infty} \log(|a - a_k - x^{m_k}|^{\alpha_k} + |x|^{\beta_k}).$$

If the f_a are locally uniformly bounded in $L^1(D)$ and for every fixed $a \in D$ we have

$$\lim_{j \to \infty} \int_D \sum_{k=j}^{\infty} \log(|a - a_k - x^{m_k}|^{\alpha_k} + |x|^{\beta_k}) dx d\bar{x} = 0 \tag{4.6}$$

then $f(x, a)$ is a Siu function in $D \times D$. In this example D is the unit disk.

PROOF. Let

$$f^j(x, a) = \sum_{k=1}^{j} \log(|a - a_k - x^{m_k}|^{\alpha_k} + |x|^{\beta_k}), \quad j = 1, 2, \ldots.$$

Then $f^j(x, a)$ are all Siu functions with $\exp f^j$ and $\exp f$ continuous. The $f_a - f_a^j$ are locally uniformly bounded in $L^1(D)$ for fixed j. The sequence $(dd^c f_a^j)_j$ of positive currents is increasing. We also have the following equality

$$dd^c f_a(x) = dd^c f_a^j(x) + dd^c \sum_{k=j+1}^{\infty} \log(|a - a_k - x^{m_k}|^{\alpha_k} + |x|^{\beta_k}).$$

Therefore the equality (4.6) implies that $dd^c f_a^j$ tends weakly to $dd^c f_a$ for every fixed $a \in D$. According to Theorem 4.11 the function $f(x, a)$ is a Siu function in $D \times D$.

5. Relations between partial and directional Lelong numbers.

As in section 4 let $D \subset \mathbf{C}^n$, $A \subset \mathbf{C}^k$ be two open sets and $f \in \mathrm{PSH}(D \times A)$. We shall write $f_a(x) = f(x, a)$, $(x, a) \in D \times A$. In this section we shall study the relation between the directional Lelong number $\nu(f, (x, a), (1, y))$ of a function f at the point (x, a) in the direction $(1, y) = (1, \ldots, 1, y, \ldots, y) \in \mathbf{R}^n_+ \times \mathbf{R}^k_+$, and the partial Lelong number $\nu(f_a, x)$. We also study the relation between the directional Lelong number $\nu(f, a), (h, 1))$, $(h, 1) = (h, \ldots, h, 1, \ldots, 1) \in \mathbf{R}^n_k \times \mathbf{R}^k_+$, and the partial Lelong number $\nu(f_a, x)$.

We know that

$$\nu(f, (x, a), (1, y)) = \lim_{t \to -\infty} \frac{1}{t} \sup_{|z_i| = 1, |w_j| = 1} f(x + ze^t, a + we^{yt}),$$

$$\nu(f_a, x) = \lim_{t \to -\infty} \frac{1}{t} \sup_{|z_i| = 1} f(x + ze^t, a).$$

THEOREM 5.1. *Given a point $(x_0, a_0) \in D \times A$ we have the following equalities*

(a) $$\lim_{y \to +\infty} \nu(f, (x_0, a_0), (1, y)) = \nu(f_{a_0}, x_0);$$

(b) $$\nu(f, (x_0, a_0), (h, 1)) = \nu(f_{a_0}, x_0)h + o(h),$$

provided that $\exp f$ *is upper Hölder regular with respect to A.*

PROOF. It is obvious that for $t \ll 0$

$$\sup_{|z| = 1} f(x_0 + z(e^t + |a - a_0|^r), a_0) \leqq \nu(f_{a_0}, x_0) \log(e^t + |a - a_0|^r).$$

The fact that $\exp f$ is upper Hölder regular with respect to A gives the following estimates

$$\begin{aligned} \sup_{|z| = 1} f(x_0 + ze^t, a) &\leqq \log [|a - a_0|^r + \exp \sup_{|z| = 1} f(x_0 + z(e^t + |a - a_0|^r), a_0)] \\ &\leqq \log [|a - a_0|^r + (e^t + |a - a_0|^r)^{\nu(f_{a_0}, x_0)}]. \end{aligned}$$

Therefore we get

$$\begin{aligned} \nu(f, (x_0, a_0), (1, y)) &= \lim_{t \to -\infty} \frac{1}{t} \sup_{|z_i| = 1, |w_j| = 1} f(x_0 + ze^t, a_0 + we^{yt}) \\ &\geqq \lim_{t \to -\infty} \frac{1}{t} \log [e^{ryt} + (e^t + e^{ryt})^{\nu(f_{a_0}, x_0)}] \qquad (5.1) \\ &= \min (ry, ry\nu(f_{a_0}, x_0), \nu(f_{a_0}, x_0)). \end{aligned}$$

If $\nu(f_{a_0}, x_0) = +\infty$, we have

$$\nu(f, (x_0, a_0), (1, y)) \geqq ry$$

which implies

$$\lim_{y \to +\infty} \nu(f, (x_0, a_0), (1, y)) = +\infty.$$

If on the other hand $\nu(f_{a_0}, x_0) < +\infty$ it follows from (5.1) that

$$\nu(f, (x_0, a_0), (1, y)) \geqq \nu(f_{a_0}, x_0),$$

when y is large. The opposite inequality is obvious, so we have proved

$$\lim_{y \to +\infty} \nu(f, (x_0, a_0), (1, y)) = \nu(f_{a_0}, x_0).$$

By Proposition 2.6 we get that

$$\nu(f, (x_0, a_0), (1, y)) = y\nu(f, (x_0, a_0), (1/y, 1)).$$

Let $1/y = h$. Then we get

$$\lim_{h \to 0} \frac{1}{h} \nu(f, (x_0, a_0), (h, 1)) = \nu(f_{a_0}, x_0).$$

by using (a). Therefore we have proved (b).

THEOREM 5.2. *If $\nu(f, (x, a), (1, y))$ tends to $\nu(f_a, x)$ locally uniformly as $y \to +\infty$ and $\nu(f_a, x) < +\infty$, then f is a Siu function*, i.e.,

$$X_c(f) = \{(x, a) \in D \times A; \nu(f_a, x) \geqq c\} = \bigcup_{a \in A} E_c(f_a) \times \{a\}.$$

is an analytic variety for every $c > 0$.

PROOF. Because analyticity is a local property we can work within the intersection of $X_c(f)$ and a compact set K. By assumption, for every point $(x, a) \in X_c(f) \cap K$ and every number ε with $0 < \varepsilon < c$ we can find a neighborhood of (x, a) and a number y'_ε such that in this neighborhood we have

$$\nu(f, (x, a), (1, y)) \geqq \nu(f_a, x) - \varepsilon; \quad y \geqq y'_\varepsilon.$$

Therefore we can find a number y_ε such that

$$\nu(f, (x, a), (1, y)) \geqq \nu(f_a, x) - \varepsilon; \quad y \geqq y_\varepsilon, \quad (x, a) \in X_c(f) \cap K.$$

By this inequality and the following inequality:

$$\nu(f, (x, a), (1, y)) \leqq \nu(f_a, x)$$

we obtain

$$X_c(f) = \bigcap_{0<\varepsilon<c} E_{c-\varepsilon}(f, y_\varepsilon)$$

which proves the theorem in view of Theorem 2.7.

REMARK 5.3. Generally Theorem 4.10 and Theorem 5.2 do not imply each other. But in the case that $\exp f$ is upper Hölder regular with respect to A and $\nu(f_a, x)$ is locally uniformly bounded we can deduce that f is a Siu function from Theorem 5.1 and Theorem 5.2.

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DEPARTMENT OF MATHEMATICS
UPPSALA UNIVERSITY
THUNBERGSVÄGEN 3
S-752 38 UPPSALA
SWEDEN