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PRO-p GROUPS WITH FEW RELATIONS AND UNIVERSAL KOSZULITY

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Abstract

Let p be a prime. We show that if a pro-p group with at most 2 defining relations has quadratic $\mathbb{F}_p$-cohomology algebra, then this algebra is universally Koszul. This proves the "Universal Koszulity Conjecture" formulated by J. Mináč et al. in the case of maximal pro-p Galois groups of fields with at most 2 defining relations.

1. Introduction

Let k be a field, and $A_\bullet = \bigoplus_{n\geq 0} A_n$ a k-algebra graded by $\mathbb{N}$. The algebra $A_\bullet$ is *quadratic* if it is 1-*generated*, i.e., every element is a combination of products of elements of A_1, and its relations are generated by homogeneous relations of degree 2. For example, symmetric algebras and exterior algebras are quadratic.

A quadratic algebra is called a *Koszul algebra* if k admits a resolution of free $\mathbb{N}$-graded right $A_\bullet$-modules such that for every $n \geq 0$ the subspace of degree n of the n-th term of the resolution is finitely generated, and this subspace generates the respective module (see §2.2). Koszul algebras were introduced by S. Priddy in [19], and they have exceptionally nice behavior in terms of cohomology (see [15, Chapter 2]). The Koszul property is very restrictive; still it arises in various areas of mathematics, such as representation theory, algebraic geometry, combinatorics. Hence, Koszul algebras have become an important object of study.

Recently, some stronger versions of the Koszul property were introduced and investigated (see, e.g., [3], [5], [14], [4], [9], [12]). For example, the universal Koszul property (see Definition 2.6 below), which implies "simple" Koszulity. Usually, checking whether a given quadratic algebra is Koszul is a rather hard problem. Surprisingly, testing universal Koszulity may be easier, even though it is a more restrictive property.

Quadratic algebras and Koszul algebras have a prominent role in Galois theory, too. Given a field $\mathbb{K}$, for p a prime number let $\mathcal{G}_{\mathbb{K}}$ denote the maximal

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pro-p Galois group of $\mathbb{K}$: namely, $\mathcal{G}_{\mathbb{K}}$ is the Galois group of the maximal p-extension of $\mathbb{K}$. If $\mathbb{K}$ contains a root of 1 of order p (including $\sqrt{-1}$ if $p = 2$), then the celebrated Rost-Voevodsky Theorem (cf. [23], [25]) implies that the $\mathbb{F}_p$-cohomology algebra $H^\bullet(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p) = \bigoplus_{n\geq 0} H^n(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p)$ of the maximal pro-p Galois group of $\mathbb{K}$, endowed with the graded-commutative cup product

$$H^s(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p) \times H^t(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p) \xrightarrow{\cup} H^{s+t}(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p), \qquad s, t \geq 0,$$

is a quadratic $\mathbb{F}_p$-algebra. Koszul algebras were studied in the context of Galois theory by L. Positselski and A. Vishik (cf. [18], [16], see also [10]), and Positselski conjectured that the algebra $H^\bullet(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p)$ is Koszul (cf. [17]). Positselski's conjecture was strengthened by J. Mináč et al. (cf. [9, Conjecture 2]):

CONJECTURE 1.1. *Let $\mathbb{K}$ be a field containing a root of* 1 *of order p, and suppose that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is finite. Then the $\mathbb{F}_p$-cohomology algebra $H^\bullet(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p)$ of the maximal pro-p Galois group of $\mathbb{K}$ is universally Koszul.*

Here $\mathbb{K}^\times$ denotes the multiplicative group $\mathbb{K} \setminus \{0\}$, and by Kummer theory $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is finite if and only if $\mathcal{G}_{\mathbb{K}}$ is a finitely generated pro-p group.

In this paper we study universal Koszulity for the $\mathbb{F}_p$-cohomology algebra of pro-p groups with at most two *defining relations*. A pro-p group G has m defining relations, with $m \geq 0$, if there is a minimal pro-p presentation F/R of G (i.e., $G \simeq F/R$ with F a free pro-p group and R a closed normal subgroup contained in the Frattini subgroup of F) such that m is the minimal number of generators of R as closed normal subgroup of F. We prove the following.

THEOREM 1.2. *Let G be a finitely generated pro-p group with at most two defining relations. If the $\mathbb{F}_p$-cohomology algebra $H^\bullet(G, \mathbb{F}_p)$ is quadratic* (*and moreover if $a^2 = 0$ for every $a \in H^\bullet(G, \mathbb{F}_2)$, if $p = 2$*)*, then $H^\bullet(G, \mathbb{F}_p)$ is universally Koszul.*

The above result settles positively Conjecture 1.1 for fields whose maximal pro-p Galois group has at most two defining relations.

COROLLARY 1.3. *Let $\mathbb{K}$ be a field containing a root of* 1 *of order p* (*including $\sqrt{-1}$ if $p = 2$*)*, and suppose that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is finite. If $\mathcal{G}_{\mathbb{K}}$ has at most two defining relations then the $\mathbb{F}_p$-cohomology algebra $H^\bullet(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p)$ is universally Koszul.*

Note that the condition on the number of defining relations of $\mathcal{G}_{\mathbb{K}}$ may be formulated both in terms of the dimension of $H^2(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p)$ and in terms of the Brauer group of $\mathbb{K}$: namely, $\mathcal{G}_{\mathbb{K}}$ has at most two defining relations if and only if $\dim(H^2(\mathcal{G}_{\mathbb{K}}, \mathbb{F}_p)) \leq 2$, and if and only if the p-part of the Brauer group of $\mathbb{K}$ has rank at most two.

Theorem 1.2 can not be extended to pro-p groups with quadratic $\mathbb{F}_p$-cohomology with more than two defining relations, as there are finitely generated pro-p groups with three defining relations whose $\mathbb{F}_p$-cohomology algebra is quadratic and Koszul, but not universally Koszul (see Example 3.8). Still, such examples are expected not to contradict Conjecture 1.1, as these pro-p groups are conjectured not to occur as maximal pro-p Galois of groups of fields (see Remark 4.3).

2. Quadratic algebras and Koszul algebras

Throughout the paper every graded algebra $A_\bullet = \bigoplus_{n\in\mathbb{Z}} A_n$ is tacitly assumed to be a unitary associative algebra over the finite field $\mathbb{F}_p$, and non-negatively graded of finite-type, i.e., $A_0 = \mathbb{F}_p$, $A_n = 0$ for $n < 0$ and $\dim(A_n) < \infty$ for $n \geq 1$. For a complete account on graded algebras and their cohomology, we direct the reader to the first chapters of [15] and of [8], and to [10, §2].

2.1. Quadratic algebras

A graded algebra $A_\bullet = \bigoplus_{n\geq 0} A_n$ is said to be *graded-commutative* if one has

$$b \cdot a = (-1)^{ij} a \cdot b \qquad \text{for every } a \in A_i, b \in A_j.$$

In particular, if p is odd then one has $a^2 = 0$ for all $a \in A_\bullet$, whereas if $p = 2$ then a graded-commutative algebra is commutative. Furthermore, if $p = 2$ we call a commutative algebra $A_\bullet$ which satisfies $a^2 = 0$ for all $a \in A_\bullet$ a *wedge-commutative* $\mathbb{F}_2$-algebra.

For a graded ideal I of a graded algebra $A_\bullet$, I_n denotes the intersection $I \cap A_n$ for every $n \geq 0$, i.e., $I = \bigoplus_{n\geq 0} I_n$. For a subset $\Omega \subseteq A_\bullet$, $(\Omega) \trianglelefteq A_\bullet$ denotes the two-sided graded ideal generated by Ω. Also, A_+ denotes the *augmentation ideal* of $A_\bullet$, i.e., $A_+ = \bigoplus_{n\geq 1} A_n$. Henceforth all ideals are assumed to be graded.

Given a finite vector space V, let $T_\bullet(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ denote the tensor algebra generated by V. The product of $T_\bullet(V)$ is induced by the tensor product, i.e., $ab = a \otimes b \in V^{\otimes s+t}$ for $a \in V^{\otimes s}$, $b \in V^{\otimes t}$.

Definition 2.1. A graded algebra $A^\bullet$ is said to be quadratic if one has an isomorphism

$$A_\bullet \simeq T_\bullet(A_1)(\Omega)$$

for some subset $\Omega \subseteq A_1 \otimes A_1$. In this case we write $A_\bullet = Q(V, \Omega)$.

Example 2.2. Let V be a finite vector space.

(a) The tensor algebra $T_\bullet(V)$ and the trivial quadratic algebra $Q(V, V^{\otimes 2})$ are quadratic algebras.

(b) The symmetric algebra $S_\bullet(V)$ and the exterior algebra $\Lambda_\bullet(V)$ are quadratic, as one has $S_\bullet(V) = Q(V, \Omega_S)$ and $\Lambda_\bullet(V) = Q(V, \Omega_\wedge)$ with

$$\Omega_S = \{u \otimes v - v \otimes u \mid u, v \in V\} \quad \text{and} \quad \Omega_\wedge = \{u \otimes v + v \otimes u \mid u, v \in V\}.$$

(c) Let $\mathbb{F}_p\langle X\rangle$ be the free algebra generated by the indeterminates $X = \{X_1, \ldots, X_d\}$. Then $\mathbb{F}_p\langle X\rangle$ is a graded algebra, with the grading induced by the subspaces of homogeneous polynomials. If $\Omega = \{f_1, \ldots, f_m\} \subseteq \mathbb{F}_p\langle X\rangle$ is a set of homogeneous polynomials of degree 2, then $\mathbb{F}_p\langle X\rangle/(\Omega)$ is a quadratic algebra.

EXAMPLE 2.3. Let $A_\bullet = Q(A_1, \Omega_A)$ and $B_\bullet = Q(B_1, \Omega_B)$ be two quadratic algebras. The direct product of $A_\bullet$ and $B_\bullet$ is the quadratic algebra

$$A_\bullet \sqcap B_\bullet = Q(A_1 \oplus B_1, \Omega),$$

with $\Omega = \Omega_A \cup \Omega_B \cup (A_1 \otimes B_1) \cup (B_1 \otimes A_1)$.

EXAMPLE 2.4. Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a finite combinatorial graph (without loops), namely $\mathscr{V} = \{v_1, \ldots, v_d\}$ is the set of vertices of Γ, and

$$\mathscr{E} \subseteq \{\{v, w\} \mid v, w \in \mathscr{V}, v \neq w\} = \mathscr{P}_2(\mathscr{V}) \setminus \Delta(\mathscr{V})$$

is the set of edges of Γ, and let V be the space with basis $\mathscr{V}$. The *exterior Stanley-Reisner algebra* $\Lambda_\bullet(\Gamma)$ associated to Γ is the quadratic algebra

$$\Lambda_\bullet(\Gamma) = \frac{\Lambda_\bullet(V)}{(v \wedge w \mid \{v, w\} \notin \mathscr{E})}.$$

In particular, $\Lambda_\bullet(\Gamma)$ is graded-commutative (wedge-commutative if $p = 2$), and if Γ is complete (i.e., $\mathscr{E} = \mathscr{P}_2(\mathscr{V}) \setminus \Delta(V)$) then $\Lambda_\bullet(\Gamma) \simeq \Lambda_\bullet(V)$, whereas if $\mathscr{E} = \varnothing$, then $\Lambda_\bullet(\Gamma) \simeq Q(V, V^{\otimes 2})$.

2.2. *Koszul algebras and universally Koszul algebras*

A quadratic algebra $A_\bullet$ is said to be *Koszul* if it admits a resolution

$$\cdots \longrightarrow P(2)_\bullet \longrightarrow P(1)_\bullet \longrightarrow P(0)_\bullet \longrightarrow \mathbb{F}_p$$

of right $A_\bullet$-modules, where for each $i \in \mathbb{N}$, $P(i)_\bullet = \bigoplus_{n\geq 0} P(i)_n$ is a free *graded* $A_\bullet$-module such that $P(n)_n$ is finitely generated for all $n \geq 0$, and $P(n)_n$ generates $P(n)_\bullet$ as graded $A_\bullet$-module (cf. [15, Definition 2.1.1] and [10, §2.2]). Koszul algebras have an exceptionally nice behavior in terms of cohomology. Indeed, if a quadratic algebra $A_\bullet = Q(V, \Omega)$ is Koszul, then one has an isomorphism of quadratic algebras

$$\bigoplus_{n\geq 0} \mathrm{Ext}^{n,n}_{A_\bullet}(\mathbb{F}_p, \mathbb{F}_p) \simeq Q(V^*, \Omega^\perp), \tag{2.1}$$

where V^* denotes the $\mathbb{F}_p$-dual of V, and $\Omega^\perp \subseteq (V \otimes V)^*$ is the orthogonal of $\Omega \subseteq V \otimes V$ (cf. [19]; since V is finite, we identify $(V^*)^{\otimes 2} = (V^{\otimes 2})^*$), whereas $\mathrm{Ext}^{i,j}_{A_\bullet}(\mathbb{F}_p, \mathbb{F}_p) = 0$ for $i \neq j$ (in fact, this is an equivalent definition of the Koszul property).

On the one hand, by (2.1) it is very easy to compute the $\mathbb{F}_p$-cohomology of a Koszul algebra. On the other hand, in general it is extremely hard to establish whether a given quadratic algebra is Koszul. For this reason, some "enhanced forms" of Koszulity, which are stronger than "simple" Koszulity, but at the same time easier to check, have been introduced by several authors. We give now the definition of *universal Koszulity* as introduced in [9].

Given two ideals I, J of a graded algebra $A_\bullet$, the *colon ideal* $I : J$ is the ideal

$$I : J = \{a \in A_\bullet \mid a \cdot J \subseteq I\}.$$

REMARK 2.5. Note that for every two ideals I, J of $A_\bullet$, the colon ideal $I : J$ contains all $a \in A_\bullet$ such that $a \cdot J = 0$, as $0 \in I$. Moreover, if $A_\bullet$ is graded-commutative (and wedge-commutative, if $p = 2$), then for every for every $b \in J$ one has $b \in I : J$, as $b \cdot b = 0$.

For a quadratic algebra $A_\bullet$, let $\mathscr{L}(A_\bullet)$ denote the set of all ideals of $A_\bullet$ generated by a subset of A_1, namely,

$$\mathscr{L}(A_\bullet) = \{I \in A_\bullet \mid I = A_\bullet \cdot I_1\}.$$

Note that both the trivial ideal (0) and the augmentation ideal A_+ belong to $\mathscr{L}(A_\bullet)$.

DEFINITION 2.6. A quadratic algebra $A_\bullet$ is said to be *universally Koszul* if for every ideal $I \in \mathscr{L}(A_\bullet)$, and every $b \in A_1 \setminus I_1$, one has $I : (b) \in \mathscr{L}(A_\bullet)$.

Universal Koszulity is stronger than Koszulity, since every quadratic algebra which is universally Koszul is also Koszul (cf. [9, §2.2]).

EXAMPLE 2.7. (a) Let V be a vector space of finite dimension. Then both the trivial algebra $Q(V, V^{\otimes 2})$ (by definition) and the exterior algebra $\Lambda_\bullet(V)$ (by [9, Proposition 31]) are universally Koszul.

(b) If $A_\bullet$ and $B_\bullet$ are two quadratic universally Koszul algebras, then also the direct product $A_\bullet \sqcup B_\bullet$ is universally Koszul (cf. [9, Proposition 30]).

EXAMPLE 2.8. For V a finite vector space of even dimension d and basis $\{v_1, \dots, v_d\} \subseteq V$, let $A_\bullet$ be the quadratic algebra $A_\bullet = \Lambda_\bullet(V)/(\Omega)$, where

$$\Omega = \left\{ \begin{array}{c} v_1 \wedge v_2 - v_i \wedge v_{i+1}, \text{ for } i = 1, 3, \dots, d-1, \\ v_i \wedge v_j, \text{ for } i < j, (i, j) \neq (1, 2), (3, 4), \dots, (d-1, d) \end{array} \right\} \subseteq \Lambda_2(V)$$

In particular, A_2 is generated by the image of $v_1 \wedge v_2$, and $A_n = 0$ for $n \geq 3$. Then $A_\bullet$ is isomorphic to the $\mathbb{F}_p$-cohomology algebra of a d-generated Demushkin pro-p group (cf. [11, Definition 3.9.9]), and thus it is universally Koszul by [9, Proposition 29].

EXAMPLE 2.9. Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a combinatorial graph without loops, and let $\Lambda_\bullet(\Gamma)$ be the exterior Stanley-Reisner algebra associated to Γ. Then $\Lambda_\bullet(\Gamma)$ is Koszul (cf. [19], see also [13, §3.2] and [26, §4.2.2]). Moreover, by [2, Theorem 4.6] the algebra $\Lambda_\bullet(\Gamma)$ is also universally Koszul if and only if Γ has the diagonal property, i.e., for any four vertices $v_1, \dots, v_4 \in \mathscr{V}$ such that

$$\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\} \in \mathscr{E},$$

then one has $\{v_1, v_3\} \in \mathscr{E}$ or $\{v_2, v_4\} \in \mathscr{E}$ (see, e.g., [6]).

3. Two-relator pro-p groups

Henceforth, every subgroup of a pro-p group is meant to be closed with respect to the pro-p topology, and generators are topological generators. For (closed) subgroups H, H_1, H_2 of a pro-p group G and for every $n \geq 1$, H_n is the subgroup of G generated by n-th powers of the elements of H, whereas $[H_1, H_2]$ is the subgroup of G generated by commutators

$$[g_1, g_2] = g_1^{-1} \cdot g_1^{g_2} = g_1^{-1} g_2^{-1} g_1 g_2,$$

with $g_1 \in H_1$ and $g_2 \in H_2$.

3.1. *Cohomology of pro-p groups*

For a pro-p group G we set

$$G_{(2)} = G^p[G, G], \qquad G_{(3)} = \begin{cases} G^p[G, [G, G]] & \text{if } p \neq 2, \\ G^4[G, G]^2[G, [G, G]] & \text{if } p = 2, \end{cases}$$

namely, $G_{(2)}$ and $G_{(3)}$ are the second and the third elements of the p-Zassenhaus filtration of G (cf. [10, §3.1]). In particular, $G_{(2)}$ coincides with the Frattini subgroup of G.

A short exact sequence of pro-p groups

$$\{1\} \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow \{1\} \tag{3.1}$$

with F a free pro-p group, is called a *presentation* of the pro-p group G. If $R \subseteq F_{(2)}$, then the presentation (3.1) is *minimal*; roughly speaking, F and G have the "same" minimal generating system. For a minimal presentation (3.1) of G, a set of elements of F which generates minimally R as normal subgroup is called a set of *defining relations* of G.

For a pro-p group G we shall denote the $\mathbb{F}_p$-cohomology groups $H^n(G, \mathbb{F}_p)$ simply by $H^n(G)$ for every $n \geq 0$. In particular, one has

$$H^0(G) = \mathbb{F}_p \qquad \text{and} \qquad H^1(G) \simeq (G/G_{(2)})^* \tag{3.2}$$

(cf. [11, Proposition 3.9.1]). Moreover, a minimal presentation (3.1) of G induces an exact sequence in cohomology

$$0 \longrightarrow H^1(G) \longrightarrow \inf^1_{F,R} H^1(F) \xrightarrow{\mathrm{res}^1_{F,R}} H^1(R)^F \xrightarrow{\mathrm{trg}_{F,R}} H^2(G) \xrightarrow{\inf^2_{F,R}} H^2(F)$$

(cf. [11, Proposition 1.6.7]). Here, if V is a continuous G-module for a pro-p group G, then V^G denotes the subspace of G-invariants. Since (3.1) is minimal, by (3.2) the map $\inf^1_{F,R}$ is an isomorphism. Moreover, also the map $\mathrm{trg}_{F,R}$ is an isomorphism, as $H^2(F) = 0$ (see Proposition 3.1 below), and its inverse induces an isomorphism ϕ of vector spaces

$$H^2(G) \xrightarrow{\phi} \left((R/R_{(2)})^*\right)^F = (R/R^p[R, F])^* \tag{3.3}$$

By (3.2), $\dim(H^1(G))$ is the minimal number of generators of G, and by (3.3) $\dim(H^2(G))$ is the number of defining relations of G. If $H^1(G)$ and $H^2(G)$ have both finite dimension, then G is said to be *finitely presented*.

The $\mathbb{F}_p$-cohomology of a pro-p group comes endowed with the bilinear *cup-product*

$$H^i(G) \times H^j(G) \xrightarrow{\cup} H^{i+j}(G),$$

which is graded-commutative (cf. [11, Chapter I, §4]). The maximum positive integer n such that $H^n(G) \neq 0$ and $H^{n+1}(G) = 0$ is called the cohomological dimension of G, and it is denoted by $\mathrm{cd}(G)$ (cf. [11, Definition 3.3.1]).

By (3.3), if G is a free pro-p group then $H^2(G) = 0$. Also the converse is true (cf. [11, Proposition 3.5.17]).

PROPOSITION 3.1. *A pro-p group G is free if and only if* $\mathrm{cd}(G) = 1$.

Let G be a finitely generated pro-p group with minimal presentation (3.1). We may identify $H^1(G)$ and $H^1(F)$ via the isomorphism $\inf^1_{F,R}$. Also, we may identify a basis $\mathscr{X} = \{x_1, \ldots, x_d\}$ of F with its image in G. Let $\mathscr{B} =$

$\{a_1, \ldots, a_d\}$ be the basis of $H^1(G)$ dual to $\mathscr{X}$, i.e., $a_i(x_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, d\}$. For every $r \in F_{(2)}$ one may write

$$r = \begin{cases} \prod_{i<j} [x_i, x_j]^{\alpha_{ij}} \cdot r', & \text{if } p \neq 2, \\ \prod_{i=1}^{d} x_i^{2\alpha_{ii}} \prod_{i<j} [x_i, x_j]^{\alpha_{ij}} \cdot r', & \text{if } p = 2, \end{cases} \tag{3.4}$$

for some $r' \in F_{(3)}$, with $0 \leq \alpha_{ij} < p$, and these numbers are uniquely determined by r. The shape of the defining relations of a pro-p group and the behavior of the cup-product are related by the following (cf. [24, Proposition 1.3.2]).

Proposition 3.2. *Let G be a finitely presented pro-p group with minimal presentation* (3.1)*, and let $\mathscr{X} = \{x_1, \ldots, x_d\}$ and $\mathscr{B} = \{a_1, \ldots, a_d\}$ be as above. Given a set of defining relations $\{r_1, \ldots, r_m\} \subseteq F_{(2)}$, for every $h = 1, \ldots, m$ the isomorphism ϕ (see* (3.3)*) induces a morphism*

$$\operatorname{tr}_h \colon H^2(G) \longrightarrow \mathbb{F}_p,$$
$$\operatorname{tr}_h(b) = \phi(b)(r_h),$$

such that, for every $1 \leq i \leq j \leq d$, one has $\operatorname{tr}_h(a_i \cup a_j) = -\alpha_{ij}$, where the α_{ij}'s are the numbers in (3.4) *with $r = r_h$.*

Example 3.3. Let G be a finitely generated one-relator pro-p group, with minimal presentation (3.1) and defining relation r, and with $\mathscr{X} = \{x_1, \ldots, x_d\}$ and $\mathscr{B} = \{a_1, \ldots, a_d\}$ as above. Since $\dim(H^2(G)) = 1$ by (3.3), the algebra $H^\bullet(G)$ is quadratic, and wedge-commutative if $p = 2$, if and only if $H^2(G)$ is generated by some non-trivial $a_i \cup a_j$ (and also $a_i \cup a_i = 0$ for all $i = 1, \ldots, d$, if $p = 2$) and $H^3(G) = 0$: by Proposition 3.2 this occurs if and only if $\alpha_{ij} \neq 0$ for some i, j, with the α_{ij}'s as in (3.4) (i.e., $r \notin F_{(3)}$), and also $\alpha_{ii} = 0$ for every $i = 1, \ldots, d$ if $p = 2$ (see [20, Proposition 4.2] for the details).

In this case, one may choose $\mathscr{X}$ such that $\alpha_{1,2} = \alpha_{3,4} = \ldots = \alpha_{s-1,s} = 1$, for some even $s \leq d$, and $\alpha_{ij} = 0$ for all other couples (i, j), so that one has an isomorphism of quadratic algebras

$$H^\bullet(G) \simeq A_\bullet \sqcup Q(V, V^{\otimes 2}),$$

where $A_\bullet$ is the quadratic algebra as in Example 2.8, (with A_1 generated by $a_1, \ldots, a_s$, and A_2 generated by $a_1 \cup a_2 = \ldots = a_{s-1} \cup a_s$), and V a finite (possibly trivial) vector space, generated by a_i with $s + 1 \leq i \leq d$ (cf. [20, Proposition 4.6]). In particular, $H^\bullet(G)$ is universally Koszul by Example 2.7(b).

3.2. Two-relator pro-p groups and cohomology

Henceforth G will be a finitely generated two-relator pro-p group, with minimal presentation (3.1). Also, the set $\mathscr{X} = \{x_1, \ldots, x_d\}$ will denote a basis of F (identified with its image in G), with $d = \dim(H^1(G))$, and $\mathscr{B} = \{a_1, \ldots, a_d\}$ will be the associated dual basis of $H^1(G)$. For simplicity, we will omit the symbol $\cup$ to denote the cup-product of two elements of $H^\bullet(G)$. For our convenience, we slightly modify the definition given in [1, §1].

DEFINITION 3.4. A two-relator pro-p group G is *quadratically defined* if the cup-product induces an epimorphism $H^1(G)^{\otimes 2} \to H^2(G)$, and also $a \cdot a = 0$ for every $a \in H^1(G)$ in the case $p = 2$.

By Proposition 3.2, G is quadratically defined if and only if $r_1, r_2 \in F_{(2)} \setminus F_{(3)}$ for any set of defining relations $\{r_1, r_2\} \subseteq F_{(2)}$, and also $\alpha_{ii} = 0$ for every $i = 1, \ldots, d$ in the case $p = 2$, where the α_{ii}'s are the numbers in (3.4) with $r = r_1, r_2$ (see also [10, Theorem 7.3]).

Set $I = \{1, \ldots, d\}$, and let $\succ$ denote the lexicographic order on $I^2 = \{(i, j) \mid 1 \leq i, j \leq d\}$; namely, $(i, j) \succ (h, k)$ if $i > h$ or if $i = h$ and $j > k$. If G is quadratically defined, by Proposition 3.2 and [21, Remark 2.5] one may choose defining relations $r_1, r_2 \in F_{(2)}$ such that

$$\begin{aligned} r_1 &\equiv [x_1, x_2] \cdot \prod_{\substack{1 \leq i < j \leq d \\ (i,j) \succ (1,2)}} [x_i, x_j]^{\alpha_{ij}} \mod F_{(3)}, \\ r_2 &\equiv [x_s, x_t] \cdot \prod_{\substack{1 \leq i < j \leq d \\ (i,j) \succ (s,t)}} [x_i, x_j]^{\beta_{ij}} \mod F_{(3)}, \end{aligned} \tag{3.5}$$

for some $(s, t) \succ (1, 2)$, and $0 \leq \alpha_{ij}, \beta_{ij} \leq p - 1$, with $\alpha_{st} = 0$. By Proposition 3.2, one has

$$\operatorname{tr}_1(a_1 a_2) = 1 \quad \text{and} \quad \operatorname{tr}_1(a_i a_j) = \alpha_{ij},$$

for $i \leq j$ and $(i, j) \succ (1, 2)$, and likewise

$$\operatorname{tr}_2(a_s a_t) = 1 \qquad \text{and} \qquad \operatorname{tr}_2(a_i a_j) = \begin{cases} 0, & \text{if } (i, j) \prec (s, t), \\ \beta_{ij}, & \text{if } (i, j) \succ (s, t). \end{cases}$$

Altogether, $\{a_1 a_2, a_s a_t\}$ is a basis of $H^2(G)$, and one has relations

$$\begin{aligned} a_i a_i &= 0, \\ a_j a_i &= -a_i a_j, \\ a_i a_j &= \alpha_{ij}(a_1 a_2) + \beta_{ij}(a_s a_t), \qquad i < j, \end{aligned} \tag{3.6}$$

where we set implicitly $\alpha_{1,2} = \beta_{st} = 1$, and $\beta_{ij} = 0$ for $(i, j) \prec (s, t)$. Finally, one has the following (cf. [1, Theorem 1–2]).

PROPOSITION 3.5. *Let G be a finitely generated quadratically defined two-relator pro-p group. Then* $\mathrm{cd}(G) = 2$.

As a consequence we obtain the following.

PROPOSITION 3.6. *Let G be a finitely generated two-relator pro-p. The following are equivalent:*

(i) *$H^\bullet(G)$ is quadratic (and wedge-commutative, if $p = 2$);*

(ii) *G is quadratically defined.*

PROOF. Assume first that $H^\bullet(G)$ is quadratic (and wedge-commutative, if $p = 2$). Then one has $H^n(G) = 0$ for $n \geq 3$, while the cup-product induces epimorphisms

$$H^1(G)^{\otimes 2} \longrightarrow \Lambda_2(H^1(G)) \longrightarrow H^2(G), \qquad \text{if } p \neq 2,$$

$$H^1(G)^{\otimes 2} \longrightarrow \frac{S_2(H^1(G))}{\langle a^2 \mid a \in H^1(G)\rangle} \longrightarrow H^2(G), \qquad \text{if } p = 2,$$

and thus G is quadratically defined.

Assume now that G is quadratically defined, and let r_1, r_2 be defining relations as in (3.5). By Proposition 3.5, for $n \geq 3$ one has $H^n(G) = 0$, whereas $H^2(G)$ is generated by a_1a_2 and a_sa_t, so that $H^\bullet(G)$ is 1-generated. In fact, by the relations (3.6) one has epimorphisms of graded algebras $\psi_*\colon \Lambda_\bullet(H^1(G)) \to H^\bullet(G)$, with $\mathrm{Ker}(\psi_n) = \Lambda_n(H^1(G))$ for $n \geq 3$; if $p = 2$, with an abuse of notation we set

$$\Lambda_\bullet(H^1(G)) = \frac{S_\bullet(H^1(G))}{(a^2 \mid a \in H^1(G))}.$$

We claim that

$$\mathrm{Ker}(\psi_2) \wedge H^1(G) = \Lambda_3(H^1(G)),$$

which implies that $\mathrm{Ker}(\psi_*)$ is the two-sided ideal of $\Lambda_\bullet(H^1(G))$ generated by $\mathrm{Ker}(\psi_2)$.

By (3.6), $\mathrm{Ker}(\psi_2)$ is the subspace of $\Lambda_2(H^1(G))$ generated by the elements

$$b_{ij} := a_i \wedge a_j - \alpha_{ij}(a_1 \wedge a_2) - \beta_{ij}(a_s \wedge a_t), \qquad 1 \leq i < j \leq d.$$

First, suppose that $s = 1$. Then one has $b_{ij} = a_i \wedge a_j - a_1 \wedge (\alpha_{ij}a_2 + \beta_{ij}a_t)$ for every $i < j$, and thus

$$\begin{aligned} a_1 \wedge b_{2,t} &= a_1 \wedge a_2 \wedge a_t - a_1 \wedge a_1 \wedge (\alpha_{2,t}a_2 + \beta_{2,t}a_t) \\ &= a_1 \wedge a_2 \wedge a_t - 0, \end{aligned}$$

so that $a_1 \wedge a_2 \wedge a_t \in \mathrm{Ker}(\psi_2) \wedge H^1(G)$. Now, for any value of $s \geq 1$ and for every $h \geq 3$, one has

$$\begin{aligned} a_2 \wedge b_{1,h} &= a_2 \wedge a_1 \wedge a_h - \alpha_{1,h} \cdot a_2 \wedge (a_1 \wedge a_2) - \beta_{1,h} \cdot a_2 \wedge (a_s \wedge a_t) \\ &= -a_1 \wedge a_2 \wedge a_h - 0 - \beta_{1,h}(a_2 \wedge a_s \wedge a_t). \end{aligned} \tag{3.7}$$

If $s = 1$ then by (3.7) $a_1 \wedge a_2 \wedge a_h$ lies in $\mathrm{Ker}(\psi_2) \wedge H^1(G)$, as $a_2 \wedge a_1 \wedge a_t$ does. If $s \geq 2$ then $(1, h) \prec (s, t)$; hence $\beta_{1,h} = 0$, and (3.7) yields $a_1 \wedge a_2 \wedge a_h \in \mathrm{Ker}(\psi_2) \wedge H^1(G)$. Similarly, for every $h = 1, \dots, d$, $h \neq s, t$, one has

$$\begin{aligned} a_t \wedge b_{s,h} &= a_t \wedge a_s \wedge a_h - \alpha_{s,h} \cdot a_t \wedge (a_1 \wedge a_2) - \beta_{s,h} \cdot a_t \wedge (a_s \wedge a_t) \\ &= -a_s \wedge a_t \wedge a_h - \alpha_{s,h}(a_1 \wedge a_2 \wedge a_t) - 0. \end{aligned}$$

Thus, $a_s \wedge a_t \wedge a_h \in \mathrm{Ker}(\psi_2) \wedge H^1(G)$, as $a_1 \wedge a_2 \wedge a_t \in \mathrm{Ker}(\psi_2) \wedge H^1(G)$ by 3.7. Finally, for every $1 \leq h < k < \ell \leq d$ one has

$$\begin{aligned} a_h \wedge b_{k\ell} &= a_h \wedge a_k \wedge a_\ell - \alpha_{k\ell} \cdot a_h \wedge (a_1 \wedge a_2) - \beta_{k\ell} \cdot a_h \wedge (a_s \wedge a_t) \\ &= a_h \wedge a_k \wedge a_\ell - \alpha_{k\ell}(a_1 \wedge a_2 \wedge a_h) - \beta_{k\ell}(a_s \wedge a_t \wedge a_h), \end{aligned}$$

and thus $a_h \wedge a_k \wedge a_\ell \in \ker(\psi_2) \wedge H^1(G)$. Therefore, $\mathrm{Ker}(\psi_2) \wedge H^1(G) = \Lambda_3(H^1(G))$, and this yields the claim.

3.3. Universally Koszul cohomology

The next result shows that the $\mathbb{F}_p$-cohomology of a quadratically defined two-relator pro-p group is universally Koszul.

THEOREM 3.7. *Let G be a finitely generated quadratically defined two-relator pro-p group. Then $H^\bullet(G)$ is universally Koszul.*

PROOF. Set $A_\bullet = H^\bullet(G)$, and $d = \dim(A_1)$. By Proposition 3.6, $A_\bullet$ is quadratic and graded-commutative (wedge-commutative if $p = 2$), and $A_n = 0$ for $n \geq 3$. Let $\mathscr{B} = \{a_1, \dots, a_d\}$ be a basis of A_1 as in §3.2. Thus, $\{a_1a_2, a_sa_t\}$ is a basis of A_2, and one has the relations (3.6).

Let I be an ideal of $A_\bullet$ lying in $\mathscr{L}(A_\bullet)$, $I \neq A_+$, and $b \in A_1 \setminus I_1$, and set $J = I : (b)$. Since $A_n = 0$ for $n \geq 3$, one has $A_2 \cdot (b) = 0 \subseteq I$, and thus $J_2 = A_2$. In order to show that $J \in \mathscr{L}(A_\bullet)$, we need to show that A_2 is generated by J_1, i.e., $J_1 \cdot A_1 = A_2$. Since $b^2 = 0$, one has $b \in J_1$. One has three cases: $\dim(b \cdot A_1) = 0$, 1 or 2.

Suppose first that $\dim(b \cdot A_1) = 0$, i.e., $ba = 0$ for every $a \in A_1$. Then $b \cdot A_1 = 0 \subseteq I_2$, and hence $A_1 \subseteq J_1$. Therefore, $J_1 \cdot A_1 = A_1 \cdot A_1 = A_2$.

Suppose now that $\dim(b\cdot A_1)=1$, i.e., $b\cdot A_1$ is generated by $\alpha a_1a_2+\beta a_sa_t$, for some $\alpha,\beta\in\mathbb{F}_p$, with α,β not both 0. Then

$$a_1b=\lambda_1(\alpha a_1a_2+\beta a_sa_t),\qquad a_2b=\lambda_2(\alpha a_1a_2+\beta a_sa_t)$$

for some $\lambda_1,\lambda_2\in\mathbb{F}_p$. If $\lambda_1=0$ then $a_1b=0\in I_2$, and $a_1\in J_1$; similarly, if $\lambda_2=0$ then $a_2\in J_1$. If $\lambda_1,\lambda_2\neq 0$, then $(a_1-\lambda_1/\lambda_2a_2)b=0\in I_2$, and $(a_1-\lambda_1/\lambda_2a_2)\in J_1$. In both cases, $a_1a_2\in J_1\cdot A_1$. Likewise,

$$a_sb=\mu_1(\alpha a_1a_2+\beta a_sa_t),\qquad a_tb=\mu_2(\alpha a_1a_2+\beta a_sa_t)$$

for some $\mu_1,\mu_2\in\mathbb{F}_p$. If $\mu_1=0$ then $a_sb=0\in I_2$, and $a_s\in J_1$; similarly, if $\mu_2=0$ then $a_t\in J_1$. If $\mu_1,\mu_2\neq 0$, then $(a_s-\mu_1/\mu_2a_t)b=0\in I_2$, and $(a_s-\mu_1/\mu_2a_t)\in J_1$. In both cases, $a_sa_t\in J_1\cdot A_1$. Therefore, $A_2\subseteq J_1\cdot A_1$, and this concludes the case $\dim(b\cdot A_1)=1$.

Finally, suppose that $\dim(b\cdot A_1)=2$, i.e., $b\cdot A_1=A_2$. Since $b\in J_1$, one has $A_2=b\cdot A_1\subseteq J_1\cdot A_1$. This concludes the proof.

Now we may prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a finitely generated pro-p group with at most two defining relations, and suppose that $H^\bullet(G)$ is quadratic, and moreover $a^2=0$ for every $a\in H^1(G)$ if $p=2$. If G has no relations, then it is a free pro-p group, and by Proposition 3.1 $H^\bullet(G)\simeq Q(V,V^{\otimes 2})$ for some finite vector space V. Hence, Example 2.7(a) yields the claim. If G is one-relator, then by [20, Proposition 4.2] G is as in Example 3.3, and thus $H^\bullet(G)$ is universally Koszul. Finally, if G is two-relator, we apply Theorem 3.7.

The next example shows that one may not extend Theorem 1.2 to finitely generated pro-p groups with quadratic $\mathbb{F}_p$-cohomology, cohomological dimension equal to 2 and more than two defining relations.

Example 3.8. Let G be a pro-p group with presentation

$$G=\langle x,y,z,t\mid [x,y]=[y,z]=[z,t]=1\rangle\,.$$

Then G is the pro-p completion of the *right-angled Artin pro-p group* associated to the graph Γ with vertices $\mathscr{V}=\{x,y,z,t\}$ and edges $\mathscr{E}=\{\{x,y\},\{y,z\},\{z,t\}\}$, i.e., Γ is a path of length 3, and Γ does not have the diagonal property. The cohomology algebra $H^\bullet(G)$ of G is isomorphic to the exterior Stanley-Reisner algebra $\Lambda_\bullet(\Gamma)$ (cf. [21, Theorem F]). In particular, $H^\bullet(G)$ is Koszul and $H^3(G)=0$, but $H^\bullet(G)$ is not universally Koszul by Example 2.9.

4. Maximal pro-p Galois groups

For a field $\mathbb{K}$, let $\mathcal{G}_{\mathbb{K}}$ denote the maximal pro-p Galois group of $\mathbb{K}$. If $\mathbb{K}$ contains a root of 1 of order p, then by Kummer theory one has an isomorphism

$$\mathbb{K}^\times/(\mathbb{K}^\times)^p \simeq H^1(\mathcal{G}_{\mathbb{K}}), \tag{4.1}$$

where $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ denotes the multiplicative group of $\mathbb{K}$. Moreover, let $\mathrm{Br}_p(\mathbb{K})$ denote the p-part of the Brauer group $\mathrm{Br}(\mathbb{K})$ of $\mathbb{K}$, i.e, $\mathrm{Br}_p(\mathbb{K})$ is the subgroup of $\mathrm{Br}(\mathbb{K})$ generated by all elements of order p. Then one has an isomorphism

$$\mathrm{Br}_p(\mathbb{K}) \simeq H^2(\mathcal{G}_{\mathbb{K}}) \tag{4.2}$$

(cf. [11, Theorem 6.3.4]).

The *mod-p Milnor K-ring* of $\mathbb{K}$ is the graded $\mathbb{F}_p$-algebra

$$K_\bullet^M(\mathbb{K})_{/p} = \bigoplus_{n\geq 0} K_n^M(\mathbb{K})_{/p} = \frac{T_\bullet(\mathbb{K}^\times)}{(\Omega)} \otimes_{\mathbb{Z}} \mathbb{F}_p,$$

where $T_\bullet(\mathbb{K}^\times)$ is the tensor algebra over $\mathbb{Z}$ generated by $\mathbb{K}^\times$, and (Ω) the two-sided ideal of $T_\bullet(\mathbb{K}^\times)$ generated by the elements $\alpha \otimes (1-\alpha)$ with $\alpha \in \mathbb{K}^\times \setminus \{1\}$ (see, e.g., [11, Definition 6.4.1]). Thus $K_\bullet^M(\mathbb{K})_{/p}$ is quadratic, with $K_1^M(\mathbb{K})_{/p} = \mathbb{K}^\times/(\mathbb{K}^\times)^p$. If $\mathbb{K}$ contains a root of 1 of order p, by the Rost-Voevodsky Theorem the isomorphism (4.1) induces an isomorphism of $\mathbb{F}_p$-algebras

$$K_\bullet^M(\mathbb{K})_{/p} \xrightarrow{\sim} H^\bullet(\mathcal{G}_{\mathbb{K}}) \tag{4.3}$$

(cf. [23], [25], see also [7, §24.3]), and thus $H^\bullet(\mathcal{G}_{\mathbb{K}})$ is quadratic. The following is a well-known fact for $\mathbb{F}_2$-cohomology of maximal pro-2 Galois groups of fields.

LEMMA 4.1. *Let $\mathbb{K}$ be a field containing $\sqrt{-1}$. Then the $\mathbb{F}_2$-cohomology algebra $H^\bullet(\mathcal{G}_{\mathbb{K}})$ of the maximal pro-2 Galois group of $\mathbb{K}$ is wedge-commutative.*

PROOF. By (4.3), it suffices to show that $K_\bullet^M(\mathbb{K})_{/2}$ is wedge-commutative. For $\alpha, \beta \in \mathbb{K}^\times$, set $\{\alpha\} = \alpha(\mathbb{K}^\times)^2 \in \mathbb{K}^\times/(\mathbb{K}^\times)^2$ and let $\{\alpha, \beta\}$ denote the image of $\alpha \otimes \beta$ in $K_2^M(\mathbb{K})_{/2}$. By definition, for every $\alpha \in \mathbb{K}^\times$ one has $\{\alpha\}\cdot\{\alpha\} = \{\alpha, -1\} \in K_2^M(\mathbb{K})_{/2}$. Hence, if $\sqrt{-1} \in \mathbb{K}$ then $-1 \in (\mathbb{K}^\times)^2$, i.e., $\{-1\}$ is trivial in $K_1^M(\mathbb{K})_{/2} = \mathbb{K}^\times/(\mathbb{K}^\times)^2$, and hence $\{\alpha\}\cdot\{\alpha\} = \{\alpha, -1\}$ is trivial in $K_2^M(\mathbb{K})_{/2}$.

From Theorem 1.2 one deduces the following.

COROLLARY 4.2. *Let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$), and suppose that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is finite. If $\mathrm{rk}(\mathrm{Br}_p(\mathbb{K})) \leq 2$, then $H^\bullet(\mathcal{G}_{\mathbb{K}})$ is universally Koszul.*

PROOF. By (4.1) and (3.2), the pro-p group $\mathcal{G}_{\mathbb{K}}$ is finitely generated. Moreover, by (4.2) and (3.3) $\mathcal{G}_K$ has at most two defining relations. Also, $H^\bullet(\mathcal{G}_K)$ is quadratic by the Rost-Voevodsky Theorem.

If $\mathrm{Br}_p(\mathbb{K})$ is trivial, then $H^n(\mathcal{G}_{\mathbb{K}}) = 0$ for all $n \geq 2$, namely, $\mathrm{cd}(\mathcal{G}_{\mathbb{K}}) = 1$, and $\mathcal{G}_{\mathbb{K}}$ is a free pro-p group by Proposition 3.1. If $\mathrm{rk}(\mathrm{Br}_p(\mathbb{K})) = 1$, then $\mathcal{G}_{\mathbb{K}}$ is one-relator, and if $p = 2$ then $H^\bullet(\mathcal{G}_{\mathbb{K}})$ is wedge-commutative by Lemma 4.1.

Finally, if $\mathrm{rk}(\mathrm{Br}_p(\mathbb{K})) = 2$, then $\mathcal{G}_{\mathbb{K}}$ is a two-relator pro-p group. If $p \neq 2$, then $\mathcal{G}_{\mathbb{K}}$ is quadratically defined by Proposition 3.6. If $p = 2$, then $H^\bullet(\mathcal{G}_{\mathbb{K}})$ is wedge-commutative by Lemma 4.1, and by Proposition 3.6 $\mathcal{G}_{\mathbb{K}}$ is quadratically defined. Altogether, Theorem 1.2 yields the claim.

The isomorphisms (3.3) and (4.2) imply that Corollary 4.2 is equivalent to Corollary 1.3.

REMARK 4.3. Let G be the pro-p group as in Example 2.9. It was recently shown by I. Snopce and P. Zalesskii that G does not occur as maximal pro-p Galois group $\mathcal{G}_{\mathbb{K}}$ for any field $\mathbb{K}$ containing a root of 1 of order p (cf. [22]). Therefore, Example 3.8 does not provide a counterexample to Conjecture 1.1.

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