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ULTRA-IRREDUCIBILITY OF INDUCED REPRESENTATIONS

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Abstract.

We study non-unitary representations of a Lie group, induced from a finite dimensional irreducible representation of a subgroup, and find criteria for such a representation to be ultra-irreducible. We apply our criteria to semi-direct products and to nilpotent groups.

I. Introduction.

In the present paper we prove irreducibility criteria for representations of Lie groups induced from finite dimensional representations. The representations need not be unitary and the representation spaces of the induced representations may in our setup be chosen among a variety of different function and distribution spaces. The notion of irreducibility that we primarily consider is that of ultra-irreducibility which is stronger than complete and topological irreducibility.

Let G be Lie group and let μ be a continuous irreducible representation of a closed subgroup K of G on a finite dimensional complex vector space F. Then G acts by left translations on the distribution space

$$\mathscr{D}'_\mu(G;F) := \{u \in \mathscr{D}'(G;F) \mid u(gk) = \mu(k^{-1})[u(g)] \text{ for } k \in K,\, g \in G\}.$$

By a representation of G induced from μ we mean the restriction of the left regular representation of G on $\mathscr{D}'_\mu(G;F)$ to an invariant subspace with a locally convex topology for which certain natural continuity and density conditions are satisfied (See Definition III.2).

We find general criteria for ultra-irreducibility (Theorem IV.2 and Corollary IV.3) and scalar irreducibility (Corollary IV.3) of such induced representations. (See Section II for irreducibility definitions). We apply the criteria to semi-direct products and to nilpotent groups.

If $G = NH$ is a semi-direct product of closed subgroups N and H, where N is normal and connected, we let the μ and K from above be of the form: $\mu = \chi \times \nu$

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and $K = NH_\chi$, where χ is a homomorphism of N into $\mathbf{C}\setminus\{0\}$ and ν is a finite dimensional irreducible representation of the stabilizer subgroup H_χ for χ in H. This is the setup considered, for compact H, by Thieleker [Th], Williams [Wi] and Rais [Ra 1,2], who studied the representations of G induced from μ with respect to topological irreducibility on the spaces $L^2_\mu(G;F) \simeq L^2_\nu(H;F)$ and $C_\mu(G;F) \simeq C_\nu(H;F)$ of square integrable and continuous functions, respectively.

Here we prove a result (Theorem V.1) which extends the fundamental criterion in [Wi; Theorem 4.11] on three accounts: It concludes ultra-irreducibility, not just topological irreducibility, it applies to more representation spaces, including $L^2_\nu(H;F)$ for compact H, and it does not require compactness of H. It is also more general than [St; Theorem III.3], which requires the representation space to contain $C^\infty_\mu(G;F)$.

We next turn to nilpotent Lie groups: For various reasons the Schrödinger representations, which are induced representations of the $(2n+1)$-dimensional Heisenberg group, have been studied and shown to be topologically (and in some case even ultra-) irreducible on the function and distribution spaces C^∞, C, C^∞_0, $\mathscr{D}'$, $\mathscr{S}'$, $B_{p,k}$ and L^p $(1 \leqq p < \infty)$ over $\mathbf{R}^n$. See Petrosyan [Pe], Litvinov [Li], Litvinov and Lomonosov [LL1], [LL2], Hole [Ho] and Poulsen [Po; Example 5.1]. In the setting of general nilpotent Lie groups Jacobsen and Stetkær [JS] and Jacobsen [Ja] proved topological and operator-irreducibility on the spaces $\mathscr{D}'_\mu(G;\mathbf{C})$ and $C^\infty_\mu(G;\mathbf{C})$, where μ is a character satisfying a polarization condition. The works [Ho] and [JS] solve particular cases of Helgason's general program as described in Chapter II §4 of [He].

These fragmentary facts are generalized by Theorems VI.1 and VI.2 below. Our results show in particular that the above representations all are ultra-irreducible. We proceed by outlining the two theorems:

Let G and its subgroup K be connected and simply connected nilpotent Lie groups. Then the representation μ reduces to a homomorphism of K into $\mathbf{C}\setminus\{0\}$. The Kirillov theory [Ki] tells that the unitary representation of G induced from a unitary μ is irreducible (topologically and hence ultra-), if and only if the Lie algebra of K is a polarization at the differential of μ (extended to a linear functional on the Lie algebra of G).

We show that this polarization condition, properly modified when μ is no longer unitary, suffices for irreducibility on a variety of other locally convex representation spaces: Scalar irreducibility holds for all the considered induced representation spaces and ultra-irreducibility if (apart from continuity and completeness conditions) either,

(a) the representation space is a $C^\infty(G/K)$-module (Theorem VI.1), or

(b) μ is unitary and the representation space consists of tempered distributions and contains the space of those smooth functions in $\mathscr{D}'_\mu(G;\mathbf{C})$ which decrease rapidly at infinity modulo K (Theorem VI.2).

Our proof of Theorem VI.1 builds on properties of the differentiated representation, established in the unitary case by Kirillov [Ki], while our proof of Theorem VI.2 uses properties of the integrated representation due to Howe [Hw].

The weakly tempered topologically irreducible locally convex G-modules considered in [Cl; Remarque 5.17] are representations of the type covered by our Theorem VI.2. In particular these modules are indeed even ultra-irreducible.

II. The ultraweak topology.

By a locally convex space we will always mean a locally convex, Hausdorff topological vector space over the field of complex numbers $\mathbf{C}$. The topological dual E' of a locally convex space E will be equipped with the strong dual topology, and the value of $e' \in E'$ at $e \in E$ is denoted $\langle e, e' \rangle$. If τ is a topology on E then E_τ denotes E with that topology.

If E and F are locally convex spaces we let $L(E, F)$, resp. $S(E, F)$, resp. $B(E, F)$ denote the vector space of those linear mappings of E into F which are continuous, resp. weakly continuous, resp. map bounded sets into bounded sets. As is well-known $L(E, F) \subset S(E, F) \subset B(E, F)$. Furthermore $L(E, F) = S(E, F)$ if E has the Mackey topology [Sc; IV.7.4], as in the case when E is barrelled [Sc; IV.3.4]; in particular if E is reflexive [Sc; IV.5.6]. We abbreviate $L(E, E)$ to $L(E)$ etc.

The *ultraweak topology* (uw) on $B(E, F)$ is the weak topology defined by the linear functionals

$$A \mapsto \sum_{i=1}^{\infty} \lambda_i \langle Au_i, u_i' \rangle \quad \text{on } B(E, F),$$

where the sequence $\{\lambda_i\}$ ranges over $l^1(\mathbf{N})$, the sequence $\{u_i\}$ over the bounded sequences in E, and $\{u_i'\}$ over the bounded sequences in F'. The ultraweak topology is clearly stronger than the weak operator topology (w) on $B(E, F)$, i.e. the weakest topology making the linear functionals $A \to \langle Au, u' \rangle$, $u \in E$, $u' \in F'$, continuous.

DEFINITION 1. A representation T of a group G on a locally convex space E, i.e. a homomorphism of G into the group of invertible elements of $S(E)$, is said to be *ultra-irreducible*, if the algebra spanned by $T(G)$ is dense in $S(E)_{\mathrm{uw}}$.

Ultra-irreducibility is a strong notion in the sense that it entails other forms of irreducibility: An ultra-irreducible representation is for example topologically completely irreducible and hence also topologically irreducible, operator irreducible and scalar irreducible (by the last notion we mean that the intertwining operators in $S(E)$ are multiples of the identity operator I_E on E).

The following lemmas will be convenient tools in dealing with the ultraweak topology. Their proofs are standard functional analysis and use the fact that strongly and weakly bounded subsets of E' coincide when E is a semi-complete (i.e. Cauchy sequences converge) locally convex space [Bo; III. §4.3].

LEMMA 2. *Let E and F be locally convex spaces.*

(α) *If F is barrelled then any continuous linear map of F into $B(E)_w$ will also be continuous into $B(E)_{uw}$.*

(β) *If E is semi-complete then any sequence in $S(E)$ which converges in $B(E)_w$ will also converge in $B(E)_{uw}$.*

If G is a Lie group we let $\mathscr{D}(G)$ denote the vector space of compactly supported C^∞-functions on G. When we consider integrated representations of $\mathscr{D}(G)$ it will be wrt. to some given left Haar measure on G.

LEMMA 3. *Let T be a strongly continuous representation of a Lie group G on a semi-complete locally convex space E. Then we have for the integrated representation T of $\mathscr{D}(G)$ that*

(α) $T(\phi) \in B(E)$ *for all* $\phi \in \mathscr{D}(G)$,

(β) span $\{T(g) \mid g \in G\}$ *and* $T(\mathscr{D}(G))$ *have the same closure in* $B(E)_{uw}$.

III. The induced representations.

Throughout this section we let G denote a Lie group which is countable at infinity, K a closed subgroup of G and μ a continuous representation of K on a finite dimensional complex vector space F. A special case is $\mu = \tau :=$ the trivial representation of K on $\mathbf{C}$. By $C^\infty(G; F)$ we denote the Fréchet space of F-valued C^∞-functions on G, by $\mathscr{D}(G; F)$ the LF-space of compactly supported functions in $C^\infty(G; F)$, and by $\mathscr{D}'(G; F)$, the space $L(\mathscr{D}(G), F)$ of F-valued distributions on G, equipped with the topology of bounded convergence. If $F = \mathbf{C}$ we will delete the symbol F from the notation. Function spaces are continuously imbedded into distribution spaces by means of fixed left Haar measures dg on G and dk on K.

The left and right actions of G on functions and distributions are denoted by L and R respectively, and the actions on distributions are defined so as to extend the ones on functions, i.e. $[L(g)u](\phi) := u(L(g^{-1})\phi)$ and $[R(g)u](\phi) := u(\Delta_G(g^{-1})R(g^{-1})\phi)$ for all $g \in G$, $u \in \mathscr{D}'(G; F)$, $\phi \in \mathscr{D}(G)$, where Δ_G is the modular function on G.

By help of μ we define an action of K on $\mathscr{D}'(G; F)$ by $[\mu(k)u](\phi) := \mu(k)[u(\phi)]$ for $k \in K$, $u \in \mathscr{D}'(G; F)$, $\phi \in \mathscr{D}(G)$, and introduce the space

$$\mathscr{D}'_\mu(G; F) := \{u \in \mathscr{D}'(G; F) \mid R(k)u = \mu(k^{-1})u \text{ for all } k \in K\}.$$

We put $C^\infty_\mu(G; F) := C^\infty(G; F) \cap \mathscr{D}'_\mu(G; F)$ and let $\mathscr{D}_\mu(G; F)$ denote the functions in $C^\infty_\mu(G; F)$ of compact support modulo K.

The spaces $\mathscr{D}'_\mu(G; F)$ and $C^\infty_\mu(G; F)$ are given the topologies from $\mathscr{D}'(G; F)$ and

$C^\infty(G;F)$ respectively. $\mathscr{D}_\mu(G;F)$ carries the inductive limit topology from the family of subspaces $\{\phi \in C_\mu^\infty(G;F) \mid \operatorname{supp} \phi \subset CK\}$ of $C^\infty(G;F)$, where C ranges over the compact subsets of G. Note for later use that $\mathscr{D}_\mu(G;F)$ is reflexive as a strict inductive limit of a sequence of reflexive Fréchet spaces [Sc; IV.5.8], and that $\mathscr{D}_\mu = C_\mu^\infty$ if G/K is compact. The spaces are all complete. Furthermore they are invariant under L, and L restricts to continuous representations of G on them.

We shall usually abbreviate $\mathscr{D}'_\mu(G;F)$ by $\mathscr{D}'_\mu$, $C_\mu^\infty(G;F)$ by C_μ^∞ etc. We will also write $C_\tau^\infty(G) = C^\infty(G/K)$ and $\mathscr{D}_\tau(G) = \mathscr{D}(G/K)$, viewing the functions on the right hand sides as functions on G.

We adapt the notion of a normal space of distributions (Cf. [Tr; Def 28.1 p. 302]) to the situation at hand:

DEFINITION 1. A *normal* subspace of $\mathscr{D}'_\mu$ is a locally convex space E such that $\mathscr{D}_\mu \subset E \subset \mathscr{D}'_\mu$ with weakly continuous and linear inclusions and with $\mathscr{D}_\mu$ dense in E.

DEFINITION 2. By a *representation of G induced from* μ we mean a restriction L_E of the left regular representation L of G to an L-invariant normal subspace E of $\mathscr{D}'_\mu$ with the property that L_E is a strongly continuous representation of G on E.

So we require of a representation space that it must contain $\mathscr{D}_\mu$, which then is minimal among the representation spaces.

There is a natural module structure in the given setup: For each $\psi \in C^\infty(G/K)$ the operator $M(\psi)$ of multiplication by ψ is a continuous endomorphism of $\mathscr{D}'_\mu$, that restricts to continuous endomorphisms of C_μ^∞ and $\mathscr{D}_\mu$. If $\psi \in \mathscr{D}(G/K)$, then $M(\psi)$ maps C_μ^∞ continuously into $\mathscr{D}_\mu$.

Many of the important spaces of distributions, e.g. all the local ones, are invariant under multiplication by C^∞-functions. Here we may regularize the distributions prior to multiplying. Technically what we need is

DEFINITION 3. A normal subspace E of $\mathscr{D}'_\mu$ is said to be a C^∞-*semimodule*, if

(α) for each $\psi \in C^\infty(G/K)$ and $\phi \in \mathscr{D}(G)$ the map $M(\psi)L(\phi)$ leaves E invariant and belongs as an operator on E to $B(E)$,

(β) the corresponding map $\psi \mapsto M(\psi)L(\phi)$ is continuous from $C^\infty(G/K)$ into $B(E)_w$ for each fixed $\phi \in \mathscr{D}(G)$.

In general it is a severe restriction on E to demand that it is a C^∞-semimodule, because the condition (α) excludes space like L^p with growth conditions at infinity. The continuity condition (β) is by the closed graph theorem automatically satisfied for the commonly met spaces of distributions.

A normal subspace E of $\mathscr{D}'_\mu$ is a C^∞-semimodule if for instance $C_\mu^\infty \subset E$ with weakly continuous inclusion. This condition is always satisfied if G/K is compact, since in that case $\mathscr{D}_\mu = C_\mu^\infty$.

We conclude this section by identifying the dual space of $\mathscr{D}_\mu(G;F)$ and introducing a certain canonical bilinear form. To that end we need the continuous representation μ' of K on F' defined by $\mu'(k) := \chi_\Delta(k^{-1})\mu(k^{-1})^t$ for $k \in K$, where χ_Δ denotes the quotient between the modular functions on K and G.

There is a natural topological isomorphism of the strong dual $\mathscr{D}(G;F)'$ onto $\mathscr{D}'(G;F')$ (See [Wa; Appendix 2.31]). We denote the corresponding bilinear form on $\mathscr{D}(G;F) \times \mathscr{D}'(G;F')$ by $\langle\cdot,\cdot\rangle_G$. It is L-invariant.

Let τ_μ denote the continuous linear map of $\mathscr{D}(G;F)$ onto $\mathscr{D}_\mu(G;F)$ given by

$$(\tau_\mu(\phi))(g) = \int_K \mu(k)\,\phi(gk)\,dk \quad \text{for } \phi \in \mathscr{D}(G;F) \text{ and } g \in G,$$

(Cf. [Wa; 5.1.1.4]).

Proposition 4. *There is an L-invariant bilinear form $\langle\cdot,\cdot\rangle_\mu$ on $\mathscr{D}_\mu(G;F) \times \mathscr{D}'_{\mu'}(G;F')$, which identifies $\mathscr{D}_\mu(G;F)$ and $\mathscr{D}'_{\mu'}(G;F')$ with the strong duals of one another and is characterized by*

$$\langle\tau_\mu(\phi),u\rangle_\mu = \langle\phi,u\rangle_G \quad \text{for } \phi \in \mathscr{D}(G;F) \text{ and } u \in \mathscr{D}'_{\mu'}(G;F').$$

We call $\langle\cdot,\cdot\rangle_\mu$ the *canonical bilinear form.*

Proof. Let $m_\theta: \mathscr{D}_\mu(G;F) \mapsto \mathscr{D}(G;F)$ denote the continuous linear map of multiplication by a C^∞-function $\theta: G \mapsto [0,\infty[$ satisfying (i) $CK \cap \operatorname{supp}\theta$ is compact for each compact subset C of G, and (ii) $\int_K \theta(gk)\,dk = 1$ for all $g \in G$, (Cf [Wa; A.1.1]).

Easy computations show that $\tau_\mu m_\theta$ is the identity on $\mathscr{D}_\mu(G;F)$, that the adjoint τ_μ^t maps $\mathscr{D}_\mu(G;F)'$ into $\mathscr{D}'_{\mu'}(G;F')$, when $\mathscr{D}(G;F)'$ is identified with $\mathscr{D}'(G,F')$, and that $\langle m_\theta\tau_\mu\phi,u\rangle_G = \langle\phi,u\rangle_G$ for all $\phi \in \mathscr{D}(G;F)$ and $u \in C^\infty_{\mu'}(G;F')$. Since $C^\infty_{\mu'}(G,F')$ is dense in $\mathscr{D}'_{\mu'}(G;F')$ this last identity holds for all $u \in \mathscr{D}'_{\mu'}(G;F')$.

This proves that τ_μ^t is a topological isomorphism of $\mathscr{D}_\mu(G;F)'$ onto $\mathscr{D}'_{\mu'}(G;F')$ with inverse given by the restriction of m_θ^t. The desired form $\langle\cdot,\cdot\rangle_\mu$ may now be defined by

$$\langle\phi,u\rangle_\mu := \langle\theta\phi,u\rangle_G = \langle\phi,(\tau_\mu^t)^{-1}u\rangle_G \quad \text{for } \phi \in \mathscr{D}_\mu(G;F) \text{ and } u \in \mathscr{D}'_{\mu'}(G;F').$$

It is L-invariant, since $\langle\cdot,\cdot\rangle_G$ is L-invariant and τ_μ commutes with L. It identifies $\mathscr{D}'_{\mu'}(G;F')$ with the dual of $\mathscr{D}_\mu(G;F)$ by continuity of τ_μ^t and m_θ^t, and it identifies $\mathscr{D}_\mu(G,F)$ with the strong dual of $\mathscr{D}'_{\mu'}(G;F')$ by reflexivity of $\mathscr{D}_\mu(G;F)$.

The canonical form $\langle\cdot,\cdot\rangle_\mu$ restricts to a separately continuous L-invariant bilinear form on $\mathscr{D}_\mu \times \mathscr{D}_{\mu'}$. If up to a constant scalar factor this restriction is the only such form we say that the canonical form is *unique*.

Proposition 5. (α) *The canonical form is unique if and only if every continuous* *linear L-intertwining map from $\mathscr{D}_\mu$ into $\mathscr{D}'_\mu$ is proportional to the inclusion map i: $\mathscr{D}_\mu \to \mathscr{D}'_\mu$.*

(β) *If the canonical form is unique then every representation of G induced from μ is scalar irreducible.*

PROOF. (α) There is a linear bijection $B \mapsto A$ of the space of separately continuous L-invariant bilinear forms on $\mathscr{D}_\mu \times \mathscr{D}_{\mu'}$ onto the space $S(\mathscr{D}_\mu, \mathscr{D}'_\mu)^G = L(\mathscr{D}_\mu, \mathscr{D}'_\mu)^G$ of intertwining operators given by $B(\phi, \psi) = \langle \psi, A\phi \rangle_{\mu'}$ for all $\phi \in \mathscr{D}_\mu$ and $\psi \in \mathscr{D}_{\mu'}$. Moreover $\langle \psi, \phi \rangle_{\mu'} = \langle \phi, \psi \rangle_\mu$ for all $\phi \in \mathscr{D}_\mu$ and $\psi \in \mathscr{D}_{\mu'}$.

(β) By restricting an intertwining operator $T \in S(E)$ to the dense subspace $\mathscr{D}_\mu$ of the representation space E we get a map $T_0 \in L(\mathscr{D}_\mu, \mathscr{D}'_\mu)^G$. By ($\alpha$) T_0 is a multiple of i, so T is a multiple of the identity on E.

IV. Irreducibility criteria

Let G, K and μ be as in Section III. The operator $L(\phi)$, $\phi \in \mathscr{D}(G)$, maps $\mathscr{D}'_\mu$ continuously into C^∞_μ, so for $\psi \in \mathscr{D}(G/K)$ the composite operator $M(\psi)L(\phi)$ maps $\mathscr{D}'_\mu$ continuously into $\mathscr{D}_\mu$. The technical key to our irreducibility results is the following theorem on the space Λ spanned by such operators.

THEOREM 1. *Let the representation μ be irreducible, and put*

$$\Lambda := \operatorname{span}\{M(\psi)L(\phi) \mid \psi \in \mathscr{D}(G/K) \text{ and } \phi \in \mathscr{D}(G)\} \subset L(\mathscr{D}'_\mu, \mathscr{D}_\mu).$$

Then

(α) *Λ is dense in $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)_{\mathrm{uw}}$.*

(β) *If E is a normal subspace of $\mathscr{D}'_\mu$ and the identity operator I_E on E belongs to the closure of $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$ in $S(E)_{\mathrm{uw}}$, then the set of operators $\Lambda|_E$ is a dense subspace $S(E)_{\mathrm{uw}}$.*

(γ) *Any $A \in S(\mathscr{D}_\mu, \mathscr{D}'_\mu) = L(\mathscr{D}_\mu, \mathscr{D}'_\mu)$ which commutes with each of the operators from Λ is a constant multiple of the inclusion map i: $\mathscr{D}_\mu \to \mathscr{D}'_\mu$.*

PROOF. (α) Let $\{u_i\}$ and $\{u'_i\}$ be bounded sequences of $\mathscr{D}'_\mu(G; F)$ and $\mathscr{D}'_{\mu'}(G; F') = \mathscr{D}'_\mu(G; F)'$ respectively, and let $\{\lambda_i\} \in l^1(\mathsf{N})$. We shall prove that if

(1) $$\sum_{i=1}^{\infty} \lambda_i \langle \psi L(\phi) u_i, u'_i \rangle_\mu = 0 \quad \text{for all } \psi \in \mathscr{D}(G/K),\ \phi \in \mathscr{D}(G)$$

then

(2) $$\sum_{i=1}^{\infty} \lambda_i \langle A u_i, u'_i \rangle_\mu = 0 \quad \text{for all } A \in L(\mathscr{D}'_\mu, \mathscr{D}_\mu).$$

We assert that (1) is equivalent to

(3) $$\sum_{i=1}^{\infty} \lambda_i \langle x, (L(\phi')u'_i)(g') \rangle_{F \times F'} u_i = 0 \quad \text{for all } \phi' \in \mathscr{D}(G),\ g \in G,\ x \in F,$$

where the series in (3) converges in $\mathscr{D}'_\mu$ by completeness of $\mathscr{D}'_\mu$.

Now, replacing in (1) ψ and ϕ by their translates $L(g^{-1})\psi$ and $L(g^{-1})\phi$ and using that $L(g^{-1})M(\psi)L(\phi) = M(L(g^{-1})\psi)\,L(L(g^{-1})\phi)$ we get by the L-invariance of $\langle\cdot,\cdot\rangle_\mu$ that (1) is equivalent to

$$(4) \qquad \sum_{i=1}^{\infty} \lambda_i \langle \psi L(\phi)u_i, L(\phi')u'_i\rangle_\mu = 0 \quad \text{for all } \psi \in \mathscr{D}(G/K),\ \phi, \phi' \in \mathscr{D}(G).$$

Given $f \in C^\infty_\mu(G; F)$ and $f' \in C^\infty_{\mu'}(G; F')$ the function $\langle f(\cdot), f'(\cdot)\rangle_{F\times F'}$ belongs to $C^\infty_{\tau'}(G)$, so that for any $\psi \in \mathscr{D}(G/K)$ (Cf. the proof of Proposition 4)

$$\langle \psi f, f'\rangle_\mu = \langle \theta\psi f, f'\rangle_G = \int_G \theta(g)\,\psi(g)\,\langle f(g), f'(g)\rangle_{F\times F'}\,dg = \langle \psi, \langle f(\cdot), f'(\cdot)\rangle_{F\times F'}\rangle_\tau.$$

Hence the left hand side of (4) equals

$$\sum_{i=1}^{\infty} \lambda_i \langle \psi, \langle (L(\phi)u_i)(\cdot), (L(\phi')u'_i)(\cdot)\rangle_{F\times F'}\rangle_\tau = \langle \psi, \sum_{i=1}^{\infty} \lambda_i \langle (L(\phi)u_i)(\cdot), (L(\phi')u'_i)(\cdot)\rangle_{F\times F'}\rangle_\tau,$$

where the last series converges in $C^\infty_{\tau'}(G)$. By the non-degeneracy of $\langle\cdot,\cdot\rangle_\tau$, (4) is therefore equivalent to

$$(5) \quad \sum_{i=1}^{\infty} \lambda_i \langle (L(\phi)u_i)(g), (L(\phi')u'_i)(g)\rangle_{F\times F'} = 0 \quad \text{for all } \phi,\ \phi' \in \mathscr{D}(G) \text{ and } g \in G.$$

Replacing ϕ' by $L(gkg'^{-1})\phi'$, where $k \in K$ and $g' \in G$, and using that $L(\phi')u'_i \in C^\infty_{\mu'}(G; F')$ we get that (5) is equivalent to

$$(6) \quad \sum_{i=1}^{\infty} \lambda_i \langle (L(\phi)u_i)(g), \mu'(k)(L(\phi')u'_i)(g')\rangle_{F\times F'} = 0 \quad \text{for all } \phi,\ \phi' \in \mathscr{D}(G), g, g' \in G, k \in K.$$

Via Burnside's theorem [μ' is a finite dimensional irreducible representation, since so is μ] (6) is equivalent to

$$\sum_{i=1}^{\infty} \lambda_i \langle (L(\phi)u_i)(g),\ \langle x, (L(\phi')u'_i)(g')\rangle_{F\times F'} x'\rangle_{F\times F'} = 0$$
$$\text{for all } \phi,\ \phi' \in \mathscr{D}(G), g, g' \in G, x \in F, x' \in F',$$

which again is equivalent to

$$(7) \quad \sum_{i=1}^{\infty} \lambda_i \langle x, (L(\phi')u'_i)(g')\rangle_{F\times F'} L(\phi)u_i = 0 \quad \text{for all } \phi, \phi' \in \mathscr{D}(G), g' \in G, x \in F.$$

Being continuous $L(\phi)$ may be pulled outside the summation, so (7) is equivalent to (3). This proves the asserted equivalence between (1) and (3).

Given (1) we apply $A \in L(\mathscr{D}'_{\mu'}, \mathscr{D}_\mu)$ to (3) and get

$$(8) \qquad \sum_{i=1}^{\infty} \lambda_i \langle x, (L(\phi')u'_i)(g')\rangle_{F\times F'} Au_i = 0 \quad \text{for all } \phi' \in \mathscr{D}(G),\ g' \in G,\ x \in F.$$

Since $\{Au_i\}$ is bounded in $\mathscr{D}_\mu$, and hence in $\mathscr{D}'_\mu$, we obtain from (8), via the fact that (3) implies (1), that

$$\text{(9)} \qquad \sum_{i=1}^{\infty} \lambda_i \langle \psi L(\phi) A u_i, u_i' \rangle_\mu = 0 \quad \text{for all } \psi \in \mathscr{D}(G/K),\ \phi \in \mathscr{D}(G).$$

We may choose a sequence $\{\psi_j\} \subset \mathscr{D}(G/K)$ such that $M(\psi_j) \to I$ in $L(\mathscr{D}_\mu)_{\mathrm{uw}}$. Also, the closure of $\{L(\phi) \mid \phi \in \mathscr{D}(G)\}$ in $L(\mathscr{D}_\mu)_{\mathrm{uw}}$ contains I. Hence (9) implies (2). This proves (α).

(β) Since the restriction map $A \mapsto A|_E$ is continuous from $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)_{\mathrm{uw}}$ into $S(E)_{\mathrm{uw}}$, it suffices by (α) to prove that the subspace $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)|_E$ is dense in $S(E)_{\mathrm{uw}}$.

To that end let $T \in S(E)$ be arbitrary. Now $BTC \in S(\mathscr{D}'_\mu, \mathscr{D}_\mu) = L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$ for all $B, C \in L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$. By hypothesis we may let B and C converge in $S(E)_{\mathrm{uw}}$ towards the identity operator on E. Composition in $S(E)_{\mathrm{uw}}$ is separately continuous, so T lies in the closure of $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)|_E$ in $S(E)_{\mathrm{uw}}$.

(γ) Since Λ is dense in $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)_{\mathrm{uw}}$ the relation $ACi = iCA$ holds by continuity for all $C \in L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$. This implies that A is a multiple of i: Choose $\psi \in \mathscr{D}_\mu$ and $\xi \in (\mathscr{D}'_\mu)'$ with $\langle i\psi, \xi \rangle = 1$. Let $\phi \in \mathscr{D}_\mu$ and put $C = \langle \cdot, \xi \rangle \phi \in L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$. Then $A\phi = A(\langle i\psi, \xi \rangle \phi) = ACi\psi = iCA\psi = \langle A\psi, \xi \rangle i\phi$.

An immediate corollary is the following

THEOREM 2. *General Irreducibility Criterion.*

Let the representation μ be irreducible, and let L_E be a representation induced from μ on a semi-complete space E. If

(i) *The identity I_E on E belongs to the closure of $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$ in $S(E)_{\mathrm{uw}}$, and*

(ii) *The operators $M(\psi)L(\phi)$, for all $\psi \in \mathscr{D}(G/K)$ and $\phi \in \mathscr{D}(G)$, belong to the closure of $L(\mathscr{D}(G))$ in $B(E)_{\mathrm{uw}}$,*

then L_E is ultra-irreducible.

COROLLARY 3. *Let μ be irreducible and assume that there is a dense subset $\mathscr{A}$ of $C^\infty(G/K)$ such that*

(*) $\quad M(\psi)L(\phi) \in L(\mathscr{D}(G))$ *(as operators on $\mathscr{D}'_\mu$)* for all $\psi \in \mathscr{A}$, $\phi \in \mathscr{D}(G)$.

Then any representation of G induced from μ and acting on a semi-complete C^∞-semimodule E is ultra-irreducible.

Furthermore the canonical bilinear form $\langle \cdot, \cdot \rangle_\mu$ is unique, and so any representation of G induced from μ is scalar irreducible.

PROOF. To prove ultra-irreducibility we verify (i) and (ii) of Theorem 2. Note that since E is a C^∞-semimodule the map $\psi \mapsto M(\psi)L(\phi)|_E$ is continuous from $C^\infty(G/K)$ into $B(E)_{\mathrm{uw}}$ (via Lemma II.2) for each $\phi \in \mathscr{D}(G)$.

(i): Choose a sequence $\{\psi_j\}$ in $\mathscr{D}(G/K)$ such that $\psi_j \to 1$ in $C^\infty(G/K)$. Then for each $\phi \in \mathscr{D}(G)$, $\{M(\psi_j)L(\phi)\}$ is a sequence in $L(\mathscr{D}'_\mu, \mathscr{D}_\mu)$ such that $M(\psi_j)L(\phi)|_E \to L(\phi)|_E$ in $B(E)_{\mathrm{uw}}$. Since E is semi-complete Lemma II.3 implies that the closure of

$L(\mathscr{D}(G))|_E = L_E(\mathscr{D}(G))$ in $B(E)_{uw}$ contains I_E. This proves (i).

(ii): The hypothesis (*) implies by the density of $\mathscr{A}$ that the closure of $L_E(\mathscr{D}(G))$ in $B(E)_{uw}$ contains the operators $M(\psi)L(\phi)|_E$ for all $\psi \in C^\infty(G/K) \supset \mathscr{D}(G/K)$, $\phi \in \mathscr{D}(G)$. Hence (ii) holds.

The final statement follows by Proposition III.5 if any $A \in L(\mathscr{D}_\mu, \mathscr{D}'_\mu)$ commuting with L is a multiple of the inclusion map $i: \mathscr{D}_\mu \to \mathscr{D}'_\mu$. Now, such A commutes with $L(\theta)$ for all $\theta \in \mathscr{D}(G)$, so by assumption (*) A commutes with $M(\psi)L(\phi)$ for all $\psi \in \mathscr{A}$ and $\phi \in \mathscr{D}(G)$. Since $\mathscr{A}$ is dense in $C^\infty(G/K)$, this extends to all $\psi \in C^\infty(G/K) \supset \mathscr{D}(G/K)$. The conclusion is now a consequence of Theorem 1(γ).

As the proof of Corollary 3 shows, the statement of scalar irreducibility requires only that the representation space E is invariant, not that L_E is a representation or that it is strongly continuous.

V. Applications to semi-direct products.

Let G be a Lie group, countable at infinity, and assume that $G = NH$ and $N \cap H = \{1\}$, where N and H are closed subgroups of G with N normal and connected. We let $\mathfrak{n}$ denote the Lie algebra of N.

Let $\chi: N \to \mathbf{C}\setminus\{0\}$ be a continuous homomorphism, not necessarily unitary. Let ν be a continuous irreducible representation of the stability subgroup H_χ for χ in H on a finite dimensional complex vector space F. Then $\mu := \chi \times \nu$ is a continuous, irreducible representation of the stability subgroup $G_\chi = NH_\chi$ on F.

The representation space $\mathscr{D}'_\mu(G, F)$ of the corresponding induced representation L is topologically isomorphic to

$$\mathscr{D}'_\nu(H;F) = \{u \in \mathscr{D}'(H;F) \mid u(hh_\chi) = \nu(h_\chi^{-1})[u(h)] \quad \text{for } h \in H, h_\chi \in H_\chi\},$$

and L transferred to $\mathscr{D}'_\nu(H;F)$ has the form

$$[L(nh)f](h_1) = \chi(h_1^{-1}nh_1)f(h^{-1}h_1) \quad \text{for } n \in N,\ h, h_1 \in H \text{ and } f \in \mathscr{D}'_\nu(H;F).$$

In this setting topological irreducibility of the induced representation has been studied for compact H on the subspaces $L^2_\nu(H;F)$ and $C_\nu(H;F)$ of $\mathscr{D}'_\nu(H;F)$ in e.g. [Th], [Wi], [Ra1] and [Ra2], and on other subspaces and with respect to ultra-irreducibility in [St].

The space

$$\mathscr{A}_\chi := \operatorname{span}\{h \mapsto e^{d\chi(\mathrm{Ad}(h^{-1})Z)} \mid Z \in \mathfrak{n}^{\mathbf{C}}\}$$

is an H-invariant subalgebra of $\mathbf{C}^\infty(H/H_\chi)$ containing the constants and separating the points in H/H_χ (See [Wi; Proposition 4.4]). Since it is H-invariant, it is dense in $C^\infty(H/H_\chi)$ iff it is dense in $C(H/H_\chi)$.

THEOREM 1. *Let ν be irreducible and assume that $\mathscr{A}_\chi$ is dense in $C^\infty(H/H_\chi)$.*

Then any representation of G induced from $\chi \times \nu$ *on a semi-complete* C^∞*-semimodule is ultra-irreducible. If* H/H_χ *is compact any induced representation with semi-complete representation space is ultra-irreducible.*

Furthermore the canonical bilinear form $\langle \cdot, \cdot \rangle_\mu$ *is unique, and so any representation of G induced from* $\chi \times \nu$ *is scalar irreducible.*

PROOF. Put $\chi_n(g) := \chi(g^{-1}ng)$ for $n \in N$, $g \in G$. Then $M(\chi_n)L(\phi)u = L(n)L(\phi)u = L(L(n)\phi)u$ for all $n \in N$, $u \in \mathscr{D}'_{\chi \times \nu}$. Hence the condition (*) of Corollary IV.3 holds with $\mathscr{A} := \operatorname{span}\{\chi_n \mid n \in N\}$. Since N is connected, $\mathscr{A}$ equals $\operatorname{span}\{\chi_n \mid n \in \exp(\mathfrak{n})\}$, so by analyticity of the map

$$Z \in \mathfrak{n}^{\mathrm{C}} \mapsto (g \mapsto e^{d\chi(\mathrm{Ad}(g^{-1})Z)}) \in C^\infty(G/G_\chi),$$

$\mathscr{A}$ has the same closure in $C^\infty(G/G_\chi) \simeq C^\infty(H/H_\chi)$ as $\mathscr{A}_\chi$, so $\mathscr{A}$ is dense in $C^\infty(G/G_\chi)$. If H/H_χ is compact then any semi-complete representation space is a C^∞-semimodule.

If $\nu = \tau$ the condition that $\mathscr{A}_\chi$ is dense in $C^\infty(H/H_\chi)$ is necessary for topological (but not for scalar) irreducibility of the induced representation L (Cf. Wi; Theorem 5.10]). Fell's example (See [Th; §9] or [Wi; p. 82]) shows that L is not always topologically irreducible. Criteria for the density of $\mathscr{A}_\chi$ in $C^\infty(H/H_\chi)$ can be found in [Wi; Section 5], [St; Section IV] and [Ra 1] (or the more detailed version [Ra2]). It is easy to see that $\mathscr{A}_\chi$ is self-adjoint and hence dense, if χ is unitary.

Theorem 1 generalizes, as mentioned in the introduction, the fundamental irreducibility criterion of [Wi; Theorem 4.11] and also [St; Theorem III.3].

EXAMPLE. The $(ax + b)$-group.

The $(ax + b)$-group is the semi-direct product $G = N \times_s H = \mathrm{R} \times_s \mathrm{R}^+$ with composition rule $(b_1, a_1)(b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2)$. Any continuous homomorphism χ: $N \to \mathrm{C}\setminus\{0\}$ is of the form $\chi(b, 1) = e^{\alpha b}$ for some $\alpha \in \mathrm{C}$.

If $\alpha \neq 0$, then H_χ consists of the identity alone, so ν is the trivial representation, the representation space reduces to $\mathscr{D}'_\nu(H) = \mathscr{D}'(H) = \mathscr{D}'(\mathrm{R}^+) \simeq \mathscr{D}'(\mathrm{R})$, and the corresponding representation on $\mathscr{D}'(\mathrm{R})$ is given by

$$(*) \quad [L((b, a))u](x) = e^{\alpha b e^{-x}} u(x - \log a) \quad \text{for } (b, a) \in G,\ u \in \mathscr{D}'(\mathrm{R}) \text{ and } x \in \mathrm{R},$$

an expression which is well known from the unitary theory. Moreover

$$\mathscr{A}_\chi = \operatorname{span}\{(1, a) \mapsto e^{\alpha a^{-1} z} \mid z \in \mathrm{C}\} \subset C^\infty(H/H_\chi),$$

so $\mathscr{A}_\chi$ is self-adjoint and hence dense in $C^\infty(H/H_\chi)$. By Theorem V.1, (*) therefore defines ultra-irreducible representations on the subspaces $\mathscr{D}'(\mathrm{R})$, $C_0^\infty(\mathrm{R})$, $C^\infty(\mathrm{R})$, $C(\mathrm{R})$ etc. of $\mathscr{D}'(R)$.

In the case of a unitary χ, (*) defines representations of G on the spaces $L^p(\mathrm{R})$ for

$1 \leq p < \infty$. Using Theorem IV.1 (β) it can be proved that these representations are ultra-irreducible as well. To the best of our knowledge this has been shown earlier only in the Hilbert space case $p = 2$.

VI. Applications to nilpotent Lie groups.

In this section G will denote a connected, simply connected, real and nilpotent Lie group with Lie algebra $\mathfrak{g}$ and K will be an analytic subgroup of G with Lie algebra $\mathfrak{k}$.

Let χ: $K \to \mathbf{C} \setminus \{0\}$ be a continuous homomorphism and assume that the following *maximality condition* is satisfied: There is a complex linear extension α: $\mathfrak{g}^{\mathbf{C}} \to \mathbf{C}$ of the differential $d\chi$: $\mathfrak{k} \to \mathbf{C}$ of χ such that

$$\mathfrak{k}^{\mathbf{C}} = \{X \in \mathfrak{g}^{\mathbf{C}} \mid \alpha([X, \mathfrak{k}^{\mathbf{C}}]) = \{0\}\},$$

where superscript C means complexification.

This maximality condition is known to be both necessary and sufficient for topological and scalar irreducibility of the induced representations of G on the spaces $C_\chi^\infty(G)$ and $\mathscr{D}'_\chi(G)$, cf. [Ja].

The representations induced from χ will be dealt with in their standard realizations which are constructed as follows: Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ with the property that $\mathfrak{g}_i := \operatorname{span}\{X_{i+1}, \ldots, X_n\} + \mathfrak{k}$ is an ideal in $\mathfrak{g}_{i-1}$ for each $i = 1, \ldots, n$. Then the map

$$(x_1, \ldots, x_n, k) \mapsto \exp(x_1 X_1) \ldots \exp(x_n X_n) k$$

is a diffeomorphism of $\mathbf{R}^n \times K$ onto G, and the map r: $f \mapsto f|_{\mathbf{R}^n}$ of restriction to $\mathbf{R}^n = \mathbf{R}^n \times \{e\} \subset G$ defines topological vector space isomorphisms of the spaces $\mathscr{D}'_\chi(G)$, $C_\chi^\infty(G)$ and $\mathscr{D}_\chi(G)$ onto $\mathscr{D}'(\mathbf{R}^n)$, $C^\infty(\mathbf{R}^n)$ and $\mathscr{D}(\mathbf{R}^n)$ respectively.

The induced representation L of G on $\mathscr{D}'_\chi(G)$ is via r equivalent to the representation $\tilde{L}$ on $\mathscr{D}'(\mathbf{R}^n)$ given by

$$\text{(1)} \qquad [\tilde{L}(g)u](x) = \chi^{-1}(\kappa(g^{-1}, x))u(g^{-1} \cdot x) \quad \text{for } g \in G,\ u \in \mathscr{D}'(\mathbf{R}^n),\ x \in \mathbf{R}^n,$$

where $(g, x) \mapsto g \cdot x$ is the canonical action of G on $\mathbf{R}^n \simeq G/K$ and κ: $G \times \mathbf{R}^n \to K$ is the map defined by $gx = (g \cdot x)\kappa(g, x)$ for all $g \in G$, $x \in \mathbf{R}^n \subset G$.

THEOREM 1. *Let G and χ be as described above. Then any representation of G induced from χ acting on a semi-complete C^∞-semimodule is ultra-irreducible.*

Furthermore the canonical bilinear form $\langle \cdot, \cdot \rangle_\chi$ is unique, and so any representation of G induced from χ is scalar irreducible.

PROOF. We need only verify the condition (*) of Corollary IV.3. Let $\mathscr{U}(\mathfrak{g})^{\mathbf{C}}$ denote the complexified universal enveloping algebra of $\mathfrak{g}$. If $\tilde{L}$ is a realization of $L|_{\mathscr{D}'_\chi}$ on $\mathscr{D}'(\mathbf{R}^n)$ as constructed above, then $d\tilde{L}(D)\tilde{L}(\phi) = \tilde{L}(dL(D)\phi) \in \tilde{L}(\mathscr{D}(G))$ for all $D \in \mathscr{U}(\mathfrak{g})^{\mathbf{C}}$ and $\phi \in \mathscr{D}(G)$; furthermore, the image $d\tilde{L}(\mathscr{U}(\mathfrak{g})^{\mathbf{C}})$ consists of all differential operators on R^n with polynomial coefficients (by Theorem 3.2 of [Ja]). Hence (*) is satisfied with $\mathscr{A}$ equal to the algebra of all polynomials on $\mathsf{R}^n \simeq G/K$.

The uniqueness of the canonical bilinear form was observed in [Pe] for the Schrödinger representations of the 3-dimensional Heisenberg group.

While the statement on ultra-irreducibility in Theorem 1 covers spaces which are invariant under multiplications by C^∞-functions, our next result applies to spaces like L^p. It involves tempered distributions and requires unitary of χ.

In order to present the result without reference to realizations we define the spaces $\mathscr{S}_\chi(G)$ and $\mathscr{S}'_\chi(G)$ as the preimages under the restriction map $r\colon \mathscr{D}'_\chi(G) \to \mathscr{D}'(\mathsf{R}^n)$ of the spaces $\mathscr{S}(\mathsf{R}^n)$ and $\mathscr{S}'(\mathsf{R}^n)$, respectively, with their usual topologies. If χ is unitary this definition is independent of the choice of realization, because any isomorphism θ intertwining two standard realizations of $L|_{\mathscr{D}'_\chi}$ has the form

$$[\theta u](x) = \chi^{-1}(p(x))u(q(x)), \quad u \in \mathscr{D}'(\mathsf{R}^n),\ x \in \mathsf{R}^n,$$

where $p\colon \mathsf{R}^n \to K$ and $q\colon \mathsf{R}^n \to \mathsf{R}^n$ are polynomial maps, q with polynomial inverse.

The expression (1) for $\tilde{L}$ shows that the spaces $\mathscr{S}_\chi(G)$ and $\mathscr{S}'_\chi(G)$ are invariant under the action of L and that L restricts to strongly continuous representations of G on them.

THEOREM 2. *Let G and χ be as described in the beginning of the section and assume that χ is unitary. Then any representation of G induced from χ on a semi-complete space E that satisfies $\mathscr{S}_\chi(G) \subset E \subset \mathscr{S}'_\chi(G)$ with weakly continuous inclusion maps is ultra-irreducible.*

PROOF. Identify $L = L|_{\mathscr{D}'_\chi}$ with one of its standard realizations. Then L restricts to a strongly continuous unitary representation of G on $L^2(\mathsf{R}^n)$, and by Theorem 3.4 of [Hw] there exists a linear map $\phi \mapsto K_\phi$ of $\mathscr{S}(G)$ onto $\mathscr{S}(\mathsf{R}^n \times \mathsf{R}^n)$ such that

$$[L(\phi)u](x) = \int_{\mathsf{R}^n} K_\phi(x,y)u(y)\,\mathrm{d}y = \langle K_\phi(x,\cdot), u\rangle_{\mathscr{S}\times\mathscr{S}'}$$

for all $\phi \in \mathscr{S}(G)$, $u \in L^2(\mathsf{R}^n)$ and $x \in \mathsf{R}^n$.

This formula implies that $L(\phi)$ extends to a continuous map of $\mathscr{S}'(\mathsf{R}^n)$ into $\mathscr{S}(\mathsf{R}^n)$ for each $\phi \in \mathscr{S}(\mathsf{R}^n)$. The map $\phi \mapsto K_\phi$ is continuous from $\mathscr{S}(G)$ into $\mathscr{S}(\mathsf{R}^n \times \mathsf{R}^n)$ (by the closed graph theorem), so the map $\phi \mapsto L(\phi)$ is continuous from $\mathscr{S}(G)$ into $L(\mathscr{S}'(\mathsf{R}^n), \mathscr{S}(\mathsf{R}^n))_{\mathrm{w}}$. The assumptions on E then implies that $\phi \mapsto L(\phi)|_E$ is a continuous map of $\mathscr{S}(G)$ into $S(E)_{\mathrm{w}}$, hence into $S(E)_{\mathrm{uw}}$ by Lemma II.2.

By surjectivity of the map $\phi \in \mathscr{S}(G) \mapsto K_\phi \in \mathscr{S}(\mathsf{R}^n \times \mathsf{R}^n)$, $L(\mathscr{S}(G))|_E$ contains all the operators of the form $\psi L(\phi)|_E$ where $\psi \in \mathscr{S}(\mathsf{R}^n)$ and $\phi \in \mathscr{S}(G)$, in particular those with $\psi \in \mathscr{D}(\mathsf{R}^n)$. Hence so does the closure of $L_E(\mathscr{D}(G)) = L(\mathscr{D}(G))|_E$ in $S(E)_{\mathrm{uw}}$, $\mathscr{D}(G)$ being dense in $\mathscr{S}(G)$.

We claim that the identity operator on E is contained in the closure of $L(\mathscr{D}'(\mathsf{R}^n), \mathscr{D}(\mathsf{R}^n))$ in $S(E)_{\mathrm{uw}}$. Indeed, choose $\psi \in \mathscr{D}(\mathsf{R}^n)$ with $\psi(0) = 1$ and put $\psi_i(x) = \psi(x/i)$ for $x \in \mathsf{R}^n, i \in \mathsf{N}$. Then for each $u \in \mathscr{S}(\mathsf{R}^n)$, $\psi_i u \to u$ in $\mathscr{S}(\mathsf{R}^n)$ and hence weakly in E. Therefore $\psi_i L(\phi)|_E \to L_E(\phi)$ in $S(E)_{\mathrm{w}}$ and so in $S(E)_{\mathrm{uw}}$ by Lemma II.2, for each $\phi \in \mathscr{D}(G)$. Since I_E is contained in the closure of $L_E(\mathscr{D}(G))$ in $B(E)_{\mathrm{uw}}$, the claim is verified.

The theorem now follows from the general irreducibility criterion Theorem IV.2.

EXAMPLE. *The Heisenberg group.*

The $(2n+1)$-dimensional Heisenberg group is $G = \{(x, y, z) \in \mathsf{R}^n \mid x, y \in \mathsf{R}^n, z \in \mathsf{R}\}$ with group multiplication $(x, y, z)(x', y', z) = (x + x', y + y', z + z' + x \cdot y')$.

For χ given on $K = \{(0, y, z) \mid y \in \mathsf{R}^n, z \in \mathsf{R}\}$ by $\chi((0, y, z)) = e^{-i\lambda z}$, where $\lambda \in \mathsf{C}$, the induced representation on $\mathscr{D}'_\chi(G)$ is realized as

$$[T_\lambda(x, y, z)u](t) = e^{i\lambda(z - y \cdot t)}u(t - x) \quad \text{for } (x, y, z) \in G,\ u \in \mathscr{D}'(\mathsf{R}^n),\ t \in \mathsf{R}^n.$$

For $\lambda \in \mathsf{R} \setminus \{0\}$ the restrictions of T_λ to $L^2(\mathsf{R}^n)$ are the well known unitary Schrödinger representations of G which are known to be irreducible.

Our Theorem VI.1 implies ultra-irreducibility for each $\lambda \in \mathsf{C} \setminus \{0\}$ of the strongly continuous representations of G obtained by restricting T_λ to e.g. the spaces:

$$C_0^r(\mathsf{R}^n),\ C^r(\mathsf{R}^n) \text{ for } 0 \leqq r \leqq \infty,$$

$$\mathscr{D}'(\mathsf{R}^n),\ \mathscr{E}'(\mathsf{R}^n),$$

$$L^p_{\mathrm{loc}}(\mathsf{R}^n),\ L^p_c(\mathsf{R}) \text{ for } 1 \leqq p < \infty.$$

If $\lambda \in \mathsf{R} \setminus \{0\}$, then Theorem VI.2 implies ultra-irreducibility of T_λ restricted to the spaces (for definitions see [Hö])

$$\mathscr{S}(\mathsf{R}^n),\ \mathscr{S}'(\mathsf{R}^n),$$

$$L^p(\mathsf{R}^n, k\,dx),\ B_{p,k}(\mathsf{R}^n) \quad \text{for } 1 \leqq p < \infty \text{ and } k \text{ a tempered weight function},$$

$$H_{(s)}(\mathsf{R}^n) \text{ (Sobolev spaces) for } -\infty < s < \infty.$$

The above list of representation spaces covers the ones considered in [Pe], [Li], [LL1], [LL2], [Ho] and [Po; Example 5.1].

REFERENCES

[Bo] N. Bourbaki, *Eléments de Mathématiques: Espaces Vectoriels Topologiques*, Ch. 1–5, Masson, Paris/New York, 1981.

[Cl] F. du Cloux, *Représentations tempérées des groupes de Lie nilpotents*, J. Funct. Anal. 85 (1989), 420–457.

[He] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York/London, 1984.

[Ho] A. Hole, *Representations of the Heisenberg group of dimension $2n + 1$ on eigenspaces*, Math. Scand. 37 (1975), 129–141, MR 53#3205.

[Hw] R. E. Howe, *On a Connection between Nilpotent Lie Groups and Oscillatory Integrals associated to Singularities*, Pacific J. Math 73 (1977), 329–363, MR 58#28270.

[Hö] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer-Verlag, Berlin/Heidelberg/New York, 1983.

[Ja] J. Jacobsen, *Eigenspace representations of nilpotent Lie groups, II, Math. Scand.* 52 (1983), 321–333, MR 85g:22015.

[JS] J. Jacobsen and H. Stetkær, *Eigenspace representations of nilpotent Lie groups*, Math. Scand. 48 (1981), 41–55, MR 84b:22016.

[Ki] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russ. Math. Surveys 17 (1962), 53–104, MR 25#5396.

[Li] G. L. Litvinov, *Group representations in locally convex spaces, and topological groups algebras.* (Russian), Trudy Sem. Vektor. Tenzor. Anal. 16 (1972), 267–349, MR 54#10479.

[LL1] G. L. Litvinov and V. I. Lomonosov, *Density theorems in locally convex spaces and their applications.* (Russian), Trudy Sem. Vektor. Tenzor. Anal. 20 (1981), 210–227, MR 83i#47009. English translation in: Selecta Math. Soviet. 8, No. 4 (1989), 323–339.

[LL2] G. L. Litvinov and V. I. Lomonosov, *Density theorems in locally convex spaces and irreducible representations.* Soviet Math. Dokl. 23 (1981), 372–376, translated to English from Dokl. Akad. Nauk. SSSR 257 (1981), No. 4, p. 826–830. MR82g#46087.

[Pe] V. S. Petrosyan, *Nonunitary representations of the group of third order triangular matrices.* Moscow Univ. Math. Bull. 26 (1971), 12–16, translated to English from Vestnik Moskov. Univ. Ser. I. Mat. Meh. 16 (1971), 16–22, MR 45#460.

[Po] N. S. Poulsen, *On C^∞-Vectors and Intertwining Bilinear Forms for Representations of Lie Groups*, J. Funct. Anal. 9 (1972), 87–120, MR 45#9239.

[Ra1] M. Rais, *Sur l'irréductibilité de certaines représentations induites non unitaires*, C. R. Acad. Sci. Paris, t.305, Série I, p. 713–716, 1987, MR 89a:22022.

[Ra2] M. Rais, *Sur l'irréductibilité de certaines représentations induites non unitaires*, Prépublication Janvier 1987, Département de Mathématique, Université de Potiers, France.

[Sc] H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, Berlin/Heidelberg/New York, 1980.

[St] H. Stetkær, *Ultra-irreducibility of Induced Representations of Semidirect Products*, Trans. Amer. Math. Soc. (to appear in 1991).

[Th] E. Thieleker, *On the irreducibility of nonunitary induced representations of certain semidirect products*, Trans. Amer. Math. Soc. 164 (1972), 353–369, MR 45#2097.

[Tr] F. Treves, *Topological Vector Spaces, Distribtions and Kernels*, Academic Press, New York/London, 1967.

[Wa] G. Warner, *Harmonic Analysis on Semi-simple Lie Groups I*, Springer-Verlag, Berlin/Heidelberg/New York, 1972.

[Wi] F. L. Williams, *Topological irreducibility of nonunitary representations of group extensions.* Trans. Amer. Math. Soc. 233 (1977), 69–84, MR 57#3316.

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