ON SIMPLE GERMS WITH NON-ISOLATED SINGULARITIES

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§ 1. Introduction.

Let $\mathcal{O} = \mathcal{O}_n$ denote the local ring of germs of analytic functions $f: (\mathbb{C}^n, 0) \to \mathbb{C}$ and $m$ its maximal ideal. For an analytic germ $f \in \mathcal{O}$ we denote by $J_f$ its Jacobi ideal, namely $J_f = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$. For an ideal $I \subset \mathcal{O}$ we consider as in [8], [9]:

- the primitive ideal $\mathfrak{f} I$, defined by $\mathfrak{f} I = \{ f \in \mathcal{O} \mid (f) + J_f \subset I \}$; we have $I^2 \subset \mathfrak{f} I \subset I$;
- the group $\mathcal{D}_I$ of local analytic isomorphisms $h: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $h^*(I) = I$; it is a subgroup of the group of all germs of local analytic isomorphisms of $(\mathbb{C}^n, 0)$.

$\mathcal{D}_I$ acts on $\mathfrak{f} I$ and we shall consider the $\mathcal{R}_I$ (right-equivalence) relation on $\mathfrak{f} I$.

In the next section we prove the following.

THEOREM 1. Let $I \subset \mathcal{O}$ be a radical ideal defining a germ of a quasihomogeneous complete intersection in $(\mathbb{C}^n, 0)$ with isolated singularity. Suppose that there exist $\mathcal{R}_I$-simple germs in $\mathfrak{f} I$. Then in some coordinates $(z_1, \ldots, z_n)$ of $(\mathbb{C}^n, 0)$ we have either

a) there exists $k \in \{1, \ldots, n\}$ such that $I = (z_1, \ldots, z_k)$, or
b) there exists $k \in \{1, \ldots, n\}$ and a quasihomogeneous isolated singularity $g = g(z_1, \ldots, z_k) \in \mathcal{O}_k$ such that $I = (g, z_{k+1}, \ldots, z_n)$.

A. Némethi has proved a similar result in [7] for the case when $I = (f^s)$ where $s \geq 1$ and $f \in \mathcal{O}$ is an isolated singularity. When $n = 3$, D. Siersma has considered a similar problem for the inner modality (see [12]).

In the last section we derive the list of $\mathcal{R}_I$-simple germs for $I = (z_1, z_2)$.

§ 2. Proof of Theorem 1.

We recall from [8], [9] that for an ideal $I \subset \mathcal{O}$ and for $f \in \mathfrak{f} I$, the tangent space at

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\( f \) to the \( R_t \)-orbit of \( f \) is defined by

\[
T_t(f) = \left\{ \eta(f) \bigg| \eta = \sum_{j=1}^n \eta_j \frac{\partial}{\partial z_j} \text{ with } \eta(I) \subset I \text{ and } \eta_j \in m \text{ for } j = 1, \ldots, n \right\}
\]

and the \( I \)-codimension of \( f \) is

\[
c_t(f) = \dim_c \frac{\mathcal{I}}{T_t(f)}. \]

Let \( f_1, \ldots, f_p \) be a minimal set of quasihomogeneous generators of \( I \). Let \( q \) be the dimension of the \( C \)-vector space \((I + m^2)/m^2\). If \( q = p \), we have \( k = q = p \).

Suppose that \( q < p \). Using a linear change of coordinates, we can assume, without altering the quasihomogeneity of \( f_1, \ldots, f_p \), that \( f_j(z) = z_j + \text{higher monomials not containing } z_j \), for \( j = 1, \ldots, q \) (we assume that the weights of the coordinates are positive). Thus, we can consider, by subtracting suitable multiples of \( f_1, \ldots, f_q \), if necessary, that \( f_{q+1}, \ldots, f_p \) are quasihomogeneous polynomials, not depending on \( z_1, \ldots, z_q \). It follows that, in a suitable system \( z \) of coordinates, the ideal \( I \) is generated by \( f_1 = z_1, \ldots, f_q = z_q, f_q = z_q, f_{q+1}, \ldots, f_p \), where \( f_{q+1}, \ldots, f_p \in m^2 \) are quasihomogeneous polynomials depending only on \( z_{q+1}, \ldots, z_n \).

Since there exist \( R_t \)-simple germs in \( \mathcal{I} \), we can find \( f \in \mathcal{I} \) such that \( c_t(f) = 0 \). (The \( R_t \)-simple germs are defined similarly with the simple isolated singularities; see for example [2] or [4].) From [8], [9], we have \( \mathcal{I} = I^2 \) and we can write

\[
f = \sum_{i,j=1}^p g_{ij} f_i f_j, \text{ with } g_{ij} = g_{ji}. \]

Let \( r \) be the rank of the matrix \( (g_{ij}(0))_{i,j=1, q} \). Then \( r \) is also the rank of the Hessian matrix evaluated in \( 0, \left( \frac{\partial^2 f}{\partial z_i \partial z_j} \right)_{i,j=1, n} \). As in the proof of Morse Lemma (see for example [6]) we can obtain a system \( \tilde{z} \) of coordinates, with \( \tilde{z}_j = z_j \) for \( j > q \), such that \( I \) is generated by \( \tilde{f}_1 = \tilde{z}, \ldots, \tilde{f}_q = \tilde{z}_q, \tilde{f}_{q+1} = f_{q+1}, \ldots, \tilde{f}_p = f_p \) and such that

\[
f = \tilde{z}_1^2 + \ldots + \tilde{z}_r^2 + \sum_{i,j=r+1}^p \tilde{g}_{ij} \tilde{f}_i \tilde{f}_j,
\]

with \( \tilde{g}_{ij} = \tilde{g}_{ji} \) and with \( \tilde{g}_{ij} \tilde{f}_i \tilde{f}_j \in m^3 \). It is easy to see that for any \( i,j \geq r + 1 \), there exists \( h_{ij} = h_{ij}(\tilde{z}_{r+1}, \ldots, \tilde{z}_n) \) with \( h_{ij} \tilde{f}_i \tilde{f}_j \in m^3 \) and such that for any \( k \in \mathbb{N}, k \geq 2 \), we have that \( f \) is \( R_t \)-equivalent to \( \tilde{z}_1^2 + \ldots + \tilde{z}_r^2 + \sum_{i,j=r+1}^p h_{ij} \tilde{f}_i \tilde{f}_j + \sum_{i,j=r+1}^p \varphi_{ij} f_i f_j \), for some \( \varphi_{ij} \in (\tilde{z}_1, \ldots, \tilde{z}_r)^k \). Since \( c_t(f) < \infty \), \( f \) is \( I \)-finitely determined (see [8], [9]). Hence we can assume that in (1) the germs \( \tilde{g}_{ij} \) do not depend on \( \tilde{z}_1, \ldots, \tilde{z}_r \).

We shall write in the sequel \( z \) for \( \tilde{z} \), \( f_j \) for \( \tilde{f}_j \) and \( g_{ij} \) for \( \tilde{g}_{ij} \). Since \( c_t(f) = 0 \), we
must have $T_I(f) = \int I = I^2$; we prove that this equality implies that $r = q = p - 1$.

Let $\theta_I = \left\{ \eta = \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial z_j} \mid \eta(I) \subseteq I \right\}$ be the $\mathcal{O}$-module of logarithmic vector fields for $I$. Since $I = (f_1, \ldots, f_p)$ is a reduced quasihomogeneous complete intersection in $(\mathbb{C}^n, 0)$ with isolated singularity, the $\mathcal{O}$-module $\theta_I$ is generated by the following vector fields (see for example [3]):

(A) $f_i \frac{\partial}{\partial z_j}$, where $i = 1, \ldots, p$ and $j = 1, \ldots, n$;

(B) the "trivial vector fields"

\[
\begin{vmatrix}
\frac{\partial}{\partial z_{i_1}} & \cdots & \frac{\partial}{\partial z_{i_{p+1}}}
\hline
\frac{\partial f_1}{\partial z_{i_1}} & \cdots & \frac{\partial f_1}{\partial z_{i_{p+1}}}
\hline
\vdots & & \vdots
\hline
\frac{\partial f_p}{\partial z_{i_1}} & \cdots & \frac{\partial f_p}{\partial z_{i_{p+1}}}
\end{vmatrix}
\]

for all $(p + 1)$-tuples $(i_1, \ldots, i_{p+1})$ satisfying $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{p+1} \leq n$;

(C) the Euler vector field $E = \sum_{j=1}^{n} w_j z_j \frac{\partial}{\partial z_j}$, where $w_1, \ldots, w_n$ are the weights of the coordinates.

It is clear that $T_I(f) = \theta_I(f)$.

We recall that $f_j = z_j$ for $1 \leq j \leq q$ and $f_{q+1}, \ldots, f_p \in m^2$ do not depend on $z_1, \ldots, z_q$. Also we recall that $f = z_1^2 + \cdots + z_q^2 + \sum_{i,j=r+1}^{p} g_{ij} f_i f_j$ with $g_{ij} = g_{ji}$ not depending on $z_1, \ldots, z_r$ and with $g_{ij} f_i f_j \in m^3$.

Suppose first that $r < q$. Then a moment's thought will convince us that for any $\eta \in \theta_I$, if we consider the expansion of $\eta(f)$ in a power series, then the coefficient of $z_q^2$ is zero. Hence $z_q^2 \notin T_I(f) = I^2$, a contradiction. It follows that $r = q$.

We look now for $f_{q+1}^2, \ldots, f_p^2$. It is easy to see that if $\eta \in \theta_I$ is one of the generators from (A) or (B), then $\eta(f)$ belongs to the ideal $L = m \cdot (f_{q+1}, \ldots, f_p)^2 + (z_1, \ldots, z_q) \cdot (f_{q+1}, \ldots, f_p) + (z_1, \ldots, z_q)^2$. On the other hand, for any germ $g \in m$ we have also $(gE)(f) \in L$. Thus $\theta_l(f) = L + C \cdot E(f)$. If $p - q \geq 2$ we have the uniqueness of the weights $w_{q+1}, \ldots, w_n$ (see for example [4]), hence $f_{q+1}^2, \ldots, f_p^2$ can not belong simultaneously to $\theta_I(f)$, in contradiction with the equality $I^2 = \theta_I(f)$. It follows that $q + 1 = p$. The theorem is proved.


§3. The simple germs for \( I = (z_1, z_2) \).

D. Siersma has found the \( \mathcal{R}_I \)-simple germs when \( I = (z_1, \ldots, z_{n-1}) \subset \emptyset \) in [10] and for \( I = (z_1z_2, z_3, \ldots, z_n) \subset \emptyset \) in [12]. For the case when \( I = (z_1) \subset \emptyset \), the list of \( \mathcal{R}_I \)-simple germs follows from the work of V.I. Arnold [1] (see for example [13]).

In the sequel we derive the list of \( \mathcal{R}_I \)-simple germs for \( I = (z_1, z_2) \). We shall suppose that \( n \geq 4 \) and we shall consider only germs \( f \in I^2 \) with \( j^2f = 0 \). (The simple germs \( f \in I^2 \) with \( j^2f \neq 0 \) are suspensions of those in [13].)

We use the following classical lemma:

**Lemma.** Let \( f_t = f + t \cdot \phi \in I^2 \) be a family of germs, with \( t \in \mathbb{R} \).

a) If \( \phi \in \mathcal{F}(f_0) \) for every \( t \in \mathbb{R} \), then, for any \( t \in \mathbb{R} \), \( f_t \) is \( \mathcal{R}_I \)-equivalent with \( f_0 \).

b) If \( \phi \notin \mathcal{F}(f_0) \) for every \( t \in \mathbb{R} \), then, for any \( t \in \mathbb{R} \), \( f_t \) is not \( \mathcal{R}_I \)-simple.

If we denote the coordinates \( z_3, \ldots, z_n \) by \( u_1, \ldots, u_{n-2} \) and the Milnor number of an isolated singularity \( g \) by \( \mu(g) \), we have the following:

**Theorem 2.** Let \( I = (z_1, z_2) \subset \emptyset \) and \( f \in I^2 \) with \( j^2f = 0 \). Then \( f \) is \( \mathcal{R}_I \)-simple if and only if \( f \) is \( \mathcal{R}_I \)-equivalent to a germ in the following table.

<table>
<thead>
<tr>
<th>Normal form of ( f )</th>
<th>( c_I(f) )</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_n ) \quad ( u_1z_1^2 + u_2z_2^2 + u_3z_1z_2 )</td>
<td>3</td>
<td>( n \geq 5 )</td>
</tr>
<tr>
<td>( I_4 ) \quad ( u_1z_1^2 + u_2z_2^2 )</td>
<td>3</td>
<td>( n = 4 )</td>
</tr>
<tr>
<td>( II ) \quad ( u_1z_1^2 + u_2z_2^2 + z_1z_2 \cdot g(u_3, \ldots, u_{n-2}) )</td>
<td>( n - 2 + \mu(g) )</td>
<td>( n \geq 5; g \in \text{A–D–E} )</td>
</tr>
<tr>
<td>( III ) \quad ( u_1z_1z_2 + u_2z_1^2 + z_2(z_2 + u_4z_2 + u_5 + u_6 + \ldots + u_{n-2}) )</td>
<td>( k + n - 2 )</td>
<td>( n \geq 4; k \geq 2 )</td>
</tr>
<tr>
<td>( u_1z_1z_2 + u_2z_1^2 + z_3z_2(z_2 + u_4z_2 + u_5 + u_6 + \ldots + u_{n-2}) )</td>
<td>( k + n - 2 )</td>
<td>( n \geq 5; k \geq 3 )</td>
</tr>
<tr>
<td>( u_1z_1z_2 + u_2z_1^2 + z_3(z_2 + u_4 + u_5 + u_6 + \ldots + u_{n-2}) )</td>
<td>( n + 2 )</td>
<td>( n \geq 4 )</td>
</tr>
<tr>
<td>( IV ) \quad ( u_1z_1z_2 + u_2z_1^2 + z_3z_2(z_1 + u_4 + \ldots + u_{n-2}) )</td>
<td>( k + n - 1 )</td>
<td>( n \geq 4; k \geq 2 )</td>
</tr>
<tr>
<td>( V_a ) \quad ( u_1z_1z_2 + u_2z_1^2 + z_3z_2(z_1 + u_4 + u_5 + \ldots + u_{n-2}) )</td>
<td>( n + 3 )</td>
<td>( n \geq 4 )</td>
</tr>
<tr>
<td>( V_b ) \quad ( u_1z_1z_2 + u_2z_1^2 + z_3z_2(z_1 + u_4 + u_5 + \ldots + u_{n-2}) )</td>
<td>( n + 4 )</td>
<td>( n \geq 5 )</td>
</tr>
<tr>
<td>( VI ) \quad ( u_1z_1z_2 + u_2z_1^2 + z_3(z_2u_2 + u_4 + \ldots + u_{n-2}) )</td>
<td>( k + n - 1 )</td>
<td>( n \geq 4; k \geq 3 )</td>
</tr>
<tr>
<td>( VI^3 ) \quad ( u_1z_1z_2 + u_2z_1^2 + z_3z_2u_2 + u_4 + \ldots + u_{n-2} )</td>
<td>( n + 4 )</td>
<td>( n \geq 5 )</td>
</tr>
</tbody>
</table>

**Proof.** If \( f \in I^2 \) has \( j^2f = 0 \), then \( j^3f = u_1Q_1(z_1, z_2) + \ldots + u_{n-2}Q_{n-2}(z_1, z_2) + C(z_1, z_2) \), where \( Q_1, \ldots, Q_{n-2} \) are quadrics and \( C \) is a cubic in \( z_1, z_2 \). We suppose that \( c_I(f) < \infty \). Hence \( f \) is \( \mathcal{R}_I \)-equivalent to a jet \( j^k f \) for sufficiently large \( k \) (see [9]).
Let $V$ be the $\mathbb{C}$-vector space generated by $Q_1, \ldots, Q_{n-2}$ in the vector space of quadrics in $z_1, z_2$.

If $\dim V = 3$ then $n \geq 5$ and we can find in $\mathcal{D}_t$ a linear isomorphism of $(\mathbb{C}^n, 0)$ such that $j^3f = u_1z_1^2 + u_2z_2^2 + u_3z_1z_2$. It follows by [8], [9] that $f$ is $\mathcal{R}_g$-equivalent with $j^3f$ (f is a $D(1, 1)$-type germ) and $f$ is $\mathcal{R}_g$-simple.

If $\dim V \leq 1$ then $f$ is not $\mathcal{R}_g$-simple. Namely, any neighbourhood of $f$ contains a germ which is $\mathcal{R}_g$-equivalent to a germ $\tilde{f} = u_1z_1z_2 + z_1^2 + z_1^2(u_2^2 + \ldots + u_{n-2}^2) + z_2^2 \cdot \varphi(u_2, \ldots, u_{n-2})$ where $\varphi \in m^2$. It is easy to see that for any $\varphi$, $\tilde{f}$ is not $\mathcal{R}_g$-simple.

If $\dim V = 2$ then, using the classification of pencils of quadrics in $z_1, z_2$ we can find in $\mathcal{D}_t$ some linear isomorphisms of $(\mathbb{C}^n, 0)$ such that $j^3f$ is one of the following cubics:

$$u_1z_1^2 + u_2z_2^2; u_1z_1z_2 + u_2z_1^2; \text{ or } u_1z_1z_2 + u_2z_1^2 + z_2^3.$$

When $j^3f = u_1z_1^2 + u_2z_2^2$ it follows, directly or using the technique of global transversal from [5], that $f$ is $\mathcal{R}_g$-equivalent to $u_1z_1^2 + u_2z_2^2 + z_1z_2g(u_3, \ldots, u_{n-2})$. Now it is easy to see, for $n \geq 5$, that $f$ is $\mathcal{R}_g$-simple if and only if $g$ is a simple isolated singularity ($g$ is an $A_D$, $E$ singularity; see [2], or [4] for the normal forms).

If $j^3f = u_1z_1z_2 + u_2z_1^2 + z_2^3$, then $f$ is $\mathcal{R}_g$-equivalent to $u_1z_1z_2 + u_2z_1^2 + z_2^3(z_2 + g(u_2, \ldots, u_{n-2}))$ with $g \in m^2$. It is easy to see that $f$ is $\mathcal{R}_g$-simple if and only if $g$ is a simple boundary singularity in the sense of Arnold, the boundary being $u_2 = 0$ (see [1]).

The most difficult case is when $j^3f = u_1z_1z_2 + u_2z_1^2$. In this situation $f$ is $\mathcal{R}_g$-equivalent to $u_1z_1z_2 + u_2z_1^2 + z_2^3h(z_2, u_2, \ldots, u_{n-2})$ with $h \in m^2$. If $h$ is a simple boundary singularity with respect to $z_2 = 0$, we change the coordinates such that $h$ becomes the normal form of a $B-C-F$ singularity. Then $f$ is $\mathcal{R}_g$-equivalent to $u_1z_1z_2 + \varphi(u_2, \ldots, u_{n-2})z_1^2 + z_2^3h$, with $\varphi \in m^2$ and $m^2$, and we obtain the germs in the table by using the lemma and looking at $j^4\varphi$.

If $h$ is not a simple boundary singularity then $f$ can be deformed to a germ which is $\mathcal{R}_g$-equivalent to $u_1z_1z_2 + \varphi(u_2, \ldots, u_{n-2})z_1^2 + z_2^3h(z_2, u_2, \ldots, u_{n-2})$ where $\varphi \in m^2$ and $h$ is one of the following unimodal boundary singularities (see [2]):

$$F_{1,0}: z_2^3 + a z_2u_2^2 + u_3^2 + u_3^2 + \ldots + u_{n-2}^2, 4a^3 + 27 \neq 0$$

$$K_{4,2}: z_2^2 + a z_2u_2^2 + u_2^4 + u_3^2 + \ldots + u_{n-2}^2, a^2 \neq 4$$

or

$$L_6: z_2u_2 + a z_2u_3 + u_2^3u_3 + u_3^2 + u_4^2 + \ldots + u_{n-2}^2.$$

Using the lemma with $\mathcal{I}(f)$ replaced by $\mathcal{I}(f) + (u_2, \ldots, u_{n-2})^3(z_1)^2$ we obtain that $f$ is not $\mathcal{R}_g$-simple.
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