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ON RÖDSETH'S h-BASES $A_k = \{1, a_2, 2a_2, \ldots, (k-2)a_2, a_k\}$

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Given an integral basis

$$A_k = \{a_1, a_2, \ldots, a_k\},\ 1 = a_1 < a_2 < \ldots < a_k$$

for a positive integer h, we form all the combinations

$$\sum_{i=1}^{k} x_i a_i,\ x_i \geqq 0,\ \sum_{i=1}^{k} x_i \leqq h,$$

and ask for the smallest integer $N(h, A_k)$ which is not represented by such a combination. The number $n(h, A_k) = N(h, A_k) - 1$ is called the *h-range* of A_k. In this connection, A_k is often called an *h-basis*.

With h_0 we denote the smallest number of addends which is sufficient for the h-range to reach the largest basis element a_k:

$$h_0 = h_0(A_k) = \min\{h \in \mathsf{N} \mid n(h, A_k) \geqq a_k\}.$$

It is easily seen that

$$n(h+1, A_k) \geqq n(h, A_k) + a_k, h \geqq h_0 - 1.$$

There exists a smallest $h = h_1 \geqq h_0 - 1$ such that this holds with equality for all $h \geqq h_1 = h_1(A_k)$. We say that the h-range is *stabilized* from $h = h_1$.

The h-bases of the title were introduced by Rödseth [1]. We shall assume knowledge of his paper. In particular, double formula numbering refers to [1], without further comment. We shall use his notation throughout, with the one exception that we write a_2 when he uses d.

Rödseth (his Remark 2) determines $n(h_1, A_k)$ explicitly. In particular,

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$$n(h_1, A_k) = h_1 a_k - (a_k - 1)(s_v - s_{v+1} - 2) \tag{1}$$
$$- (a_k - \kappa a_2)\left\lceil \frac{P_{v+1} - 1}{\kappa} \right\rceil - a_2(\kappa - 1)$$

when $R_v < \kappa = k - 2$.

In his Theorem 1, Rödseth also determines h_1. However, he leaves two problems open:

I) Describe explicitly all bases with $h_1 > h_0$.

II) For these bases, determine $n(h, A_k)$ for all h with $h_0 \leqq h < h_1$.

We shall solve problem I completely. The resulting bases A_k are of two types, one with $v = 1$, and the other with $v = m$, $s_m = 1$. For the first type, we have proved formulas for $n(h_0, A_k)$, but for the "intermediate" h-ranges of problem II only when $k = 4$. For the second type, we have not tried to solve the (apparently very complicated) problem II. – Since $h_1 \leqq h_0$ for $k = 3$, we assume $k \geqq 4$ throughout.

Our results were first conjectured, using extensive numerical evidence produced by Svein Mossige on the Univac 1100 computer at the University of Bergen. We are grateful for his support.

We shall use Rödseth's Theorem 1 to determine the cases with $h_1 = h' > h_0$. A necessary condition is then $0 < R_v < \kappa$. On the other hand, Rödseth shows (p. 13) that $R_j - R_{j+1} \geqq \kappa, j = 0, 1, \ldots, m$. If we find an R_j with $0 < R_j < \kappa$, we thus have $R_{j+1} < 0$ and hence $v = j$.

The case $v = 1$ is straightforward. By (3.1), we have

$$R_1 = \kappa s_1 - P_1 + \kappa Q_1 = \kappa s_1 - q_1 + \kappa.$$

We substitute $R = \kappa - R_1$, so $0 < R_1 < \kappa$ means $0 < R < \kappa$. Since $a_k = q_1 a_2 - s_1$ and $q_1 = \kappa s_1 + R$, we thus get the bases

$$A_k = \{1, a_2, 2a_2, \ldots, \kappa a_2, (\kappa s_1 + R)a_2 - s_1\}, 0 < R < \kappa, 0 < s_1 < a_2, \tag{2}$$

as the only candidates for $h_1 > h_0$ when $v = 1$. Using $a_2 = q_2 s_1 - s_2$ and $P_2 = q_1 q_2 - 1$, it then follows easily from (3.4–5) that

$$h_0 = \begin{cases} a_2 + s_1 - 1 & \text{if } R \leqq 2 \\ a_2 + s_1 & \text{if } R > 2 \end{cases}$$

$$h' = a_2 + s_1 - 1 + \left\lceil \frac{(q_2 - 1)R - 2}{\kappa} \right\rceil.$$

Consequently, we get $h_1 = h' > h_0$ for the bases (2) just when

$$q_2 = \left\lceil \frac{a_2}{s_1} \right\rceil \geqq \begin{cases} 4 & \text{if } R = 1 \\ 3 & \text{if } R = 2 \\ \lfloor k/R \rfloor + 2 & \text{if } R > 2. \end{cases}$$

In terms of s_1, this may be written as

$$s_1 \leqq \begin{cases} \dfrac{a_2 - 1}{3} & \text{if } R = 1 \\ \dfrac{a_2 - 1}{2} & \text{if } R = 2 \\ \dfrac{a_2 - 1}{\lfloor k/R \rfloor + 1} & \text{if } R > 2. \end{cases} \tag{3}$$

In addition to these cases, Mossige's calculations indicated another type of bases A_k for which $h_1 > h_0$. If we put $a_k - a_{k-1} = a_k - \kappa a_2 = u$, there is then a basis element ta_2, $t < \kappa$, which is $\equiv 1 \pmod u$, hence

$$A_k = \{1, a_2, 2a_2, \ldots, ta_2 = \alpha u + 1, \ldots, \kappa a_2, \kappa a_2 + u\}, 0 < t < \kappa. \tag{4}$$

Since $(a_2, u) = (a_2, a_k) = 1$, Rödseth's division algorithm (p. 6) ends with $s_m = 1$, $s_{m+1} = 0$, and $P_{m+1} = s_{-1} = a_k$, $Q_{m+1} = s_0 = a_2$. We consider the indeterminate equation

$$a_2 x - a_k y = 1. \tag{5}$$

By (2.2), this has the solution $x = P_m$, $y = Q_m$, which is of course unique if we demand $0 < x < a_k = P_{m+1}$, $0 < y < a_2 = Q_{m+1}$. On the other hand, it follows from (4) that (5) also has the solution $x = \kappa\alpha + t$, $y = \alpha$. If $\alpha = qa_2 + r$, $0 < r < a_2$ ($r > 0$ since $(\alpha, a_2) = 1$), we "translate" the second solution into the first one by

$$Q_m = \alpha - qa_2 = r, \; P_m = \kappa\alpha + t - qa_k = \kappa r + t - qu.$$

From (3.1), it follows that

$$R_m = \kappa s_m - P_m + \kappa Q_m = \kappa - t + qu > 0 \Rightarrow v = m$$

(since always $R_{m+1} < 0$). The necessary condition $R_v < \kappa$ for $h_1 > h_0$ is satisfied, since

$$ta_2 = \alpha u + 1 = (qa_2 + r)u + 1 \Rightarrow qu = t - \frac{ru + 1}{a_2} < t.$$

From Rödseth's first expression for h_0 (bottom line p. 8), we find

$$h_0 = a_2 + 1 + \left\lfloor \frac{u - 2a_2}{\kappa a_2} \right\rfloor = \begin{cases} a_2 & \text{if } u < 2a_2 \\ a_2 + 1 & \text{if } u \geqq 2a_2. \end{cases}$$

On the other hand, we have from (3.5) that

$$h' = s_m - s_{m+1} - 2 + \left\lceil \frac{P_{m+1} + R_m - 1}{\kappa} \right\rceil = a_2 + \left\lceil \frac{(q+1)u - (t+1)}{\kappa} \right\rceil.$$

Consequently, we get $h_1 = h' > h_0$ for the bases (4) just when

$$\text{(6)} \qquad \frac{t+1}{q+1} < u \leqq 2a_2 - 1, \quad \text{or} \quad u > \max\left\{2a_2 - 1, \frac{\kappa + t + 1}{q+1}\right\},$$

where $q = \lfloor \alpha/a_2 \rfloor$.

We make three remarks:

1) There may be cases of (4) $\wedge$ (6) with $v = 1$, hence overlap with (2) $\wedge$ (3). The simplest example is given by

$$A_4 = \{1, 4, 8, 11\},\ h_0 = 4,\ h_1 = 5.$$

2) In a basis of the type (4), the choice of basis element ta_2 is not necessarily unique. Probably the simplest example (with $h_1 > h_0$) is given by

$$A_7 = \{1, 7, 14, 21, 28, 35, 38\},\ h_0 = 7,\ h_1 = 8.$$

It has $u = 3$, but we may choose either $t = 1$, $\alpha = 2$, or $t = 4$, $\alpha = 9$.

3) It is fairly easy to show by Rödseth's methods that $h_1 \leqq h_0$ when $u = 1, 2$. The choice $u = 3$ in the above examples is thus smallest possible.

We shall now prove our main result:

THEOREM. *The bases*

$$A_k = \{1, a_2, 2a_2, \ldots, (k-2)a_2, a_k\},\ k \geqq 4,$$

with $h_1 > h_0$ are either of the form (2) *satisfying* (3), *or of the form* (4) *satisfying* (6).

PROOF. Since the case $v = 1$ is completely settled, we may assume $v > 1$, and must show that the only possibility for $h_1 > h_0$ is then given by (4). By Rödseth's Theorem 1, we may also assume $0 < R_v < \kappa$.

Using (3.4–5), we form

$$h_0 - h' = a_2 - 1 + \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil - s_v + s_{v+1} + 2 - \left\lceil \frac{P_{v+1} + R_v - 1}{\kappa} \right\rceil,$$

which should "usually" be $\geqq 0$.

By (2.4), we can substitute $a_2 = s_0 = Q_{v+1}s_v - Q_v s_{v+1}$. We then assume the "worst cases"

$$s_{v+1} \leqq s_v - 1, \left\lceil \frac{P_{v+1} + R_v - 1}{\kappa} \right\rceil \leqq \frac{P_{v+1} + R_v + \kappa - 2}{\kappa}.$$

In the resulting expression, we substitute

$$P_{v+1} = q_{v+1}P_v - P_{v-1},\ Q_{v+1} = q_{v+1}Q_v - Q_{v-1},\ P_v = \kappa s_v + \kappa Q_v - R_v$$

by (2.1) and (3.1), and get

$$h_0 - h' \geqq q_{v+1}\{s_v Q_v - Q_v - s_v\} - (s_v - 1)(Q_v + Q_{v-1}) - 1 + \frac{P_{v-1} - \kappa Q_{v-1} + (q_{v+1} - 1)R_v + 2}{\kappa} + \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil. \tag{7}$$

By (9) below and (3.1), we have $P_v - \kappa Q_v = \kappa s_v - R_v \geqq 1$.

(i) $\kappa s_v - R_v < \kappa$. Since we assume $R_v < \kappa$, we must have $s_v = 1$ and thus $v = m$, hence the situation described in connection with (5). Again with $a_k = \kappa a_2 + u$, we have $(a_2, u) = 1$, and thus solutions of the indeterminate equation in t and α:

$$1 = a_2 t - u\alpha = a_2(t + \kappa\alpha) - a_k\alpha. \tag{8}$$

We choose the known solution $\alpha = Q_m$, $t + \kappa\alpha = P_m$, for which

$$R_m = \kappa s_m - P_m + \kappa Q_m = \kappa - t.$$

But $R_m = R_v$, and our assumption $0 < R_v < \kappa$ implies $0 < t < \kappa$. Since $ta_2 = \alpha u + 1$ by (8), we thus get the form (4) of A_k.

The proof of the Theorem will then be complete if we can show that $h_0 - h' \geqq 0$, hence $h_0 \geqq h_1$, in the remaining case:

(ii) $\kappa s_v - R_v \geqq \kappa$. Since we assume $R_v > 0$, we must now have $s_v \geqq 2$. With $v > 1$, we also have $Q_v \geqq 2$, and so $\{\ \} \geqq 0$ in (7). We note that

$$\frac{P_i}{Q_i} \geqq \frac{a_k}{a_2} > \kappa \Rightarrow P_i - \kappa Q_i \geqq 1 \tag{9}$$

$$q_1 = \left\lceil \frac{a_k}{a_2} \right\rceil \geqq \kappa + 1 \geqq 3 \Rightarrow \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil \geqq 1.$$

The first $\geqq$ of (9) stems from the fact that the convergents P_j/Q_j decrease for increasing j, cf. (2.2).

Substituting $q_{v+1} \geqq 2$ and $P_{v-1} - \kappa Q_{v-1} \geqq 1$ in (7), we get

$$h_0 - h' \geqq (s_v - 1)(Q_v - Q_{v-1}) - 2s_v - 1 + \frac{R_v + 3}{\kappa} + \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil.$$

If $Q_v - Q_{v-1} \geqq 3$, then

$$h_0 - h' \geqq s_v - 4 + \frac{R_v + 3}{\kappa} + \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil \geqq -1 + \frac{R_v + 3}{\kappa} > -1,$$

so $h_0 - h' \geqq 0$ then. It thus remains to consider the possibilities $Q_v - Q_{v-1} = 1$ or 2.

If $Q_v - Q_{v-1} = 1$, we must have $Q_i - Q_{i-1} = 1$ for $i = 1, 2, \ldots, v$. This is possible if and only if $q_i = 2$, $i = 2, 3, \ldots, v$, giving

$$P_i = iq_1 - (i - 1),\ Q_i = i;\ i = 1, 2, \ldots, v. \tag{10}$$

In (7), we substitute $R_v = \kappa s_v - P_v + \kappa Q_v$, and take P_{v-1}, Q_{v-1}, P_v and Q_v from (10). Still using $q_{v+1} \geqq 2$, the terms with s_v then cancel, and we are left with

$$h_0 - h' \geqq \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil - \frac{q_1 - 2}{\kappa} + \frac{1}{\kappa} - 1 > -1 \Rightarrow h_0 - h' \geqq 0.$$

If $Q_v - Q_{v-1} = 2$, we substitute $q_{v+1} \geqq 2$, $Q_{v-1} = Q_v - 2$ and $R_v = \kappa s_v - P_v + \kappa Q_v$ in (7). The terms with Q_v then cancel, and we are left with

$$h_0 - h' \geqq s_v - 1 + \frac{P_{v-1} - P_v + 2}{\kappa} + \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil. \tag{11}$$

We noted that $Q_v - Q_{v-1} = 1$ implies (10). Similarly, it is easily seen that $Q_v - Q_{v-1} = 2$ implies $q_2 = 3$, and $q_i = 2$ for $i = 3, 4, \ldots, v$ if $v > 2$. For all $v \geqq 2$, this gives

$$P_i = (2i - 1)q_1 - (i - 1),\ Q_i = 2i - 1;\ i = 1, 2, \ldots, v. \tag{12}$$

Substituting in (11), we get

$$h_0 - h' \geqq s_v - 1 + \frac{-2q_1 + 3}{\kappa} + \left\lceil \frac{q_1 - 2}{\kappa} \right\rceil \geqq s_v - 1 - \frac{q_1 - 1}{\kappa} > -1.$$

The last inequality follows from $q_1 \leqq \kappa s_v$. To see that this holds, we note that assuming $q_1 \geqq \kappa s_v + 1$, and using (12), we easily get the contradiction $R_v = \kappa s_v - P_v + \kappa Q_v < 0$.

This concludes the proof of our Theorem.

As remarked earlier, we are still left with the problem of determining $n(h, A_k)$ for $h_0 \leqq h < h_1$ in the cases when $h_1 > h_0$.

For the bases A_k of (4), numerical evidence indicates that this is a very complicated problem, which we have not tried to sort out.

For the bases A_k of (2), we can determine $n(h_0, A_k)$:

$$n(h_0, A_k) = \begin{array}{ll} (3\kappa s_1 + 2)a_2 - (2s_1 + 2), & R = 1,\ q_2 \geqq 3 \\ (2\kappa s_1 + 2)a_2 - (s_1 + 2), & R = 2,\ q_2 \geqq 2 \\ (q_2\kappa s_1 + k)a_2 - ((q_2 - 1)s_1 + 2), & R \geqq 3,\ q_2 \geqq \lfloor k/R \rfloor + 1. \end{array}$$

The straightforward but rather tedious proof is found in [2].

With equalities for q_2, we see from (3) that we then have the largest q_2 for which $h_1 = h_0$. In these cases, the results follow from (1). It is rather striking that the formulas for $n(h_0, A_k)$ are valid also for all larger q_2, when $h_1 > h_0$. This fact was first observed numerically, from Mossige's computations.

The above formulas do not cover the cases

$$n(h_0, A_k) = \begin{cases} (3\kappa s_1 + 3 - \kappa)a_2 - (2s_1 + 2), & R = 1,\ q_2 = 2 \\ ((q_2 + 1)\kappa s_1 + R + 2)a_2 - (q_2 s_1 + 2), & R \geqq 3,\ 2 \leqq q_2 \leqq \lfloor k/R \rfloor. \end{cases}$$

They follow from (1), since here $h_1 = h_0$.

Still considering the bases (2), we now know $n(h_0, A_k)$, and also $n(h_1, A_k)$ by (1). But we are left with the problem of determining $n(h, A_k)$ if $h_0 < h < h_1$.

Let us first look at $k = 4$:

$$A_4 = \{1, a_2, 2a_2, (2s_1 + 1)a_2 - s_1\}, \quad 0 < s_1 < a_2$$

$$h_0 = a_2 + s_1 - 1, h_1 = a_2 + s_1 - 2 + \lfloor q_2/2 \rfloor$$

$$h_1 > h_0 \Leftrightarrow q_2 \geqq 4 \Leftrightarrow a_2 > 3s_1.$$

With $R = 1$, the earlier formulas give

$$n(h_0, A_4) = (6s_1 + 1)a_2 - (2s_1 + 2), \quad q_2 = 2$$

$$n(h_0, A_4) = (6s_1 + 2)a_2 - (2s_1 + 2), \quad q_2 > 2.$$

If $h_1 > h_0$, it is fairly easy to prove that

$$\Delta = n(h, A_4) - n(h - 1, A_4) = 2a_4, \quad h = h_0 + 1, \ldots, h_1,$$

except for q_2 even, when the last difference equals

$$\Delta_1 = n(h_1, A_4) - n(h_1 - 1, A_4) = 2a_4 - a_2.$$

To see how the series of "jumps" Δ behaves for larger k, Mossige performed extensive calculations for $k = 5, 6, 7$. We give the *observed* result for $k = 5$:

$$R = 1\colon \Delta = 3a_5, \quad \text{but} \quad \Delta_1 = \begin{cases} 2a_5 - a_2, & q_2 \equiv 1 \pmod 3 \\ 3a_5 - a_3, & q_2 \equiv 2 \pmod 3 \end{cases}$$

$$R = 2\colon \Delta \text{ alternates between } a_5 + 2 \text{ and } 2a_5 - a_2,$$

$$\text{but } \Delta_1 = 2a_5 - a_3 \text{ if } q_2 \equiv 1 \pmod 3.$$

The increasingly complicated observed patterns for $k = 6, 7$ are given in [2]. The only simple rule seems to be $\Delta = \kappa a_k$ for $R = 1$, but even then the exceptions for Δ_1 behave rather irregularly.

Because of the complexity, we have not tried to prove any of the "jump patterns" for $k > 4$.

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