GENERALIZATION OF THE GENERAL DIOPHANTINE APPROXIMATION THEOREM OF KRONECKER

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1. Introduction

Let \( G \) be the abelian group

\[
G = \mathbb{Z}^r = \{(h_1, \ldots, h_r) : h_1, \ldots, h_r \in \mathbb{Z}\}
\]

We consider the additive characters

\[
\chi_1(h_1, \ldots, h_r) \equiv \beta_{1,1} h_1 + \ldots + \beta_{1,r} h_r \pmod{1}
\]

\[
\chi_p(h_1, \ldots, h_r) \equiv \beta_{p,1} h_1 + \ldots + \beta_{p,r} h_r \pmod{1}
\]

where \( \beta_{i,j} \in \mathbb{R} \). The necessary and sufficient condition that the \( p \) inequalities

\[
|\chi_1(h_1, \ldots, h_r) - a_1| \leq \varepsilon \pmod{1}
\]

\[
|\chi_p(h_1, \ldots, h_r) - a_p| \leq \varepsilon \pmod{1}
\]

where \( a_1, \ldots, a_p \in \mathbb{R} \), for every \( \varepsilon > 0 \) have at least one solution \((h_1, \ldots, h_r)\) is that for arbitrary integers \( n_1, \ldots, n_p \) with

\[
n_1 \chi_1(h_1, \ldots, h_r) + \ldots + n_p \chi_p(h_1, \ldots, h_r) \equiv 0 \pmod{1}
\]

or equivalently

\[
n_1 \beta_{1,1} + \ldots + n_p \beta_{p,1} \equiv 0 \pmod{1}
\]

\[
n_1 \beta_{1,r} + \ldots + n_p \beta_{p,r} \equiv 0 \pmod{1}
\]

also

\[
n_1 a_1 + \ldots + n_p a_p \equiv 0 \pmod{1}
\]

This is Kronecker's general diophantine approximation theorem from [5].

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We shall generalize this result in two directions. In both cases we replace the group \( \mathbb{Z}^* \) by an arbitrary topological abelian group \( G \), and the characters \( \chi_1, \ldots, \chi_p \) by arbitrary continuous additive characters of \( G \).

2. The first direction.

The first direction we take consists in proving the

**Theorem 1.** A necessary and sufficient condition that the set \( C \) of solutions \( x \) of the inequalities

\[
|\chi_1(x) - a_1| \leq \varepsilon \pmod{1}
\]

\[
|\chi_p(x) - a_p| \leq \varepsilon \pmod{1}
\]

where \( a_1, \ldots, a_p \in \mathbb{R} \), for every \( \varepsilon > 0 \) be not empty, is that for arbitrary integers \( n_1, \ldots, n_p \) with

\[
n_1 \chi_1(x) + \ldots + n_p \chi_p(x) \equiv 0 \pmod{1} \text{ for } x \in G
\]

also

\[
n_1 a_1 + \ldots + n_p a_p \equiv 0 \pmod{1}
\]

(A simpler proof than mine can be given by help of theorem 1.8.3 in [6] but my method leads to the deeper-lying Theorem 5.)

It is obvious that the condition is necessary. In order to prove the sufficiency, we prove somewhat more. Let \( 0 < \varepsilon < \frac{1}{2} \) and let (2) \( \Rightarrow \) (3). Then we prove that \( C \) for \( \varepsilon \) small is \( W \)-almost periodic and that its mean value \( MC \) is positive. Then of course \( C \) is not empty. We do it by proving that \( MC \), when \( (2) \Rightarrow (3) \), and \( \varepsilon \) is small, does not depend on \( a_1, \ldots, a_p \) (and consider \( a_1 = \ldots = a_p = 0 \)).

A generalization of this fact is used in the second direction we take, which consists in finding formulas for the mean value of \( C \) and more general sets.

For orientation about \( W \)-almost periodic (\( W \)-ap) functions on a topological group \( G \), about the Bohr compactification \( BG \) of \( G \), and about the \( W_1 \)-ap functions on \( BG \) we refer to [2], [3], [4].

As stated above, \( G \) is a topological abelian group. Let \( \chi : G \to \mathbb{R}/\mathbb{Z} = H \) be a continuous additive character of \( G \). We consider \( H \) as the interval \( 0 \leq y \leq 1 \) with identification of 0 and 1, modulo 1 addition, and usual circle topology.

We shall need the following

**Theorem 2.** Let the values of \( \chi(x) \) be everywhere dense on \( H \) and let \( g : H \to \mathbb{C} \) be a Riemann-integrable function on \( H \) with the Fourier series

\[
g(y) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi iny}
\]
Then \( g(\chi(x)) \) is a \( W \)-ap function on \( G \) with the Fourier series

\[
g(\chi(x)) \sim \sum_{n = -\infty}^{\infty} c_n e^{2\pi i n \chi(x)}
\]

where

\[
e^{2\pi i n \chi(x)}
\]

for each integer \( n \) is a certain continuous multiplicative character:
\( G \to \{ z : z \in \mathbb{C}, |z| = 1 \} \).

**proof.** Let

\[
A = (\alpha_1, \ldots, \alpha_Q; x'_1, \ldots, x'_Q)
\]

where \( \alpha_q > 0 \) and \( \sum_{q=1}^{Q} \alpha_q = 1 \) and \( x'_q \in G \).

We know that the Riemann-integral

\[
\int_{0}^{1} \left| g(y) - \sum_{n = -N}^{N} c_n e^{2\pi i n y} \right| dy \to 0 \text{ for } N \to \infty
\]

We shall show that

\[
0 \leq \mathcal{M} \left| g(\chi(x)) - \sum_{n = -N}^{N} c_n e^{2\pi i n \chi(x)} \right|
\]

\[
\inf_{A} \sup_{x} \sum_{q=1}^{Q} \alpha_q \left| g(\chi(x + x'_q)) - \sum_{n = -N}^{N} c_n e^{2\pi i n \chi(x + x'_q)} \right| \leq
\]

\[
\int_{0}^{1} \left| g(y) - \sum_{n = -N}^{N} c_n e^{2\pi i n y} \right| dy \to 0 \text{ for } N \to \infty
\]

For a Riemann-integrable function \( f(y) \) on \( H \), viz.

\[
f(y) = \left| g(y) - \sum_{n = -N}^{N} c_n e^{2\pi i n y} \right|
\]

we know that

\[
\sum_{q=1}^{Q} f(y + \eta_q)((y + y_q) - (y + y_{q-1})) \to \int_{0}^{1} f(y) dy
\]

uniformly in \( y \) for \( |D| \to 0 \) where \( D: y_0 = 0 \leq y_1 \leq y_2 \leq \ldots \leq y_Q = 1 \) and \( |D| = \max(y_q - y_{q-1}) \) and \( y_{q-1} \leq \eta_q \leq y_q \). Thus to \( \varepsilon > 0 \) we can find \( \delta > 0 \) such that
\[
\sum_{q=1}^{Q} |g(y + \eta_q) - \sum_{n=-N}^{N} c_n e^{2\pi i n (y + \eta_q)}| (y_q - y_{q-1}) \leq \int_{0}^{1} f(y) dy + \varepsilon
\]

for \( y \in H \) and \(|D| < \delta \). We take \( y = \chi(x) \) and let \( y_0 = 0 = \chi(x_0) < y_1 = \chi(x_1) < y_2 = \chi(x_2) < \ldots < y_Q = \chi(x_Q) = 1 \) where \( x_0 = x_Q = 0 \), and \( \chi(x_q - 1) \leq \eta_q = \chi(\xi_q) \leq \chi(x_q) \) and \(|D| \leq \delta \). Here we have used that the values of \( \chi(x) \) are everywhere dense on \( H \). Then we get

\[
\sup_{x} \sum_{q=1}^{Q} (\chi(x_q) - \chi(x_{q-1})) |g(\chi(x + \xi_q)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x + \xi_q)}| \\
\leq \int_{0}^{1} |g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y}| dy + \varepsilon
\]

Since \( \chi(x_q) - \chi(x_{q-1}) > 0 \) and \( \sum_{q=1}^{Q} (\chi(x_q) - \chi(x_{q-1})) = 1 \)

we see that

\[
\frac{M}{Q} \left| g(\chi(x)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x)} \right| = \\
\inf \sup_{A} \sum_{q=1}^{Q} \alpha_q \left| g(\chi(x + x'_q)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x + x'_q)} \right| \\
\leq \int_{0}^{1} |g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y}| dy + \varepsilon
\]

for every \( \varepsilon > 0 \) and hence (6) follows.

(6) shows that

\[
\frac{M}{Q} \left| g(\chi(x)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x)} \right| \to 0 \text{ for } N \to \infty
\]

and hence \( g(\chi(x)) \) is \( W \)-ap with the Fourier series

\[
g(\chi(x)) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \chi(x)}
\]

when we note that the multiplicative characters

\[
e^{2\pi i n \chi(x)}, \ n \in \mathbb{Z}
\]

are continuous on \( G \) since \( e^{2\pi i n y} \) is continuous on \( H \) and \( y = \chi(x) \) is continuous on \( G \), and they are different, because otherwise, for a certain \( n \neq 0 \)
\[ n \chi(x) \equiv 0 \pmod{1} \text{ for } x \in G \]

which contradicts that the values of \( \chi(x) \) are everywhere dense on \( H \).

A special consequence of (4) and (5) is that

\[ M_g(\chi(x)) = c_0 = \int_0^1 g(y) \, dy \]

which for \( G = \mathbb{Z} \) gives a result of Weyl, viz. that for an irrational number \( \alpha \) the sequence \( \alpha n, n = 1, 2, \ldots \) satisfies

\[ \frac{1}{N} \sum_{n=1}^{N} g(\alpha n \pmod{1}) \to \int_0^1 g(y) \, dy \text{ for } N \to \infty \]

and thus, in particular, is equi-distributed modulo 1.

Next we assume that each of the first \( t, 0 \leq t \leq p \), of the characters in (1) do not have values that are everywhere dense on \( H \), while each of the rest do, and intend to prove the following

**Theorem 3.** Let (2) \( \Rightarrow \) (3) and let \( 0 < \varepsilon < \frac{1}{2} \). Then the set \( C_1 \) of solutions of the \( p \) relations

\[ \chi_1(x) \equiv a_1 \pmod{1} \]
\[ \cdots \cdots \cdots \]
\[ \chi_t(x) \equiv a_t \pmod{1} \]

(7)

\[ |\chi_{t+1}(x) - a_{t+1}| \leq \varepsilon \pmod{1} \]
\[ \cdots \cdots \cdots \cdots \cdots \cdots \]
\[ |\chi_p(x) - a_p| \leq \varepsilon \pmod{1} \]

is \( W \)-ap and \( MC_1 > 0 \). Further, \( MC_1 \) does not depend on \( a_1, \ldots, a_p \). For \( \varepsilon \) small, say \( 0 < \varepsilon < \varepsilon_1 < \frac{1}{2} \), \( C_1 = C \), so that Theorem 1 follows.

**Proof.** Let \( T_j \) for \( j = 1, \ldots, t \) be the smallest positive integer with

\[ T_j \chi_j(x) \equiv 0 \pmod{1} \text{ for } x \in G \]

Let \( g_j(y) \) be the characteristic function of the one-point set \( \{a_j \pmod{1}\} \) on \( H \) for \( j = 1, \ldots, t \) and of the set \( a_j - \varepsilon \leq y \leq a_j + \varepsilon \pmod{1} \) on \( H \) for \( j = t + 1, \ldots, p \). Then the characteristic function of \( C_1 \) is

(8)

\[ C_1(x) = g_1(\chi_1(x)) \cdots g_p(\chi_p(x)) \]
Since (2) \(\Rightarrow\) (3) we have \(T_j a_j \equiv 0 \pmod{1}\) for \(j = 1, \ldots, t\), abbreviated \(Ta \equiv 0 \pmod{1}\), and hence for \(\varepsilon\) small, say \(0 < \varepsilon < \varepsilon_1 < \frac{1}{2}\), the sets \(C\) and \(C_1\) are identical. Similarly we write \(g\) for \(g_j\) and \(\chi\) for \(\chi_j\) when \(j = 1, \ldots, t\).

Now

\[
\frac{1}{T} \sum_{n=0}^{T-1} e^{2\pi i \frac{S}{T} n} = \begin{cases} 1 & \text{for } S = 0 \\ 0 & \text{for } S = 1, \ldots, T - 1 \end{cases}
\]

and if

\[
a \equiv \frac{S_0}{T} \pmod{1}
\]

we have

\[
\frac{1}{T} \sum_{n=0}^{T-1} e^{2\pi i \frac{S - S_0}{T} n} = \begin{cases} 1 & \text{for } S \equiv S_0 \pmod{T} \\ 0 & \text{otherwise} \end{cases}
\]

\[
= \frac{1}{T} \sum_{n=0}^{T-1} e^{-2\pi in} e^{2\pi i \frac{S}{T} n} = \frac{1}{T} \sum_{n=0}^{T-1} e^{-2\pi in} e^{2\pi i \chi(x)}
\]

\[
= \begin{cases} 1 & \text{for } \chi(x) \equiv a \pmod{1} \\ 0 & \text{otherwise} \end{cases} = g(\chi(x))
\]

a trigonometric polynomial, where the \(T\) continuous multiplicative characters

\[
e^{2\pi in\chi(x)}, n = 0, \ldots, T - 1
\]

are different.

\(g_{t+1}(\chi_{t+1}(x)), \ldots, g_p(\chi_{p}(x))\) are \(W\)-ap on account of Theorem 2, that also gives us their Fourier series. For \(j = t + 1, \ldots, p\), if

\[
g_j(y) \sim \sum_{n=-\infty}^{\infty} c_{j,n} e^{2\pi iny}
\]

we have \(c_{j,0} = \int_{a_j-\varepsilon}^{a_j+\varepsilon} 1 \, dy = 2\varepsilon\), and for \(n \neq 0\)

\[
c_{j,n} = \int_{a_j-\varepsilon}^{a_j+\varepsilon} e^{-2\pi iny} \, dy = e^{-2\pi ina_j} \frac{\sin 2\pi nc}{\pi n}
\]

We let it mean \(2\varepsilon\) if \(n = 0\).

The multiplication theorem for \(W\)-ap functions on topological abelian groups is proved as for the usual almost periodic functions in [1], so the Fourier series of
the $W$-ap function $C_1(x)$ in (8) is obtained by formal multiplication of its factors. In particular, since $(2) \Rightarrow (3),

\[ MC_1 = \frac{1}{T_1 \ldots T_t} \sum \frac{\sin 2\pi n_{t+1} \epsilon}{\pi n_{t+1}} \ldots \frac{\sin 2\pi n_p \epsilon}{\pi n_p} \]

\[ n_1 \chi_1(x) + \ldots + n_p \chi_p(x) \equiv 0 \pmod{1} \text{ for } x \in G \]

\[ 0 \leq n_1 < T_1, \ldots, 0 \leq n_t < T_t \]

where the series, as for the usual almost periodic functions, is absolutely convergent.

We see that $MC_1$, when $(2) \Rightarrow (3)$, is independent of $a_1, \ldots, a_p$.

For $a_1 = \ldots = a_p = 0$, trivially $(2) \Rightarrow (3)$, and the set $C_0$ of solutions of (7) in this case has $MC_0 = MC_1$. We shall show that $MC_0 > 0$. We divide the cube $H^{p-t}$ in $q^{p-t}$ closed and congruent cubes, where the positive integer $q$ satisfies $q^t \leq \epsilon$. We consider the points

\[ (\chi_1(x), \ldots, \chi_p(x)) \in \left\{ 0, \frac{1}{T_1}, \ldots, \frac{T_1 - 1}{T_1} \right\} \times \ldots \]

\[ \times \left\{ 0, \frac{1}{T_t}, \ldots, \frac{T_t - 1}{T_t} \right\} \times H^{p-t} \text{ for } x \in G. \]

There are at most $T_1 \ldots T_t q^{p-t}$ possibilities for the simultaneous choice (i.e. for a given $x$) of the values of $\chi_1(x), \ldots, \chi_t(x)$ and a small cube in which $(\chi_{t+1}(x), \ldots, \chi_p(x))$ is lying. Let $x_1, \ldots, x_j, \ldots, x_s$ give all the possibilities where $s \leq T_1 \ldots T_t q^{p-t}$. For an arbitrary $x \in G$ we can therefore find an $x_j$ such that

\[ \chi_1(x - x_j) \equiv 0 \pmod{1} \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ \chi_t(x - x_j) \equiv 0 \pmod{1} \]

\[ |\chi_{t+1}(x - x_j)| \leq \epsilon \pmod{1} \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ |\chi_p(x - x_j)| \leq \epsilon \pmod{1} \]

Thus

\[ MC_1 = MC_0 = MC_0 \geq \inf_x \frac{1}{s} \sum_{j=1}^s C_0(x - x_j) \]

\[ \geq \frac{1}{s} \geq \frac{1}{T_1 \ldots T_t} \left( \frac{1}{q} \right)^{p-t} > 0. \]
We remark that Theorem 1, in contrast to Theorem 2 and 3, is contained in its discrete form (we give $G$ the discrete topology). We could also say that the characters do not have to be continuous in the original topology. Let us use the discrete form to the discrete abelian group $G_1$ of the continuous additive characters $\chi$ of the topological abelian group $G$. Since

$$x(\chi) = \chi(x), \ x \in G, \ \chi \in G_1$$

is an additive character of $G_1$, we get the

**Theorem 4.** Let $x_1, \ldots, x_p \in G$ and $a_1, \ldots, a_p \in \mathbb{R}$. A necessary and sufficient condition that the $p$ inequalities

$$|\chi(x_1) - a_1| \leq \varepsilon \pmod 1$$

$$\cdots \cdots \cdots$$

$$|\chi(x_p) - a_p| \leq \varepsilon \pmod 1$$

have a solution $\chi \in G_1$ is that for arbitrary integers $n_1, \ldots, n_p$ with

(9) \hspace{1cm} \chi(n_1 x_1 + \ldots + n_p x_p) \equiv 0 \pmod 1 \hspace{1cm} \text{for } \chi \in G_1

also

$$n_1 a_1 + \ldots + n_p a_p \equiv 0 \pmod 1$$

If, in particular, $G$ is maximally almost periodic, (9) means

$$n_1 x_1 + \ldots + n_p x_p = 0$$

**3. The second direction.**

In the following we deal with continuous additive characters $\chi$ of the topological abelian group $G$, i.e. $\chi: G \to \mathbb{R}/\mathbb{Z}$, but we prefer to let $\chi$ be multi-valued with values in $\mathbb{R}$, only determined modulo 1. We build on a generalized form of Theorem 3, but since its proof goes in nearly the same way, we omit it. By $H$ we understand the interval $0 \leq y < 1$.

Let exactly $t, 0 \leq t < p$, of the continuous additive characters $\chi_1(x), \ldots, \chi_p(x)$ be non-dense on $\mathbb{R}$. We know the structure of the closed subgroups of $\mathbb{R}^p$ and

$$K = \{ (\chi_1(x), \ldots, \chi_p(x)) : x \in G \} = \{ (\chi_1(x), \ldots, \chi_p(x)) : x \in BG \}$$

is such a subgroup, and it contains the subgroup $\mathbb{Z}^p$.

Except when $K = \mathbb{R}^p$, by eventually rearranging the characters, we get

$$K = R \oplus F$$

where $R$ is a vector space of the form

$$y_{s+1} = l(y_1, \ldots, y_s), \ldots, y_p = m(y_1, \ldots, y_s)$$
with linear forms \(l, \ldots, m\) which have rational coefficients, and \(F\) is a discrete infinite subgroup of the \(y_{s+1}, \ldots, y_p\)-space for which \(F/\mathbb{Z}^{p-s}\) is finite, with, say, \(N\) elements.

This follows from Theorem 1 by which when
\[
J = \{(n_1, \ldots, n_p) : n_1, \ldots, n_p \in \mathbb{Z}, n_1 \chi_1(x) + \ldots + n_p \chi_p(x) \equiv 0 \pmod{1} \text{ for } x \in G\}
\]
then
\[
K = \{(a_1, \ldots, a_p) : a_1, \ldots, a_p \in \mathbb{R}, n_1 a_1 + \ldots + n_p a_p \equiv 0 \pmod{1} \text{ for } (n_1, \ldots, n_p) \in J\}
\]
and since the dimension of \(J\) is \(< p\), the largest vector space contained in \(K\) has dimension \(s \geq 1\), i.e. it is not the 0-space. The rationality of the coefficients of \(l, \ldots, m\) stems from the fact that \(J\) is a subgroup of \(\mathbb{Z}^p\).

When \(K \neq \mathbb{R}^p\) we denote by \(P\) the set of points \((y_1, \ldots, y_p)\) given by

\[
\begin{align*}
y_1, \ldots, y_s & \in H \\
l(y_1, \ldots, y_s) & \leq y_{s+1} < l(y_1, \ldots, y_s) + 1 \\
\vdots & \\
m(y_1, \ldots, y_s) & \leq y_p < m(y_1, \ldots, y_s) + 1
\end{align*}
\]

When \(K = \mathbb{R}^p\) we let \(P = H^p\). Obviously it is no restriction on \(x \in BG\) to demand that

\[
(\chi_1(x), \ldots, \chi_p(x)) \in P
\]

because the values of the characters are only determined modulo 1.

We shall prove the following

**Theorem 5.** Let \(E\) be a subset of \(P\), and if \(K = \mathbb{R}^p\), let \(E\) be Jordan-measurable. If \(K \neq \mathbb{R}^p\), let the projection on the \(y_1, \ldots, y_s\)-space

\[
\text{proj } E \cap (R + f_j) \text{ for } j = 1, \ldots, N
\]

be Jordan-measurable, where

\[
\{f_1, \ldots, f_N\} = F \cap H^{p-s}
\]

In both cases the set

\[
B = \{x : x \in G, (\chi_1(x), \ldots, \chi_p(x)) \in E\}
\]

is \(W\)-ap. If \(K = \mathbb{R}^p\) then

\[
MB = mE
\]

If \(K \neq \mathbb{R}^p\) then

\[
MB = \frac{1}{N} \sum_{j=1}^{N} m \text{proj } E \cap (R + f_j)
\]
(We shall express the theorem by the phrase:

\[(\chi_1(x), \ldots, \chi_p(x)), \ x \in G\]

is \(W\)-ap equi-distributed from \(G\) into its closure \(K\) modulo 1.)

**Proof.** Let

\[f_1 = (y_{s+1}^1, \ldots, y_p^1), \ldots, f_N = (y_{s+1}^N, \ldots, y_p^N)\]

where for instance \(f_1 = (0, \ldots, 0)\). It is understood that they lie in the \(y_{s+1}, \ldots, y_p\)-space. Obviously \(Nf_j\) has integral coordinates.

For a positive integer \(q\) we divide \(H^n\) in \(q^s\) congruent cubes of the type

\[a_1 - \frac{1}{2q} \leq y_1 < a_1 + \frac{1}{2q}, \ldots, a_s - \frac{1}{2q} \leq y_s < a_s + \frac{1}{2q}\]

In each cube we choose the center \((a_1, \ldots, a_s) \in H^n\) and consider the points

\[(a_1, \ldots, a_s, l(a_1, \ldots, a_s) + y_{s+1}^j, \ldots, m(a_1, \ldots, a_s) + y_p^j) =\]

\[(a_1, \ldots, a_s, a_{s+1}^j, \ldots, a_p^j) \in P \cap (R + f_j) \subset P \cap K\]

We note that for two different \(j\) and \(k\) at least one of the numbers

\[|a_{s+1}^j - a_{s+1}^k|, \ldots, |a_p^j - a_p^k| \leq \frac{1}{N}\]

We also note that for a fixed \(j\) and a fixed \(Y = (y_{s+1}, \ldots, y_p) \in F \setminus H^{p-s}\) at least one of the numbers \(|y_{s+1}^j - y_{s+1}|, \ldots, |y_p^j - y_p|\) is \(\geq \frac{1}{N}\).

Let \(\varepsilon_0 = \min \left(\frac{1}{N}, \varepsilon_1\right)\). In generalization of Theorem 3 the \(Nq^s\) sets

\[C = C_j = \left\{x : x \in BG, a_1 - \frac{1}{2q} \leq \chi_1(x) < a_1 + \frac{1}{2q}, \ldots, a_s - \frac{1}{2q} \leq \chi_s(x) < a_s + \frac{1}{2q}, \ a_s + \frac{1}{2q}, a_{s+1}^j - \frac{\varepsilon_0}{3} \leq \chi_{s+1}(x) \leq a_{s+1}^j + \frac{\varepsilon_0}{3}, \ldots, a_p^j - \frac{\varepsilon_0}{3} \leq \chi_p(x) \leq a_p^j + \frac{\varepsilon_0}{3} \right\} \text{ (mod 1)}\]

are \(W_1\)-ap and since

\[n_1a_1 + \ldots + n_s a_s + n_{s+1}a_{s+1}^j + \ldots + n_p a_p^j \equiv 0 \pmod{1}\]

for \((n_1, \ldots, n_p) \in J\), this generalization tells us that the \(Nq^s\) sets \(C\) all have the same \(W_1\)-mean value \(M_1^1\).
Since $l, \ldots, m$ are uniformly continuous on $H^+$ the $Nq^*$ sets $C$ will, for $q$ sufficiently large have a disjoint union which is equal to
\[ \left\{ x : x \in BG, (\chi_1(x), \ldots, \chi_p(x)) \in P \cap \bigcup_{j=1}^{N} (R + f_j) = P \cap K \right\} = BG \]
Thus
\[(*) \quad Nq^* M_1^q = M_1 BG = 1 \]

Now
\[ E \cap K = \bigcup_{j=1}^{N} E \cap (R + f_j) \text{ (disjoint union)} \]
where
\[ \text{proj } E \cap (R + f_j), \quad j = 1, \ldots, N \]
is Jordan-measurable. Let
\[ \bigcup' C_j = B'_j \quad \text{and} \quad \bigcup'' C_j = B''_j \]
where $\bigcup'$ denotes union over those centers $(a_1, \ldots, a_s)$ whose $A_j$ cubes
\[ a_1 - \frac{1}{2q} \leq y_1 < a_1 + \frac{1}{2q}, \ldots, a_s - \frac{1}{2q} \leq y_s < a_s + \frac{1}{2q} \]
are contained in
\[ \text{proj } E \cap (R + f_j) \]
while $\bigcup''$ denotes union over those centers $(a_1, \ldots, a_s)$ whose $A''_j$ cubes have points in common with this projection.

Since the projection is Jordan-measurable, both
\[ \frac{A'_j}{q^*} \quad \text{and} \quad \frac{A''_j}{q^*} \rightarrow m \text{proj } E \cap (R + f_j) \quad \text{for} \quad q \rightarrow \infty \]

The set
\[ B_0 = \{ x : x \in BG, (\chi_1(x), \ldots, \chi_p(x)) \in E \} \]
satisfies
\[ B_0 \supset B'_1 \cup \ldots \cup B'_N \quad \text{(disjoint union)} \]
and
\[ B_0 \subset B''_1 \cup \ldots \cup B''_N \quad \text{(disjoint union)} \]
Here $B'_j$ and $B''_j$ are $W_1$-ap and by (*)

$$M_1B'_j = A'_j M'_1 = \frac{A'_j}{Nq^s}$$

and

$$M_1B''_j = A''_j M''_1 = \frac{A''_j}{Nq^s}$$

Hence $B_0$ is $W_1$-ap and

$$M_1B_0 = \frac{1}{N} \sum_{j=1}^{N} m \text{proj } E \cap (R + f_j)$$

From this follows that the “restriction” of $B_0$ to $G$, viz.

$$B = \{x : x \in G, (\chi_1(x), \ldots, \chi_p(x)) \in E\}$$

is $W$-ap on $G$ and

$$MB = M_1B_0 = \frac{1}{N} \sum_{j=1}^{N} m \text{proj } E \cap (R + f_j)$$

The case $K = \mathbb{R}^p$ can be treated in a similar, but simpler, way.

For $E$ lying in a fundamental domain modulo 1 of the form

$$c_1 \leq y_1 < c_1 + 1, \ldots, c_p \leq y_p < c_p + 1$$

instead of $P$, we assume, for $K \neq \mathbb{R}^p$, that

$$\text{proj } E \cap (R + f)$$

is Jordan-measurable for $f \in F$. Then

$$B = \{x : x \in G, (\chi_1(x), \ldots, \chi_p(x)) \in E\}$$

is $W$-ap with

$$MB = \frac{1}{N} \sum_{f \in F} m \text{proj } E \cap (R + f)$$

where only a finite number of terms are different from zero. For $K = \mathbb{R}^p$ we assume that $E$ is Jordan-measurable. Then $B$ is $W$-ap and

$$MB = mE$$
If \( t = p \), and the set \( E \) is contained in a fundamental domain modulo 1, then the set \( B \) is almost periodic, and

\[
MB = \frac{|E \cap K|}{|H^p \cap K|}
\]

where \( |\cdot| \) denotes "number of elements in". This follows from the proof of Theorem 3.

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