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THE CONVERSE TO ALEXANDRE FRODA'S IRRATIONALITY CRITERION

DAVID SANKER

Abstract.

Let $x_1, \ldots, x_n, \ldots, y_1, \ldots, y_n, \ldots$ be positive integers with $x_{n+1} > x_n$ for all n. In 1910 Viggo Brun proved that if y_n/x_n is a monotone increasing sequence converging to α, and $(y_{n+1} - y_n)/(x_{n+1} - x_n)$ is monotone *decreasing*, then the number α is irrational [1]. This was considerably generalized in 1963 by Alexandre Froda by including a sequence of positive integer parameters q_n: Let $Y_n = y_n/q_n$ and $X_n = x_n/q_n$. Then apply Brun's criterion to Y_n/X_n. Because Y_n and X_n are no longer integers, some modifications have to be made. The most significant change is that we are now required to find a decreasing subsequence of "fractional parts". See [3].

Froda proved that if we can find an allowable set of parameters q_n and a decreasing subsequence of fractional parts, then the number α is irrational. In this paper we will prove the converse: if α is irrational (and written as a convergent monotone sequence) then we can always choose an allowable set of parameters and a subsequence of fractional parts.

Notation.

We make the following notational changes:

1) Let $t_r = \dfrac{y_r}{x_r} - \dfrac{y_{r-1}}{x_{r-1}}, \left(t_1 = \dfrac{y_1}{x_1}\right)$ so that

$$\frac{y_r}{x_r} = \sum_{n=1}^{r} t_n \text{ and } \alpha = \sum_{n=1}^{\infty} t_n.$$

2) Define $f\colon \mathsf{N} \to \mathsf{R}$ by $f(r) = \dfrac{x_r}{q_r}$. With this definition, choosing a sequence of q's is equivalent to defining a function f. Because of the requirements on the q's, we see that $f(r) \to \infty$ and that f is monotone increasing.

3) Also let $\beta_r = \dfrac{f(r-1)}{f(r)}$ for $r \geqq 2$. Note that $0 < \beta_r < 1$. It is also convenient to define $\beta_1 = 1$.

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4) Froda defined $p_{r+1} = q_{r+1}/q_r$. We now translate Froda's condition

$$\text{(1)} \qquad \frac{y_{r+1} - p_{r+1}y_r}{x_{r+1} - p_{r+1}x_r} > \frac{y_{r+2} - p_{r+2}y_{r+1}}{x_{r+2} - p_{r+2}x_{r+1}}.$$

into the new notation. We make one modification. Froda required strict inequality in (1), but allowed inclusive inequality for the subsequence of fractional parts. It is actually more natural to switch roles, requiring strict inequality on the fractional parts, but allowing inclusive inequality in (1). In the new notation Froda's condition (1) becomes

$$\text{(2)} \qquad \frac{t_r}{t_{r+1}} \frac{\beta_r(1 - \beta_{r+1})}{(1 - \beta_r)} \geqq 1.$$

A function f on the positive integers for which $f(r) \to \infty$ and has corresponding β's satisfying $0 < \beta_r < 1$ and (2) will be called a *Froda* function. With these definitions, there exists a sequence of parameters q_n satisfying Froda's conditions if and only if there is a Froda function f.

Assume f is a Froda function for the series $\sum_{n=1}^{\infty} t_r$. It is shown in [5] that for every $s > r$ we have the bounds

$$\text{(3)} \qquad \frac{\beta_r t_r}{\beta_r \sum_{n=r}^{s} t_n - \sum_{n=r+1}^{s} t_n} \leqq \frac{f(s)}{f(r)} \leqq \frac{\sum_{n=r+1}^{\infty} t_n}{\sum_{n=s+1}^{\infty} t_n}$$

Now let's look at "fractional parts". Given an α and a Froda function f, we want to know if we can find a descending sequence of fractional parts. With our new notation the fractional parts are the quantities

$$\zeta_r = \alpha(\lceil f(r) \rceil - f(r)) + \operatorname{frac}\left(\frac{y_r}{x_r} f(r)\right),$$

where $\operatorname{frac}(z) = z - [z]$. As we can see, this would be considerably simplified if we could assume that $f(r)$ were an integer because this would eliminate the first quantity. Our construction, in fact, will produce Froda functions whose images contain the positive integers; we will then restrict our attention to those r-values for which $f(r)$ is an integer. This effectively eliminates consideration of the first quantity.

Preliminaries.

Our hope would be to prove a strong converse to Froda's Irrationality theorem: Let $\alpha = \sum_{n=1}^{\infty} t_n$, and let f be *any* Froda function for the series. If α is irrational, can we find a descending subsequence of fractional parts? An elementary counter-example in [5] shows that is strong converse does not hold, so the best we can hope for is to prove

THEOREM. *Let $\alpha = \sum_{n=1}^{\infty} t_n$, and assume α is irrational. Then there exists a Froda function f for the series which produces a descending sequence of fractional parts.*

To prove this, we will prove two other theorems. First, for any convergent series $\sum_{n=1}^{\infty} t_n$ we will construct a Froda function whose image contains the natural numbers. Second, if $\alpha = \sum_{n=1}^{\infty} t_n$ is *irrational* and f is a Froda function of the type constructed above, then we will show that f produces a descending sequence of fractional parts.

With explicit upper and lower bounds on the growth of Froda functions in (3) we now want to reverse our point of view. Given a convergent series $\sum_{n=1}^{\infty} t_n$, we will want to construct a Froda function for it. Since the process will be recursive, we will have "partial" Froda functions which we will extend to full Froda functions on the positive integers. A function f on a segment $\{1,2,\ldots,s\}$ of the positive integers is a *partial Froda function* for the series $\sum_{n=1}^{\infty} t_n$ if

$$\frac{\beta_{r-1}(t_{r-1}+t_r)-t_r}{\beta_{r-1}t_{r-1}} \geqq \beta_r \geqq \frac{\sum_{n=r+1}^{\infty} t_n}{\sum_{n=r}^{\infty} t_n} \quad \text{for } r=2,3,\ldots,s. \tag{4}$$

In order to extend partial Froda functions we need

PROPOSITION (Growth Lemma). *Let f be a partial Froda function on $\{1,2,\ldots,r\}$ for the series $\sum_{n=1}^{\infty} t_n$. Then for any $s>r$ we have*

$$\frac{\beta_r t_r}{\beta_r \sum_{n=r}^{s} t_n - \sum_{n=r+1}^{s} t_n} \leqq \frac{\sum_{n=r+1}^{\infty} t_n}{\sum_{n=s+1}^{\infty} t_n}, \tag{5}$$

and for any real number y with

$$\frac{\beta_r t_r}{\beta_r \sum_{n=r}^{s} t_n - \sum_{n=r+1}^{s} t_n} \leqq \frac{y}{f(r)} \leqq \frac{\sum_{n=r+1}^{\infty} t_n}{\sum_{n=s+1}^{\infty} t_n}, \tag{6}$$

we can extend f to a partial Froda function on $\{1, 2, \ldots, s\}$ with $f(s) = y$.

PROOF. Let's first establish inequality (5). Since f is a partial Froda function on $\{1, 2, \ldots, r\}$, we know that $\beta_r \geqq \sum_{n=r+1}^{\infty} t_n \Big/ \sum_{n=r}^{\infty} t_n$ and thus $\beta_r t_r + (\beta_r - 1) \sum_{n=r+1}^{\infty} t_n \geqq 0$. Multiply by t_{r+1} and then add $\beta_r t_r \sum_{n=r+2}^{\infty} t_n$ to get

$$[\beta_r t_r + t_{r+1}(\beta_r - 1)] \sum_{n=r+1}^{\infty} t_n \geqq \beta_r t_r \sum_{n=r+2}^{\infty} t_n.$$

This immediately yields inequality (5) with $s = r + 1$. Now, using induction, suppose that (5) holds for a given value of s. We will pass to the next inequality if we can show that

$$\frac{\beta_r \sum_{n=r}^{s} t_n - \sum_{n=r+1}^{s} t_n}{\beta_r \sum_{n=r}^{s+1} t_n - \sum_{n=r+1}^{s+1} t_n} \leqq \frac{\sum_{n=s+1}^{\infty} t_n}{\sum_{n=s+2}^{\infty} t_n}.$$

That is, it suffices to show that

$$1 + \frac{(1 - \beta_r) t_{s+1}}{\beta_r \sum_{n=r}^{s+1} t_n - \sum_{n=r+1}^{s+1} t_n} \leqq 1 + \frac{t_{s+1}}{\sum_{n=s+2}^{\infty} t_n}$$

or

$$\frac{1 - \beta_r}{\beta_r \sum_{n=r}^{s+1} t_n - \sum_{n=r+1}^{s+1} t_n} \leqq \frac{1}{\sum_{n=s+2}^{\infty} t_n}.$$

This inequality also follows from $\beta_r \geqq \sum_{n=r+1}^{\infty} t_n \Big/ \sum_{n=r}^{\infty} t_n$ by multiplying by $\sum_{n=r}^{\infty} t_n$:

$$\beta_r \left(\sum_{n=r}^{s+1} t_n + \sum_{n=s+2}^{\infty} t_n \right) \geqq \sum_{n=r+1}^{s+1} t_n + \sum_{n=s+2}^{\infty} t_n.$$

Then

$$\beta_r \sum_{n=r}^{s+1} t_n - \sum_{n=r+1}^{s+1} t_n \geqq (1 - \beta_r) \sum_{n=s+2}^{\infty} t_n.$$

Dividing by $\left(\sum_{n=s+2}^{\infty} t_n\right)\left(\beta_r \sum_{n=r}^{s+1} t_n - \sum_{n=r+1}^{s+1} t_n\right)$ yields the desired inequality and completes the induction.

Since we have established (5), there exist y values satisfying (6). Choose an s and a y value in the range given by (6). We will now show how to extend f. Surprisingly, we can choose to make the extension in a rather simple way. (And we shall *always* choose to extend in this way!) We take minimal growth for $\beta_{r+2}, \ldots, \beta_s$ [see (3)] so that

$$\beta_s = \frac{\beta_{s-1}(t_{s-1} + t_s) - t_s}{\beta_{s-1} t_{s-1}} \tag{7}$$

$$\vdots$$

$$\beta_{r+2} = \frac{\beta_{r+1}(t_{r+1} + t_{r+2}) - t_{r+2}}{\beta_{r+1} t_{r+1}},$$

and we will solve for β_{r+1}. We want to choose β_{r+1} with

$$\frac{\beta_r(t_r + t_{r+1}) - t_{r+1}}{\beta_r t_r} \geqq \beta_{r+1} \geqq \frac{\sum_{n=r+2}^{\infty} t_n}{\sum_{n=r+1}^{\infty} t_n}$$

and such that

$$\beta_{r+1}\beta_{r+2}\cdots\beta_s = \frac{f(r)}{y}.$$

To compute β_{r+1} we need a formula for $\beta_{r+1}\beta_{r+2}\cdots\beta_s$ in terms of β_{r+1}. Use induction, starting with (7) above to get

$$\beta_{s-1}\beta_s = \frac{\beta_{s-1}(t_{s-1} + t_s) - t_s}{t_{s-1}}$$

$$= \frac{\left[\dfrac{\beta_{s-2}(t_{s-2} + t_{s-1}) - t_{s-1}}{\beta_{s-2} t_{s-2}}\right](t_{s-1} + t_s) - t_s}{t_{s-1}}$$

$$= \frac{(\beta_{s-2}t_{s-2} + \beta_{s-2}t_{s-1} - t_{s-1})(t_{s-1} + t_s) - \beta_{s-2}t_{s-2}t_s}{\beta_{s-2}t_{s-2}t_{s-1}}$$

$$= \frac{\beta_{s-2}(t_{s-2} + t_{s-1} + t_s) - (t_{s-1} + t_s)}{\beta_{s-2}t_{s-2}}$$

Therefore

$$\beta_{s-2}\beta_{s-1}\beta_s = \frac{\beta_{s-2}\sum_{n=s-2}^{s} t_n - \sum_{n=s-1}^{s} t_n}{t_{s-2}}$$

$$\vdots$$

$$\beta_{r+1}\beta_{r+2}\cdots\beta_s = \frac{\beta_{r+1}\sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n}{t_{r+1}} \tag{8}$$

We want this to equal $\dfrac{f(r)}{y}$, so we must have $\beta_{r+1}\sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n = \dfrac{f(r)}{y}\cdot t_{r+1}$, or

$$\beta_{r+1} = \frac{\sum_{n=r+2}^{s} t_n + \dfrac{f(r)}{y} t_{r+1}}{\sum_{n=r+1}^{s} t_n}$$

In order for this to be an acceptable value of β_{r+1} it must satisfy

$$\frac{\beta_r(t_r + t_{r+1}) - t_{r+1}}{\beta_r t_r} \geqq \frac{\sum_{n=r+2}^{s} t_n + \dfrac{f(r)}{y} t_{r+1}}{\sum_{n=r+1}^{s} t_n} \geqq \frac{\sum_{n=r+2}^{\infty} t_n}{\sum_{n=r+1}^{\infty} t_n}.$$

These two inequalities follow readily from the two inequalities in (6). Thus β_{r+1} satisfies (4) as required. Next, it is easy to verify that

$$\beta_{r+1} \geqq \frac{\sum_{n=r+2}^{\infty} t_n}{\sum_{n=r+1}^{\infty} t_n} \Rightarrow \frac{\beta_{r+1}(t_{r+1} + t_{r+2}) - t_{r+2}}{\beta_{r+1}t_{r+1}} \geqq \frac{\sum_{n=r+3}^{\infty} t_n}{\sum_{n=r+2}^{\infty} t_n}. \tag{9}$$

But $\beta_{r+2} = \dfrac{\beta_{r+1}(t_{r+1} + t_{r+2}) - t_{r+2}}{\beta_{r+1}t_{r+1}}$, so substituting this into the second inequality in (9) we get $\beta_{r+2} \geqq \sum_{n=r+3}^{\infty} t_n \Big/ \sum_{n=r+2}^{\infty} t_n$. This shows that β_{r+2} satisfies (4). Proceeding inductively now using (9) with $r + 1$ replaced by $r + 2, \ldots$ we establish inequality (4) for all of the β's. Therefore, we can extend f to a partial Froda function on $\{1, 2, \ldots, s\}$ by defining

$$f(r + 1) = \frac{f(r)}{\beta_{r+1}},\ f(r + 2) = \frac{f(r + 1)}{\beta_{r+2}}, \ldots,\ f(s) = \frac{f(s - 1)}{\beta_s}.$$

We have satisfied all of the conditions, and the construction certainly makes $f(s) = y$.

LEMMA (Minimal growth formula). *If $r \geqq l$, and $\beta_r, \beta_{r-1}, \ldots, \beta_l$ each have minimal growth over the preceeding* (i.e., $\beta_r = \dfrac{\beta_{r-1}(t_{r-1} + t_r) - t_r}{\beta_{r-1} t_{r-1}}$, etc.), then

$$\beta_r = 1 - \frac{(1 - \beta_{l-1})}{\beta_{l-1} \sum_{n=l-1}^{r-1} t_n - \sum_{n=l}^{r-1} t_n} \cdot t_r. \tag{10}$$

PROOF. The assertion follows by an easy induction. For $r = l$ we have

$$\beta_r = \frac{\beta_{r-1}(t_{r-1} + t_r) - t_r}{\beta_{r-1} t_{r-1}} = 1 - \frac{(1 - \beta_{r-1})}{\beta_{r-1} t_{r-1}} \cdot t_r,$$

which is what we wanted. Now suppose inductively that (10) holds for a given r. Going to β_{r+1}, we have

$$\beta_{r+1} = 1 - \frac{(1 - \beta_r)}{\beta_r t_r} \cdot t_{r+1}.$$

From (10), we have

$$1 - \beta_r = \frac{(1 - \beta_{l-1})}{\beta_{l-1} \sum_{n=l-1}^{r-1} t_n - \sum_{n=l}^{r-1} t_n} \cdot t_r,$$

and

$$\beta_r = \frac{\beta_{l-1} \sum_{n=l-1}^{r-1} t_n - \sum_{n=l}^{r-1} t_n - (1 - \beta_{l-1}) t_r}{\beta_{l-1} \sum_{n=l-1}^{r-1} t_n - \sum_{n=l}^{r-1} t_n}.$$

Thus

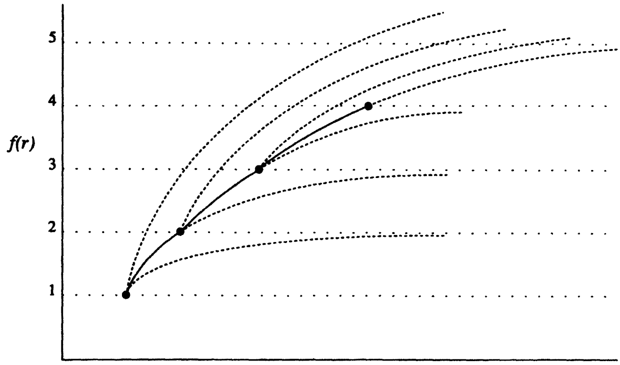
$$\begin{aligned} \beta_{r+1} &= 1 - \left(\frac{1 - \beta_r}{\beta_r}\right) \frac{t_{r+1}}{t_r} \\ &= 1 - \frac{(1 - \beta_{l-1})}{\sum_{n=l-1}^{r} t_n - \sum_{n=l}^{r} t_n} \cdot t_{r+1}. \end{aligned}$$

This completes the induction and the lemma.

The Converse, Part I.

THEOREM (Good Froda functions exist.). *If* $\sum_{n=1}^{\infty} t_n$ *converges, then there exists a Froda function for the series whose image contains the natural numbers.*

Refer to the following picture:



IDEA OF PROOF. We take $f(1) = 1$, then choose r sufficiently large so that by taking $f(r) = 2$, minimal growth after this point will be bounded by 3. In the picture, the lower dotted lines represent continued minimal growth after the chosen point, and the upper dotted lines represent maximal growth. This puts the number 3 in the "window" of possible values for $f(s)$ for all s sufficiently large. We then choose some s with $f(s) = 3$, and so large that the number 4 is in the "window" for all sufficiently large numbers. This procedure is continued recursively, giving a full Froda function for the series.

PROOF. Let $f(1) = 1$, and recall that $\beta_1 = 1$. We proceed inductively. Assume that we have a partial Froda function on $\{1, 2, \ldots, r\}$ with $f(r) = m$, and chosen so that

$$\frac{\beta_r t_r}{\beta_r \sum_{n=r}^{\infty} t_n - \sum_{n=r+1}^{\infty} t_n} < \frac{m+1}{m}.$$

Note that this is true when $r = 1$, $m = 1$, so the induction can begin. We need to show that we can extend this to a partial Froda function on $\{1, 2, \ldots, s\}$ for some $s > r$ with $f(s) = m + 1$ and such that

$$\frac{\beta_s t_s}{\beta_s \sum_{n=s}^{\infty} t_n - \sum_{n=s+1}^{\infty} t_n} < \frac{m+2}{m+1}.$$

We first show that for any s sufficiently large we may extend f to a partial Froda function on $\{1, 2, \ldots, s\}$ with $f(s) = m + 1$. According to the growth lemma, it suffices to show that

$$\frac{\beta_r t_r}{\beta_r \sum_{n=r}^{s} t_n - \sum_{n=r+1}^{s} t_n} \leqq \frac{m+1}{m} \leqq \frac{\sum_{n=r+1}^{\infty} t_n}{\sum_{n=s+1}^{\infty} t_n} \tag{11}$$

Since $\sum_{n=s+1}^{\infty} t_n \to 0$ as $s \to \infty$, the right hand expression approaches ∞, and thus the right hand inequality holds for all large s. For any finite s we certainly have

$$\frac{\beta_r t_r}{\beta_r \sum_{n=r}^{s} t_n - \sum_{n=r+1}^{s} t_n} \leqq \frac{\beta_r t_r}{\beta_r \sum_{n=r}^{\infty} t_n - \sum_{n=r+1}^{\infty} t_n} < \frac{m+1}{m}$$

because $\beta_r \leqq 1$. Therefore the left hand inequality in (11) holds for all $s > r$.

To complete our recursive construction we must show that for every s sufficiently large we have

$$\frac{\beta_s t_s}{\beta_s \sum_{n=s}^{\infty} t_n - \sum_{n=s+1}^{\infty} t_n} < \frac{m+2}{m+1}. \tag{12}$$

From formula (8) in the growth lemma we have

$$\beta_{r+1} \sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n = t_{r+1}(\beta_{r+1}\beta_{r+2} \cdots \beta_s)$$
$$= t_{r+1} \cdot \frac{f(r)}{f(s)} = t_{r+1} \cdot \left(\frac{m}{m+1}\right).$$

We also have from formula (10) in the minimal growth formula that

$$\beta_s = 1 - \frac{(1 - \beta_{r+1}) t_s}{\beta_{r+1} \sum_{n=r+1}^{s-1} t_n - \sum_{n=r+2}^{s-1} t_n}. \tag{13}$$

Let $\dfrac{m+2}{m+1} = 1 + \varepsilon$, and let $\delta = t_{r+1} \cdot \left(\dfrac{m}{m+1}\right) - \dfrac{t_{r+1} \cdot \left(\dfrac{m}{m+1}\right)}{1 + \varepsilon}$. Since $\delta > 0$, we will have $\sum_{n=s+1}^{\infty} t_n < \delta$ for all large s. Then

$$\beta_{r+1} \sum_{n=r+1}^{\infty} t_n - \sum_{n=r+2}^{\infty} t_n = \left(\beta_{r+1} \sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n\right) - (1 - \beta_{r+1}) \sum_{n=s+1}^{\infty} t_n$$

$$> \left(\beta_{r+1} \sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n\right) - \delta$$

$$= \left(\beta_{r+1} \sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n\right) - t_{r+1}\left(\frac{m}{m+1}\right)$$

$$+ \frac{t_{r+1}\left(\frac{m}{\mathrm{m}+1}\right)}{1+\varepsilon} = \frac{t_{r+1}\cdot\left(\frac{m}{m+1}\right)}{1+\varepsilon}$$

$$= \frac{\beta_{r+1} \sum_{n=r+1}^{s} t_n - \sum_{n=r+2}^{s} t_n}{1+\varepsilon}.$$

Multiply this inequality by $1 + \varepsilon$ and rearrange terms to get

$$(1+\varepsilon)\beta_{r+1} \sum_{n=r+1}^{\infty} t_n + \sum_{n=r+2}^{s} t_n > (1+\varepsilon) \sum_{n=r+2}^{\infty} t_n + \beta_{r+1} \sum_{n=r+1}^{s} t_n.$$

Now subtract $(1+\varepsilon)\beta_{r+1} \sum_{n=s}^{\infty} t_n + \beta_{r+1} \sum_{n=r+1}^{s-1} t_n + t_s + (1+\varepsilon) \sum_{n=r+2}^{s-1} t_n$ from both sides to get

$$(1+\varepsilon)\beta_{r+1} \sum_{n=r+1}^{s-1} t_n + \sum_{n=r+2}^{s-1} t_n - \beta_{r+1} \sum_{n=r+1}^{s-1} t_n - (1+\varepsilon) \sum_{n=r+2}^{s-1} t_n >$$

$$> (1+\varepsilon) \sum_{n=s}^{\infty} t_n + \beta_{r+1} t_s - t_s - (1+\varepsilon)\beta_{r+1} \sum_{n=s}^{\infty} t_n.$$

Combining terms, we get

$$\varepsilon\beta_{r+1} \sum_{n=r+1}^{s-1} t_n - \varepsilon \sum_{n=r+2}^{s-1} t_n > (1+\varepsilon) \sum_{n=s}^{\infty} t_n - t_s - (1+\varepsilon)\beta_{r+1} \sum_{n=s}^{\infty} t_n + \beta_{r+1} t_s.$$

Now, factor to get

$$\varepsilon\left(\beta_{r+1} \sum_{n=r+1}^{s-1} t_n - \sum_{n=r+2}^{s-1} t_n\right) > (1-\beta_{r+1})\cdot\left((1+\varepsilon) \sum_{n=1}^{\infty} t_n - t_s\right).$$

Dividing, we get the appropriate fractions:

$$\frac{\varepsilon}{(1+\varepsilon) \sum_{n=s}^{\infty} t_n - t_s} > \frac{1-\beta_{r+1}}{\beta_{r+1} \sum_{n=r+1}^{s-1} t_n - \sum_{n=r+2}^{s-1} t_n}.$$

If we multiply this by t_s and subtract the quantities from 1 we get

$$1 - \frac{(1-\beta_{r+1})t_s}{\beta_{r+1}\sum_{n=r+1}^{s-1} t_n - \sum_{n=r+2}^{s-1} t_n} > 1 - \frac{\varepsilon \cdot t_s}{(1+\varepsilon)\sum_{n=s}^{\infty} t_n - t_s}.$$

By (13), the left expression is β_s, so

$$\beta_s > 1 - \frac{\varepsilon \cdot t_s}{(1+\varepsilon)\sum_{n=s}^{\infty} t_n - t_s}.$$

Then

$$(1+\varepsilon)\beta_s \sum_{n=s}^{\infty} t_n - \beta_s t_s > (1+\varepsilon)\sum_{n=s}^{\infty} t_n - t_s - \varepsilon \cdot t_s = (1+\varepsilon)\sum_{n=s+1}^{\infty} t_n.$$

Therefore $(1+\varepsilon)\left(\beta_s \sum_{n=s}^{\infty} t_n - \sum_{n=s+1}^{\infty} t_n\right) > \beta_s t_s$, so

$$\frac{\beta_s t_s}{\beta_s \sum_{n=s}^{\infty} t_n - \sum_{n=s+1}^{\infty} t_n} < 1 + \varepsilon = \frac{m+2}{m+1}.$$

This establishes (12) for all large s, and completes the recursive construction. Finally, we should note that the heights are proceeding in integer steps so that we do have $f(r) \to \infty$. This completes the proof.

The Converse, Part II.

To show that these Froda finctions produce decreasing fractional parts when α is irrational, we will need the following lemma:

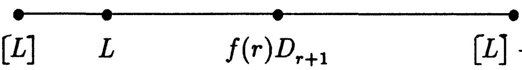
LEMMA. *If ξ is irrational, then the sequence $\xi, 2\xi, 3\xi, \ldots$ is uniformly distributed modulo* 1.

PROOF. See Niven [4].

THEOREM. *Let $\alpha = \sum_{n=1}^{\infty} t_n$, and assume α is irrational. Let f be a Froda function for the series whose image contains* N. *Then there exists a descending subsequence of fractional parts.*

PROOF. In the following it will be convenient to define $D_{r+1} = \sum_{n=r+1}^{\infty} t_n$. From (3) we see that $f(s)D_{s+1} \leqq f(r)\,D_{r+1}$. Since this sequence is monotone decreasing, there is a limit L by the completeness of the real numbers. With this L, let us define

$\varepsilon = 1 - \operatorname{frac}(L)$. (Note that $0 < \varepsilon \leqq 1$.) Take R so large that $r \geqq R \Rightarrow f(r)D_{r+1} < L + \frac{\varepsilon}{2}$. In particular, we note that for $r \geqq R$ we have $f(r)D_{r+1} < [L] + 1$, so $\operatorname{frac}(f(\mathrm{r})D_{r+1}) - \operatorname{frac}(L) = f(r)D_{r+1} - L$.
Here is the picture:



Now choose an $r_1 \geqq R$ with $1 - \frac{\varepsilon}{4} < \operatorname{frac}(\alpha \cdot f(r_1)) < 1$ and $L \leqq f(r_1)D_{r_1+1} < L + \frac{\varepsilon}{8}$. This may be done because the image of f contains $\mathbf{N}$ and the fact that integer multiples of any irrational number are uniformly distributed (see lemma). We now proceed with a recursive construction. (We will be implicitly using the Axiom of Choice.) We will show at the kth step there exists an $r_k > r_{k-1}$ with

$$(14) \qquad M - \frac{\varepsilon}{2^{2k}} < \operatorname{frac}(\alpha \cdot f(r_k)) < M \text{ and } f(r_k)D_{r_k+1} < L + \frac{\varepsilon}{2^{2k+1}}.$$

where $M = \operatorname{frac}(\alpha \cdot f(r_{k-1})) - f(r_{k-1})D_{r_{k-1}+1} + L$. We have already constructed r_1, so we proceed invductively assuming $r_1, r_2, \ldots, r_{k-1}$ have been constructed, all satisfying (14). To use Niven's Lemma and show that an r_k exists, we need to show that $I = \left[M - \frac{\varepsilon}{2^{2k}}, M\right]$ is a subinterval of $[0, 1]$. That is, we must show that $M - \frac{\varepsilon}{2^{2k}} > 0$ and $M < 1$. The inequality $M < 1$ is easy to see:

$$\begin{aligned} M = {} & \operatorname{frac}(\alpha \cdot f(r_{k-1})) - f(r_{k-1})D_{r_{k-1}+1} + \\ & + L < \operatorname{frac}(\alpha \cdot f(r_{k-1})) \\ & \quad < \operatorname{frac}(\alpha \cdot f(r_{k-2})) - f(r_{k-2})D_{r_{k-1}+1} + L \\ & \quad < \operatorname{frac}(\alpha \cdot f(r_{k-2})) \\ & \quad \vdots \\ & \quad < \operatorname{frac}(\alpha \cdot f(r_1)) \\ & \quad < 1. \end{aligned}$$

We also have

$$\begin{aligned} M - \frac{\varepsilon}{2^{2k}} &= \operatorname{frac}(\alpha \cdot f(r_{k-1})) - f(r_{k-1})D_{r_{k-1}+1} + L - \frac{\varepsilon}{2^{2k}} \\ &> \operatorname{frac}(\alpha \cdot f(r_{k-1})) - \frac{\varepsilon}{2^{2k-1}} - \frac{\varepsilon}{2^{2k}} \\ &> \operatorname{frac}(\alpha \cdot f(r_{k-2})) - f(r_{k-2})D_{r_{k-1}+1} + L - \frac{\varepsilon}{2^{2k-2}} - \frac{\varepsilon}{2^{2k-1}} - \frac{\varepsilon}{2^{2k}} \end{aligned}$$

$$
\begin{aligned}
&> \operatorname{frac}(\alpha \cdot f(r_{k-2})) - \frac{\varepsilon}{2^{2k-3}} - \frac{\varepsilon}{2^{2k-2}} - \frac{\varepsilon}{2^{2k-1}} - \frac{\varepsilon}{2^{2k}} \\
&\ \vdots \\
&> \operatorname{frac}(\alpha \cdot f(r_1)) - \frac{\varepsilon}{2^3} - \frac{\varepsilon}{2^4} - \cdots - \frac{\varepsilon}{2^{2k}} \\
&> 1 - \frac{\varepsilon}{2^2} - \frac{\varepsilon}{2^3} - \cdots - \frac{\varepsilon}{2^{2k}} \\
&= 1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{2k}} \\
&> 1 - \frac{\varepsilon}{2} > 0
\end{aligned}
$$

Since $f(r)D_{r+1} \to L$, there exists an R_k such that $r \geqq R_k \Rightarrow f(r)\,D_{r+1} - L < \frac{\varepsilon}{2^{2k+1}}$. Then there exists an $r_k \geqq \max\{R_k, r_{k-1}\}$ with $\operatorname{frac}(\alpha \cdot f(r_k)) \in I$. This r_k satisfies (14).

This completes our recursive construction. Now note that

$$\operatorname{frac}\left(\frac{y_r}{x_r} f(r)\right) + \operatorname{frac}(f(r)D_{r+1}) = \operatorname{frac}(\alpha \cdot f(r)) + \delta_r, \tag{15}$$

where $\delta_r = 0$ or 1. From our construction we will show that $\delta_r = 0$ for $r = r_1, r_2, \ldots$ Now $\operatorname{frac}\left(\frac{y_r}{x_r} f(r)\right) < 1$, and $\operatorname{frac}(f(r)D_{r+1}) < \operatorname{frac}(L) + \frac{\varepsilon}{2}$ for $r \in \{r_1, r_2, \ldots\}$ because each $r_k \geqq R$, so the left hand side of (15) is less than $1 + \operatorname{frac}(L) + \frac{\varepsilon}{2}$ for $r \in \{r_1, r_2, \ldots\}$. On the other hand we will prove by induction that

$$\operatorname{frac}(\alpha \cdot f(r_k)) > 1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{2k}} \tag{16}$$

The case $k = 1$ is clear because $\operatorname{frac}(\alpha \cdot f(r_1)) > 1 - \frac{\varepsilon}{4}$. Now assume that (16) is true for $k - 1$ and proceed. By (14) we have

$$
\begin{aligned}
\operatorname{frac}(\alpha \cdot f(r_k)) &> \operatorname{frac}(\alpha \cdot f(r_{k-1})) - f(r_{k-1})D_{r_{k-1}+1} + L - \frac{\varepsilon}{2^{2k}} \\
&> \left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{2(k-1)}} - \frac{\varepsilon}{2^{2(k-1)+1}}\right) - \frac{\varepsilon}{2^{2k}} \\
&= 1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{2k}} > 1 - \frac{\varepsilon}{2} = \operatorname{frac}(L) + \frac{\varepsilon}{2}.
\end{aligned}
$$

This completes the induction. Now back to (15). If $\delta_r = 1$, then we would have the right hand side greater than $1 + \operatorname{frac}(L) + \frac{\varepsilon}{2}$, which is impossible since the left side is less than this quantity. Therefore

$$\operatorname{frac}(\alpha \cdot f(r_k)) = \operatorname{frac}\left(\frac{y_{r_k}}{x_{r_k}} f(r_k)\right) + \operatorname{frac}(f(r_k) D_{r_k+1})$$

for $k = 1, 2, \ldots$. Now show that we have decreasing fractional parts:

$$\begin{aligned}
\operatorname{frac}\left(\frac{y_{r_k}}{x_{r_k}} f(r_k)\right) + \operatorname{frac}(f(r_k) D_{r_k+1}) = & \\
&= \operatorname{frac}(\alpha \cdot f(r_k)) \\
&< \operatorname{frac}(\alpha \cdot f(r_{k-1})) - f(r_{k-1}) D_{r_{k-1}+1} + L \\
&= \operatorname{frac}\left(\frac{y_{r_{k-1}}}{x_{r_{k-1}}} f(r_{k-1})\right) + \operatorname{frac}(f(r_{k-1}) D_{r_{k-1}+1}) - \\
&\quad - f(r_{k-1}) D_{r_{k-1}+1} + L \\
&= \operatorname{frac}\left(\frac{y_{r_{k-1}}}{x_{r_{k-1}}} f(r_{k-1})\right) + \operatorname{frac}(f(r_{k-1}) D_{r_{k-1}+1}) - \\
&\quad - \operatorname{frac}(f(r_{k-1}) D_{r_{k-1}+1}) + \operatorname{frac}(L) \\
&= \operatorname{frac}\left(\frac{y_{r_{k-1}}}{x_{r_{k-1}}} f(r_{k-1})\right) + \operatorname{frac}(L).
\end{aligned}$$

Since $\operatorname{frac}(f(r_k) D_{r_k+1}) > \operatorname{frac}(L)$, it follows that

$$\operatorname{frac}\left(\frac{y_{r_k}}{x_{r_k}} f(r_k)\right) < \operatorname{frac}\left(\frac{y_{r_{k-1}}}{x_{r_{k-1}}} f(r_{k-1})\right).$$

This completes the proof that we have descending fractional parts for this subsequence.

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HOLY NAMES COLLEGE
3500 MOUNTAIN BLVD.
OAKLAND, CA. 94619
U.S.A