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SPACES MAKING CONTINUOUS CONVERGENCE AND LOCALLY UNIFORM CONVERGENCE COINCIDE, THEIR VERY WEAK P-PROPERTY, AND THEIR TOPOLOGICAL BEHAVIOUR.

Zur Erinnerung an G. H. Wenzel, einen großartigen Kollegen und begeisternden Lehrer

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Abstract.

Although c = lu-spaces (namely, those defined by the title) were characterized internally some years ago by what can now be seen as the weakest in a range of P-properties, their general behaviour remained largely obscure. Our study of these properties casts some light on this, and enables us to place c = lu-spaces in their topological context. In addition to the obvious links with the P-property and with local compactness, some unexpected ones emerge as well, with countable box products and anti-compactness, for instance.

§0. Introduction.

In order to refine the duality between a space Z and the set CZ of all its continuous real-valued functions, the latter has often been equipped with various types of convergence such as uniform or compact convergence or more recently, continuous or locally uniform convergence. This led to what we call the c = lu-question: when do continuous convergence and locally uniform convergence coincide? In other words, what do c = lu-spaces look like? Kutzler [10], Binz [3] and Schroder [13] all gave answers. Then Schroder [14] devoted a paragraph to the topological meaning of their results, with plausible but false findings. The present paper originates in our efforts to understand the answers better, to correct the errors, and above all, to study the behaviour of c = lu-spaces.

Survey. Let us review the situation. Since we take the various topological characterizations of c = lu-spaces as the starting point for our work, we content ourselves with a rather crude picture of locally uniform convergence and continuous convergence: they both lie between their more familiar relatives, uniform and compact convergence. Details can be found in the references listed above.

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To begin with, locally compact spaces are c = lu, because they make compact convergences coincide with both locally uniform and continuous convergence. Conversely, for any completely regular Hausdorff topological space, if either locally uniform or continuous convergence coincides with compact convergence, then the space is locally compact. See Poppe [12] or Binz [3].

More than ten years ago, c = lu-spaces were characterized for the first time, in terms of Čech-Stone compactifications and P-sets. By definition, a point or set in a topological space has the *P-property* if its neighbourhood filter is s-complete. (We hope set-theorists will excuse this term, which just means that the filter is closed under countable intersections.) Naturally, P-spaces consist entirely of P-points.

0.1. THEOREM. *A completely regular Hausdorff topological space is* c = lu *iff it is a* P-*set in its Čech-Stone compactification.*

0,2. COROLLARY. *In any completely regular Hausdorff* c = lu-*space, each first-countable point has a compact neighbourhood. In particular, metrizable* c = lu-*spaces are locally compact.*

These are proved in Binz [3], Theorem 85 and Corollary 87. The more versatile form of this characterization given below can be obtained from 0.1 simply by 'chasing compact sets up and down between remainders'.

0.3. COROLLARY. *A completely regular Hausdorff topological space is* c = lu *iff it is a* P-*set in some (or equivalently, every) Hausdorff compactification.*

By direct calculation or by 0.3, all P-spaces are c = lu. Looking for a topological property common to P-spaces and to locally compact spaces, we devised a weak P-property and called it P/C. (Van Douwen [17] surveyed several other types of weak P-property, but P/C is rather weaker than most of these.)

To define P/C, take a filter ϕ on a space Z. We call ϕ sc-*complete* (again with apologies to set-theorists), if for each sequence (P_n) in ϕ there is a member P_0 of ϕ, such that the sets $P_0 \backslash P_n$ are all sub-compact. (One calls a set *sub-compact* if it has a compact superset.)

Much as above, we call a point or set P/C if its neighbourhood filter is sc-complete, and we call a space P/C if it consists entirely of P/C-points.

0.4. PROPOSITION. *Locally compact spaces are* P/C, *and so are* P-*spaces. All* P/C-*spaces are* c = lu.

To prove this, use the definition or see 2.5.

The Tychonov plank provides the raw material for two useful examples. Let ω_0 and ω_1 be the first infinite and the first uncountable ordinals respectively, and let W_0 and W_1 be the corresponding compact Hausdorff ordinal spaces. Put $W = W_0 \times W_1$ and $V = W \backslash R$, where $R = \omega_0 \times [0, \omega_1)$.

Clearly W compactifies V, with remainder R. Now as the union of any sequence of compacta in R is sub-compact in R, the space V is c = lu, by 0.3. On the other hand, no neighbourhood of $\omega_0 \times \omega_1$ in V is compact, yet the sequence converges to $(\mu \times \omega_1)$ in V.

0.5. EXAMPLE. *The* c = lu-*space* V *is neither a* P-*space nor locally compact.*

In fact, V is not just c = lu, it is P/C, by 2.6. However, c = lu-spaces without the P/C-property do exist as well. To construct one, let U be the P-space obtained from W_1 by deleting all countable limit ordinals, and put $L = W_0 \times U$. Clearly W compactifies L. Thus by 0.3, L is c = lu. However, because compact sets in U are all finite, the neighbourhood filter of $\omega_0 \times \omega_1$ in L is not sc-complete.

0.6. EXAMPLE. *The space* L *is* c = lu *but not* P/C.

In short, all four properties really are distinct.

To complete this review, we point out the main drawback of 0.3: even within its domain of complete regularity, it can only be put into use if a suitable compactification can be found. This difficulty vanished in 1976 with Schroder's internal description of c = lu-spaces, stated in 2.0 below. Our entire paper is based on this result. Its proof appears most accessibly in Schroder [15], together with a real extension of the original results.

The paper. We aim this paper at the main-stream topologist using the language of convergence to place some familiar ideas in the context that the generality of 2.0 demands and justifies. For completeness and to display our notation, we include a brief comparison of convergence and topology in §1, which the expert in convergence can pass over. We emphasize: *most of our results hold for both convergence and topology; the exceptions are marked clearly.*

To make 2.0 easier to use and understand, we express it in simpler terms, a sort of regularity and a sort of countable completeness for covers. The essence of c = lu lies in the latter, an apparently new property called cc. In fact, cc implies c = lu, and for suitably regular spaces, they are euqivalent. Within the rough bounds set by local compactness and by the P-property, P/C reflects the behaviour of c = lu quite closely – often with much easier proofs. Because of all this, we devote this paper not just to c = lu, but also to P/C and cc. We study their inter-relationships and their hereditary behaviour, we compare them with their elder relatives, local compactness and the P-property (see Misra [11] for details), and we begin to look at their interaction with other topological properties.

In their hereditary behaviour, c = lu, cc and P/C broadly resemble local compactness: they are closed-hereditary, and open-hereditary too, in suitably regular spaces. In final situations, their behaviour depends sensitively on the category in use. For example, they pass to all final limits in the category of filter convergence spaces, they pass to some pre-topological final limits, and to even fewer topological ones. See §3 and §6.

Products caused us the difficulties seen in §4 and §5. For the usual Tychonov product to be cc, almost all factors must be compact and all must be cc. Even for finite products, we do not know whether this necessary condition is also sufficient, that is, if cc × cc = cc. On the positive side, we prove that cc × P/C = cc (one of our main results), and we do know when P/C × P/C = P/C.

To our surprise, we found an infinite-dimensional Hausdorff topological vector space with the cc-property. This shows the weakness of cc compared with P and local compactness: the latter makes the space finite-dimensional, and the former makes it zero-dimensional. This example led us to a general theorem: countable box products of locally compact spaces are cc.

Among the strictly topological facts collected in §6, we found

(a) that the neighbourhood filter of the set of all points devoid of compact neighbourhoods helps determine whether a topological space is cc, and

(b) that two rather special topological inductive limits do inherit cc, even when the results of §3 do not apply to them.

The process of decompactification introduced in §7 enables us to analyse and refute the claim in Schroder [14], §5, that the c = lu-property of a space resides outside its compact sets. (Evidence for this claim comes from the definition of P/C, which 'factors out the compact sets', and from item (a) in the paragraph above.) Further, decompactification generates precisely those T_1 spaces called compact-finite in Hutton and Reilly [6] and anti-compact in Bankston [1].

§1. Topology and Convergence.

This short comparison of convergence and topology summarises the ideas used in our work and sets out our notation. We hope it will enable any general topologist to read the paper; those familiar with convergence will find little new except perhaps 1.3 and 1.5. The facts needed here appear below, and Gähler [5] provides a comprehensive treatment for spaces satisfying slightly different axioms.

Convergence. Throughout this paper, sets or spaces labeled Q to Z are non-void, filters are proper filters on some set, and as our description of the space V shows, we often identify a singleton with its point. Regarding convergence as a relation (usually symbolised as $\rightarrow$) between the filters on a set Z and the points of Z, we follow Beattie, Butzmann and Herrlich [2] and stipulate that for all z in Z,

(C_1) the ultra-filter $\hat{z}$ over z converges to z, and

(C_2) if a filter converges to z, then so does every finer filter.

One calls any such pair Z, $\rightarrow$ *a filter convergence space*, or simply, the *space* Z.

Neighbourhoods and pre-topological spaces. Consider a space Z. The intersection of all filters converging to z is its *neighbourhood filter*, denoted by $\mathrm{nb}_Z(z)$ or simply by $\mathrm{nb}(z)$. The point z is *pre-topological in* Z if $\mathrm{nb}(z) \to z$; the space nZ in which ϕ converges to z iff $\phi \supset \mathrm{nb}(z)$ is known as the *pre-topological modification of* Z; and Z is *pre-topological* if $Z = nZ$. Clearly, topological spaces are just those pre-topological spaces whose neighbourhood filters satisfy the 'fourth neighbourhood axiom'.

Closure and topology. Given a subset A of a space Z, one defines its *closure*, denoted by $\mathrm{cl}_Z(A)$ or simply by $\mathrm{cl}(A)$, just as in topology: $z \in \mathrm{cl}(A)$ iff $A \in \phi$ for some filter ϕ converging to z in Z. Similarly, one calls A *closed* if $A = \mathrm{cl}(A)$, and *open* if $A \in \mathrm{nb}(z)$ for all $z \in A$. Though closures need not be closed in general, the familiar connections between neighbourhood filters and closure, and between closed sets and open sets do persist. See 1.1 below for a reminder.

1.1. LEMMA. *Let $z \in Z$ and $A \subset Z$. Then*
(i) $z \notin \mathrm{cl}(A)$ *iff* $Z \backslash A \in \mathrm{nb}(z)$
(ii) *A is closed iff $Z \backslash A$ is open.*

The open sets form a topology on Z, defining its *topological modification* tZ. Clearly, Z is a topological space iff $Z = tZ$.

Continuity and the functor w. By definition, continuous functions preserve convergence and by convention, CZ stands for the set of all continuous real-valued functions on Z. One obtains the cr-*modification* wZ of a space Z by giving it the initial topology induced by CZ. We call any subset of Z closed in wZ, a w-*set*.

The functors n, t and w enjoy similar universal properties over pre-topological, topological and completely regular spaces respectively. For instance, a map from Z to a topological space T, is continuous iff it is continuous, from tZ to T. Finally, using $>$ for continuous inclusions, we record the obvious: $Z > nZ > tZ > wZ$.

Regularity and the functor v. Regularity and the lower separation axioms need no change from topology: a space is T_1 if all its finite sets are closed, T_2 if each filter converges to at most one point, and T_3 if it is T_1 and regular (see below). As usual, $T_3 \Rightarrow T_2 \Rightarrow T_1$.

The space rZ in which ζ converges to z iff ζ contains $\{\mathrm{cl}(P) : P \in \Phi\}$, for some ϕ converging to z in Z, is known as the *regularization of* Z, and Z is *regular* if $Z = rZ$. (Note: even in topology, these regularizations need not be regular.)

Similarly, taking closure in wZ, one gets the *w-regularization* vZ of Z. Of course, one calls Z *w-regular* if $Z = vZ$.

1.2. LEMMA. *Let A be a closed subset of a regular space Z. Suppose $\phi \to z$ and $z \notin A$. Then* $\mathrm{cl}(S) \cap A = \emptyset$, *for some* $S \in \phi$.

PROOF. Being an open set in a regular space, Z/A belongs to the filter based on the family $\{\mathrm{cl}(F): F \in \phi\}$.

COVERS. Take subsets $A, B, \ldots$ of a set Z and families $\alpha, \beta, \ldots$ of subsets of Z. Now let $\mathfrak{p}\alpha$ denote the closure of α under finite unions, and let

$$\alpha + \beta = \{A \cup B : A \in \alpha \text{ and } B \in \beta\}.$$

Further, one says α *meets* β if some set belongs to both α and β. Finally, we write $A < \beta$ if A lies inside a finite union of members of β, and we say α *refines* β *weakly* if α refines $\mathfrak{p}\beta$, or in other words, if $A < \beta$ for all $A \in \alpha$.

Let γ be a family of subsets of Z. Then by definition, γ *covers* a given set of filters on Z if it meets each filter in the set. For the sets of all convergent filters on a space Z, or of all filters converging to z, phrases such as 'γ covers z in Z', or 'γ covers Z' mean simply that γ covers the relevant set of convergent filters. Naturally, *w-covers* are covers consisting wholly of w-sets.

It seems to have escaped notice that one can use covers to characterize continuity.

1.3. PROPOSITION. *Take spaces X and Y, let s be a map from X to Y, and let $\tilde{s}$ stand for its converse, as a relation from Y to X. Then*

(i) *the family $\tilde{s}(\delta) = \{\tilde{s}(P): P \in \delta\}$ covers x in X, if s is continuous at x and δ covers $s(x)$ in Y, and*

(ii) *s is continuous at each point of $\tilde{s}(y)$ iff $\tilde{s}(\delta)$ covers each point of $\tilde{s}(y)$ whenever δ covers y in Y.*

PROOF. (i) Suppose that s is continuous at x, that δ covers $s(x)$ and that $\phi \to x$ in X. Since $s(\phi) \to s(x)$ in Y, there are sets F in ϕ and D in δ such that $s(F) \subset D$. Hence $\tilde{s}(\delta)$ meets ϕ, as desired.

(ii) Part (i) deals with one half of this. Conversely, let $\tilde{s}(\delta)$ cover each point of $\tilde{s}(y)$ whenever δ covers y in Y, take x in $\tilde{s}(y)$ and let ϕ converge to x in X. Suppose $s(\phi)$ does not converge to y. Then by the axioms for convergence, $\zeta \backslash s(\phi)$ is non-void, for each filter ζ converging to y. Thus one can find a cover δ of y in Y such that $D \notin s(\phi)$ for all D in δ. But as $\tilde{s}(\delta)$ covers x in X, one can find some D_0 in δ such that $\tilde{s}(D_0)$ belongs to ϕ. In other words, $D_0 \in s(\phi)$, a contradiction.

Initial limits. Sometimes known as inverse limits, initial limits encompass sub-spaces, products, and pull-backs.

Consider an initial family of maps, in which f_j maps a set Z to a space X_j for each j in an index set J. Now define convergence on Z as follows: $\phi \to z$ in Z iff for all j in J, the image filter $f_j(\phi)$ converges to $f_j(z)$ in X_j. The space $Z, \to$ obtained in this way is called the *initial limit* of the family.

In particular, one obtains sub-spaces and products by using the inclusion maps and the coordinate projections respectively.

If the spaces X_j are all pre-topological (or topological), then so is the limit described above, and further, it 'coincides with' the pre-topological (or topological) initial limit.

1.4. LEMMA. *Let A be a subset of Z and let α cover the sub-space* cl(A) *of Z. Then $\alpha + \{Z\backslash A\}$ covers Z.*

PROOF. Suppose $\phi \to z$ in Z. Put $\zeta = \{Q \cap A : Q \in \phi\}$ and $\eta = \{Q\backslash A : Q \in \phi\}$. Then $\phi = \zeta + \eta$ and $Z\backslash A \in \eta$, and either $\emptyset \in \zeta$ or $\zeta \to z$ in A (so that ζ meets α). Together, these facts imply that ϕ meets $\alpha + \{Z\backslash A\}$, as desired.

Final limits. Unlike initial limits, final limits depend sensitively on the category in use. For this reason, we describe them in rather more concrete detail. Consider a final family of maps g_j from spaces Y_j to the set Z, and define $\phi \to z$ in Z iff $\phi = \hat{z}$ or $g_j(y) = z$ and $\phi \supset g_j(\zeta)$, for some index j, some point y in Y_j, and some filter ζ converging to y in Y_j. The space $Z, \to$ obtained in this way is the (direct or) final limit in the category of filter convergence spaces. But obviously, it needs a shorter name: we call it the *minimal limit* of the family.

In other categories, the final limit coincides with the reflection of the minimal limit – for instance, if the spaces Y_j are all topological then the final limit is simply the topological modification of the minimal limit defined above.

Minimal quotients. Though minimal limits can be characterized by covers, the result looks rather messy in its full generality. Accordingly, we consider only the special case of minimal quotients.

Take a surjection s from a space X to a set Y. We turn Y into the minimal quotient (that is, the minimal limit of the family $\{s\}$) as follows: $\zeta \to y$ in Y iff for some x with $s(x) = y$ and some filter ϕ converging to x in X, $s(\phi) = \zeta$. Again, note our 'Bourbaki convention' under which $\tilde{s}$ denotes the converse of s. Note also 1.3, which characterizes continuity by covers.

1.5. PROPOSITION. *Let X and Y be spaces, and let s map X onto Y. Then Y is the minimal quotient iff s is continuous and for all y, $s(\gamma)$ covers y in Y whenever γ covers each point of $\tilde{s}(y)$ in X.*

PROOF. Let Y be the minimal quotient. Then trivially, s is continous. Further, let γ cover each point of $\tilde{s}(y)$, and take a filter ζ converging to y in Y. By definition of the quotient, there are x in $\tilde{s}(y)$ and ϕ converging to x in X such that $s(\phi) \supset \zeta$. Now choose C_0 in $\gamma \cap \phi$, and note that its image $s(C_0)$ belongs to both $s(\gamma)$ and ζ, as desired.

Conversely, assume the cover and continuity conditions given above. Suppose that Y is not the minimal quotient. Then one can take a filter ζ converging to y in Y such that, for each x in $\tilde{s}(y)$ and each filter ϕ converging to x in X, ζ is not finer than $s(\phi)$. Then much as above, one can find a family γ which covers each point of

$\bar{s}(y)$ in X, such that $s(C) \notin \zeta$, for all C in γ. In other words, $s(\gamma)$ fails to cover y in Y, a contradiction.

Compactness. As usual, one calls Z *compact* if each ultra-filter on Z converges to some point of Z. A subset A of Z is *compact* if it is compact, as a sub-space of Z. Further, a point or space is *locally compact* (lc, for short) if it is covered by the compact subsets.

1.6. PROPOSITION. *Let ζ be a filter on Z and Ξ be a set of ultra-filters on Z. Then Ξ contains the set of all ultra-filters finer than ζ iff $\mathfrak{p}\gamma$ meets ζ whenever γ covers Ξ.*

This piece of pure set theory, needed later, enables one to define compact sets of filters. It also enables one to retrieve the usual characterization of compactness given below: just put $\zeta = \{Z\}$ and let Ξ be the set of all convergent ultra-filters on Z.

1.7. COROLLARY. *A space Z is compact iff $Z < \gamma$, for each cover γ of Z.*

As in topology, this criterion can be weakened in regular spaces.

1.8. PROPOSITION. *Let Z be regular and D be dense in Z. Then Z is compact iff $D < \gamma$, for each cover γ of Z.*

PROOF. As noted above, if Z is compact then $D \subset Z \in \mathfrak{p}\gamma$, for each cover γ of Z. Conversely, let β cover Z. By regularity, $\{\mathrm{cl}(Q) : Q \in \gamma\}$ refines β, for some cover γ of Z, and by assumption, $D \subset Q_1 \cup \ldots \cup Q_k$, for some $Q_1, \ldots, Q_k$ in γ. Now $Z = \mathrm{cl}(D) \subset \bigcup \mathrm{cl}(Q_i) \subset Z$, and so $Z < \beta$, as required for compactness.

§2. Definitions and Basic Properties: c = lu and cc.

That brought our short course in convergence to an end, and allows us to return to the c = lu-problem. Calling a set Ψ of families of subsets of a space Z *weakly countably directed* if, for each sequence (γ_n) in Ψ, some γ in Ψ refines each γ_n weakly, Schroder [13] characterized c = lu-spaces as follows.

2.0. BASIC THEOREM. *For any space Z, the following statements are equivalent:*
(i) *Z is* c = lu, *and*
(ii) *the set of all w-covers of Z is weakly countably directed.*

Because 'real analysis' plays no (other) part in this paper, (ii) above becomes our primary definition. Since much of our work deals with individual points, we need the pointwise formulation given below. Then we 'tidy it up', by separating off the forgetful effects of the well-behaved functor v.

2.1. DEFINITION. Consider a space Z. We call z a cc-*point* in Z, if for each sequence γ_n of covers of Z, some cover γ of z refines each γ_n weakly. Naturally we call Z a cc-*space* if it consists entirely of cc-points, or equivalently, if the set of all covers of Z is weakly countably directed. Using w-covers instead of covers, we define c = lu-*points* similarly.

2.2. COROLLARY. (i) *A space is* c = lu *iff its w-regularization is* cc, *iff all its points are* c = lu.

(ii) *For both points and spaces,* cc *implies* c = lu; *conversely, in w-regular spaces,* c = lu *implies* cc.

(iii) *For both points and spaces, regularization and w-regularization preserve the* cc-*property.*

One derives 2.2 directly from the basic theorem, with the help of two obvious facts: Z and vZ have the same w-covers, and each cover of vZ can be refined by a w-cover.

2.3. DEFINITION. One calls z a *P-point* in the space Z if, whenever $\phi \to z$, some coarser s-complete filter ζ has the same limit z. Similarly, P/C-*points* are defined by sc-complete filters ζ.

2.4. LEMMA. *In any space Z, z is a P-point iff for each filter ϕ converging to z, there is a coarser filter ζ converging to z such that $\bigcap G_n \in \phi$, for each sequence (G_n) in ζ.*

Even in this general situation, 0.4 remains true: see below.

2.5. PROPOSITION. *For both points and spaces,* lc $\Rightarrow$ P/C, P $\Rightarrow$ P/C, *and* P/C $\Rightarrow$ cc.

PROOF. We omit the trivialities, and turn to 'P/C $\Rightarrow$ cc' Let z be P/C in Z. Take a sequence (γ_n) of covers of Z. For each filter ϕ with limit z, choose an sc-complete filter ζ, coarser than ϕ and with the same limit z. Now find sets G_n in $\gamma_n \cap \zeta$ and F in ζ, together with compacta K_n, such that $F \subset G_n \cup K_n$ for all n. Then $F < \gamma_n$ for all n, since $G_n \in \gamma_n$ and $K_n < \gamma_n$, by 1.7. In other words, the sets F constructed in this way cover z, and refine each γ_n weakly, as desired.

Examples 0.5 and 0.6 distinguish lc, P, P/C and c = lu, but being completely regular topological spaces, they cannot separate cc from c = lu. To do this, take any space Z without the cc-property, such that wZ is indiscrete, and in particular, compact. Steen and Seebach [16] offer several suitable examples.

Note the difference between P, P/C and lc on one hand, and cc and c = lu on the other. The former are local properties, defined pointwise, while the latter are global, defined by covers. This led us to define local versions of cc and c = lu. (It also led us to define the cover versions of P and P/C discussed in [4].)

Though the gap between cc and P/C is quite large, the cc-property does imply higher ones under certain conditions, as 2.6 and 2.7 show. The first applies to simple examples such as 0.5, and the second extends 0.2 significantly.

2.6. Proposition.

(i) *Let z be a pre-topological* cc-*point in a space Z consisting otherwise of* lc-*points. Then z and Z are both* P/C.

(ii) *Let z be a pre-topological* c = lu-*point in a w-regular space Z consisting otherwise of* lc-*points. Then z and Z are both* P/C.

Proof. Let κ stand for the family of all compact subsets of Z. By 2.5, the lc-points are all P/C. So let us try to show the sc-completeness of nb(z). Take a sequence C_n in nb(z) and for each n, cover Z with $\{C_n\} \cup \kappa$. Now use cc to find a cover γ of z refining each of these weakly, choose C from $\gamma \cap$ nb(z), and hence find $K_n \in \kappa$ with $C \subset C_n \cup K_n$ for all n. This deals with (i), while (i) and 2.2(ii) together yield (ii).

As in topology, one calls z a *first-countable point* in the space Z if, whenever $\phi \to z$, some coarser filter ζ with a countable base has the same limit z.

2.7. Proposition.

(i) *Any first countable* cc-*point in a regular space is locally compact.*

(ii) *Any first countable* c = lu-*point in a w-regular space is locally compact.*

(iii) *Any first-countable w-regular* c = lu-*space is locally compact.*

Proof. Obviously, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). So, take a regular space Z and a first-countable point z which is not locally compact in Z. Then some filter ϕ converging to z has a countable base (A_n) such that the sets A_n are not sub-compact. In particular, the sets cl(A_n) are not compact. But by regularity, the filter ζ based on the sets cl(A_n) converges to z in Z.

As sub-spaces of Z, the sets cl(A_n) are regular. Thus by 1.8, for each n there is a cover α_n of cl(A_n) *such that* $A_n \not< \alpha_n$. By 1.4, the families $\alpha_n + \{Z \backslash A_n\}$ all cover Z; they also keep z from being cc. To see this, let γ cover z, choose Q from $\gamma \cap \phi$, and find some $A_k \subset Q$. Clearly, $Q \not< \alpha_k + \{Z \backslash A_k\}$: in other words, γ does not refine α_k weakly.

Though the functors n and t tend to degrade or even destroy the cc, c = lu and other properties, they do not always do so: two cases appear below.

2.8. Lemma. *Any* P-*point in a space Z remains* P *in nZ. Further, if Z is a* P-*space then so is tZ.*

Given a space Z, one says that its convergence at z has a *countable base*, if there is a sequence of filters ϕ_i, all converging to z, such that $\phi \to z$ in Z iff ϕ is finer than some ϕ_i. (Of course, this includes the possibility of a finite base for the convergence. For example, any pre-topological space has a finite base at each point, since one can choose the constant sequence with just one term, the neighbourhood filter.)

2.9. LEMMA.

(i) *For* q = cc, c = lu *or* P/C, *let z be* q *in Z, and suppose convergence at z has a countable base. Then z is* q *in nZ.*

(ii) *Let z be* lc *in Z, and suppose convergence at z has a finite base. Then z is* lc *in nZ.*

PROOF. Consider the cc case first. Take a sequence γ_k of covers of nZ, and choose C_k from $\gamma_k \cap \mathrm{nb}(z)$ for all k. Now use the cc-property to find $D_i \in \phi_i$ with $D_i < \gamma_k$ for all k. Put $E_i = D_i \cap C_1 \cap \ldots \cap C_i$ and $E = \bigcup E_i$. Then $E \in \mathrm{nb}(z)$, as $E_i \in \phi_i$ for all i and $\mathrm{nb}(z) = \bigcap \phi_i$. Further,

$$E \subset C_k \cup E_1 \cup \ldots \cup E_{k-1} \subset C_k \cup D_1 \cup \ldots \cup D_{k-1}$$

since $E_i \subset C_k$ if $i > k$ and $E_i < \gamma_k$ if $i \leqq k$. Consequently, $E < \gamma_k$ for all k, as desired.

In the c = lu case, proceed much as above. Begin with with w-covers γ_k, choose D_i closed in wZ (thus E_i is also w-closed), let F be the closure of E in wZ, and conclude that $F \in \mathrm{nb}(z)$ and $F < \gamma_k$, as desired. (Note: one really needs closure in wnZ, but $wnZ = wZ$, since Z and nZ admit the same continuous real-valued functions.)

The remaining cases are left to the reader.

§3. Heredity.

Sub-spaces and sums provide no surprises. However, topological quotients and other topological final limits often differ from their counterparts in convergence. This allows one to see the problems in the usual topological limits from a different angle.

We consider just three kinds of final limit here: topological, pre-topological, and minimal (for filter convergence spaces). The last of these, described in §1 in the section on final limits, preserves all the properties discussed here: cc, c = lu, P/C, P and lc. Then the pre-topological and topological limits are obtained by applying the functors n and t to the minimal one. This seems simple, but... Many pre-topological cases respond well to our analysis, within the sharp bounds exposed by star spaces. Our results also explain why the often recalcitrant topological limits behave well, in several well-known situations.

Sub-spaces. Like local compactness, cc and P/C are closed-hereditary, and open-hereditary too, in suitably regular spaces; c = lu is both closed- and open-hereditary, in w-regular spaces.

3.1. PROPOSITION. *Let $z \in H \subset Z$. Then*

(i) *z is cc in H, if it is cc in Z and H is closed, or if it is cc in Z and H is open and Z is regular,*

(ii) *z is* c $=$ lu *in H, if it is* c $=$ lu *in Z and Z is* w*-regular and H is either closed or open,*

(iii) *For* q $=$ P/C *or* lc, *z is* q *in H, if it is* q *in Z and H is closed, or if it is* q *in Z and H is open and Z is* T_3, *and*

(iv) *z is* P *in H if it is* P *in Z.*

PROOF. All but the cc cases are left to the reader. However, in the open P/C case, the proof seems to need closures to be closed, at least within compact sets. This happens not only in T_3 spaces, by Gähler [5], §3.16.18, but also under much weaker conditions. We do not wish to embark on such technical diversion in this introductory paper, though.

Let H be closed, and take a sequence γ_n of covers of the space H. By 1.3, the families $\gamma_n + \{Z\backslash H\}$ all cover Z. Take a cover γ of z in Z, refining them all weakly, and put $\delta = \{Q \cap H : Q \in \gamma\}$. Clearly δ covers z in H. Further, it refines each γ_n weakly. Thus z is cc in H.

Next, let H be open and Z be regular. Again, take covers γ_n of H. Fix a filter $\phi \to z$ in H, and use 1.1 and 1.2 to find a set $S_\phi \in \phi$ with $Z\backslash S_\phi \in \mathrm{nb}(y)$ for all y outside H. The families $\gamma_n \cup \{Z\backslash S_\phi\}$ all cover Z, as H is open. Thus they can all be weakly refined by a cover γ_ϕ of z in Z. Choose D_ϕ from ϕ, and for each n, choose Q_n from $\mathfrak{p}\gamma_n$ with $D_\phi \subset Q_n \cup (Z\backslash S_\phi)$. Let $E_\phi = D_\phi \cap S_\phi$. Then $E_\phi \in \phi$ and $E_\phi \subset Q_n$ for all n. In short, the family $\{E_\phi : \phi \to z \text{ in } H\}$ so constructed covers z in H and refines each γ_n weakly, as desired.

The next result, a partial converse of 3.1, goes some way towards localizing the global nature of the cc and c $=$ lu properties.

3.2. PROPOSITION. *Take a space Z, a point z in Z and a neighbourhood N of z in Z. For* q $=$ cc, c $=$ lu, P/C, lc *or* P, *if z is* q *in N, then it is* q *in Z.*

PROOF. Consider the cc-case first. For all n, let γ_n cover Z, and put $\delta_n = \{C \cap N : C \in \gamma_n\}$. Then each γ_n covers the space N. By assumption, some cover δ of z in N refines each δ_n. But δ also covers z in Z, because $N \in \mathrm{nb}(z)$. Further, δ refines γ_n weakly, as desired.

In the c $=$ lu case, take the γ_n to be w-covers, obtain δ as above, let ω consist of the closures in wZ of the members of δ, and observe that ω does what is needed. (This relies on the continuity of the inclusion from wN to wZ.)

We leave the remaining cases to the reader.

Sums and minimal limits. Disjoint or 'topological' sums behave well, because each component X_j is clopen in $\sum X_i$. Thus to prove 3.3, just use 3.2 and 3.1.

3.3. PROPOSITION. *For* q $=$ cc, c $=$ lu, P/C, lc, *or* P, *a disjoint sum is* q *iff each of its components is* q. *More generally, a point is* q *in a sum iff it is* q *in its component.*

3.4. THEOREM. *For* q = cc, c = lu, P/C, lc, *or* P, *minimal limits inherit* q.

PROOF. To illustrate what this means, consider a final system of maps g_j from spaces Y_j to a set Z, for j in an index set J. Let M denote its minimal limit, take z in Z, and suppose that $\tilde{g}_j(z)$ consists entirely of q-points for all j. Then z is q in M.

We deal with the cc case first. Take covers γ_m of M, and let ϕ converge to z in M. If $\phi = \hat{z}$, just put $C_\phi = \{z\}$. Otherwise, take j, ζ and y as in the definition (see §1), and assume y to be cc. Since $\tilde{g}_j(\gamma_m)$ covers Y_j for all m, by continuity, some cover δ of y in Y_j refines them all weakly. Choose D from $\delta \cap \zeta$ and put $C_\phi = g_j(D)$. Now as $C_\phi \in \phi$ and $C_\phi < \gamma_m$ for all m, the sets C_ϕ form the desired cover of z that refines each γ_m weakly.

In the c = lu case, consider the spaces vY_j and let their minimal limit be N. By the cc result above, z is cc in N. Thus by 2.2, z is cc in vN. Further, $vM = vN$, by an easy exercise in universal properties, and so z is cc in vM. By 2.2 again, z is c = lu in M, as desired.

We leave the remaining cases to the reader.

Pre-topological final limits. Consider again the final system of 3.4 and its minimal limit M, and consider the pre-topological space nM as well. Clearly, if the spaces Y_j are all pre-topological, then nM is their final limit in the category of pre-topological spaces, because it has the requisite uinversal property. Thus the next result follows immediately from 2.8, 2.9 and 3.4.

3.5. PROPOSITION. Pre-topological final limits inherit
(i) cc, c = lu, *and* P/C, *when the index set* J *is finite or countable,*
(ii) *local compactness, when* J *is finite, and*
(iii) *the* P-*property, in all cases.*

Though one can still generalize and extend 3.5 (say, by allowing 'large' sets J, if they are in some suitable sense, countably generated), the pre-topological stars described below show how little room remains for real improvement.

Pre-topological stars. For each j in an index set J, let X_j be a pointed space, the centre of which is denoted by $*_J$. Now let X_J be the star-set obtained by identifying all the points $j \times *_j$ in $\sum X_j$, let $*$ be its centre and s, the identification map. Finally, let Y_J be the minimal quotient and let $Z_J = nY_J$.

3.6 EXAMPLE. Suppose each X_J is pre-topological. For q = cc, c = lu or P/C, the centre $*$ is a q-point in Z_J iff
(i) $*_j$ is a q-point in X_j for all j, and
(ii) the set N of indices j for which $*_j$ is not a P-point, is at most countable.

PROOF. Assume (i) and (ii), and let $M = J\backslash N$. Because the centre is a P-point in Z_M and a q-point in Z_N, by lemmas 2.8 and 2.9, it is a q-point in the pre-topological quotient Z_J of $Z_M + Z_N$, by lemma 2.9 again.

Conversely, let N be uncountable. Consider the cc case. For each j in N, let ν_j be the neighbourhood filter of $*_j$ in X_j. Take a decreasing sequence $(C_{n,j})$ in ν_j with

$$(*) \qquad \bigcap_n C_{n,j} \notin \nu_j.$$

Further, for all n and all $j \in M$, put $C_{n,j} = X_j$. Set $C_n = s(\sum_{j\in J} C_{n,j})$ and

$$\gamma_n = \{C_n\} \cup \{s(X_j) : j \in J\}.$$

Then the families γ_n all cover Z_J. Now take $C \in \text{nb}(*)$ and suppose that C refines each γ_n weakly. This means, there are finite subsets A_n of J, such that for all n,

$$C \subset C_n \cup \left(\bigcup \{s(X_j) : j \in A_n\} \right).$$

Thus $\tilde{s}(C) \cap X_j \subset C_{n,j}$ for all $j \in J\backslash A_n$, which implies

$$\bigcap_n C_{n,j} \supset \tilde{s}(C) \cap X_j \in \nu_j,$$

for all j in the infinite set $N\backslash \bigcup A_n$. Hence, $\bigcap_n C_{n,j} \in \nu_j$, for all such j. This contradicts $(*)$, and shows that $*$ is not cc.

Similar arguments dispose of the other two cases.

Applications in topology. Minimal quotients appear in disguise in topology, noteworthy for their good behaviour. For topological X and Y, if s is surjective, continuous and open, then Y is the minimal quotient, by 1.5. The usual topological quotient of a topological group by a normal subgroup exemplifies this. So do the projections from a product of spaces onto a sub-product, and from a box product onto the reduced product: see §5.

Pre-topological quotients also appear in toplogy. For example, if one 'glues' two T_3 toplogical spaces along homeomorphic closed sets then the topological and pre-topological quotients coincide, even though the surjection from their disjoint sum onto their topological quotients need not be open, nor need the quotient be minimal.

Stars and another case: 3.6 applies directly to topological stars as well as pre-topological ones, because if each X_j is topological, then the pre-topological star Z_J described above is actually topological.

Further, more specialised, results appear in §6.

§4. Tychonov Products.

Take another look at 0.6: there the product of two P/C-spaces (the one, compact and the other, a P-space) lost the P/C-property. This illustrates the difficulties that even finite products cause.

After recalling the two main types of product and dealing with the trivialities, we look at the problems arising with the Tychonov product in the cc case, and solve them in the P/C case. Note: Tychonov products of topological spaces are topological, even when taken in the category of filter convergence spaces.

Products and their factors. Take a collection Y_j of sets, for j in an index set J, along with its Cartesian product Y. Given filters η_j on Y_j, we consider only the extreme cases among their many box products, namely their Tychonov product $\prod \eta_j$ and their (full) box product $\square \eta_j$. The two filters coincide for finite J, but otherwise, the latter can be properly finer than the former.

One defines the *Tychonov* and *box* products of spaces Y_j similarly: $\eta \to y$ in $\prod Y_j$ iff its image η_j under the jth projection converges to y_j in Y_j for all $j \in J$, while $\eta \to y$ in $\square Y_j$ iff $\eta \to y$ in $\prod Y_j$ and $\eta \supset \square \eta_j$. Since the projections make each factor or sub-product of either type of product a minimal quotient, 3.4 implies 4.1 below.

4.1. PROPOSITION. *If a point in a Tychonov or full box product has any of the* q-*properties, then so does its image in each factor or sub-product.*

Tychonov products. Recall Tychonov theorem: a Tychonov product is locally compact iff all factors are locally compact and almost all, compact. The P-property behaves analogously [11]. More generally though, infinite products kill even the cc and c = lu properties, unless they are preserved as above by compactness or by the P-property.

4.2 PROPOSITION. (i) *No point in a Tychonov product is* cc, *if infinitely many factors are not compact.*

(ii) *No point in a Tychonov product is* c = lu, *if infinitely many w-regular factors are not compact.*

PROOF. By 4.1, we lose no generality by considering the Tychonov product Y of non-compact spaces Y_j for $j = 1, 2, \ldots$ Let γ_j cover Y_j 'without a finite sub-cover', and define

$$\delta_1 = \{Q \times Y_2 \times Y_3 \times \ldots : Q \in \gamma_1\},$$
$$\delta_2 = \{Y_1 \times Q \times Y_3 \times \ldots : Q \in \gamma_2\},$$

and so on. Clearly, each δ_n covers Y. Choose a point y in Y, let v be the Tychonov product of the ultra-filters over y_j, and suppose δ refines each δ_n weakly. Then $v \to y$ in Y, but δ does not meet v. (Suppose on the contrary: $D \in \delta \cap v$. Then D contains some $F = \prod F_j \in v$. Choose j so that $F_j = Y_j$. Now $D < \delta_j$. Project this onto Y_j. Then $D_j < \gamma_j$, while $D_j = Y_j$, a contradiction.) All in all, no cover of y can refine each δ_n weakly. This disposes of (i), while (i) and 2.2 together deal with (ii).

A stronger cc-*property*. However, we still do not know whether finite products preserve the cc-property. To study the problem in more detail, we turn it around as follows.

4.3. DEFINITION. We call a point y in a space Y *cc-productive* if $x \times y$ is cc in $X \times Y$ whenever x is cc in X; as usual, we call the space Y *cc-productive* if it consists entirely of cc-productive points. (One could call Y cc-productive, if $X \times Y$ is cc for cc-spaces X. For the present, we ignore this slightly weaker definition.)

4.4. PROPOSITION. *For both points and spaces,*

(i) cc-*productivity implies the* cc-*property, and*

(ii) cc-*productivity passes to closed sub-spaces (and to open ones too, in* T_3 *spaces), to finite products, to sums, to minimal quotients, and in fact, to all minimal limits.*

PROOF. For (i), take X to be the space with one point. For finite products, use the associative law. For closed heredity, use 3.1. For open heredity, use the proof of 3.1 as a guide – since 3.1 itself does not apply unless X is regular. For minimal limits, use 3.4 and the distributive law: $X \times \lim - = \lim(X \times -)$.

Two of our main results, 4.5 and 5.1 below, may help in the search for an internal description of cc-productivity. When found, it should lead to a significant improvement in 4.4. It should also settle the main open question: *does the* cc-*property imply* cc-*productivity*? We guess – no.

4.5. THEOREM. *All* P/C-*points and spaces are* cc-*productive.*

PROOF. Let y be P/C in Y and x be cc in X. To prove $x \times y$ is cc, consider a sequence ρ_n of covers of $X \times Y$. Without loss of generality, assume that each ρ_n consists of boxes indexed as follows by the convergent filters ϕ and ζ on X and Y:

$$U(n, \phi, \zeta) = V(n, \phi, \zeta) \times W(n, \phi, \zeta) \in \phi \times \zeta.$$

Next, take any filters σ and ν converging to x and y respectively, and assume (as P/C permits) ν to be sc-complete. One can now find a set R in $\sigma \times \nu$, refining each ρ_n weakly: this completes the three step proof.

Step 1. For each n, the sets $V(n, -, \nu)$ cover X. Thus one can find $B \in \sigma$ and finite sets b_n of filters converging in X, such that

$$B \subset \bigcup \{V(n, \phi, \nu) : \phi \in b_n\}.$$

Let

$$C_n = \bigcap \{W(n, \phi, \nu) : \phi \in b_n\}.$$

Then $C_n \in \nu$, and

(1) $$B \times C_n \subset \bigcup \{U(n, \phi, \nu) : \phi \in b_n\},$$

Step 2. By sc-completeness, one can find $T \in \nu$ and compact sets K_n such that $T \subset C_n \cup K_n$ for all n. Now for each n and ϕ, as the sets $W(n, \phi, -)$ cover Y, one can find finite sets $k(n, \phi)$ of filters converging in Y, such that

$$K_n \subset \bigcup \{W(n, \phi, \zeta) : \zeta \in k(n, \phi)\}.$$

Define

$$A(n, \phi) = \bigcap \{V(n, \phi, \zeta) : \zeta \in k(n, \phi)\}.$$

Because $A(n, \phi) \in \phi$, the families $A(n, -)$ cover X. So one can find $J \in \sigma$ and finite sets j_n of filters converging in X, such that

$$J \subset \bigcup \{A(n, \phi) : \phi \in j_n\}.$$

Thus

(2) $$J \times K_n \subset \bigcup \{U(n, \phi, \zeta) : \phi \in j_n \text{ and } \zeta \in k(n, \phi)\}.$$

Step 3. Put $S = B \cap J$ and $R = S \times T$, and use (1) and (2) to show that $R < \rho_n$ for all n.

P/C *in Tychonov products.* This completes our incomplete set of results on cc and c = lu. On the other hand, we know all about P/C, largely because the set-theoretic difference of two boxes in a product of two sets, is the union of three disjoint boxes. Bearing this in mind, the reader can easily check the proof of 4.6 below.

4.6. PROPOSITION. For any spaces X and Y, the point $x \times y$ is P/C in $X \times Y$ in and only in the following cases:

(i) *both x and y are* P, OR

(ii) both x and y are lc, OR

(iii) *both x and y are* P/C, *but one of them is neither* P *nor* lc, *while the other is both* P *and* lc.

PROOF. Suppose x is not locally compact and y is not P. To complete the proof, it suffices to show that $x \times y$ is not P/C. First, choose a filter ϕ with limit x, such that ϕ does not meet the family of compact subsets of X. Second, see lemma 2.4 and choose a filter ζ with limit y, such that for each η converging to y, if η is coarser than ζ then there is a decreasing sequence (H_n) in η such that $\bigcap H_n \notin \zeta$. Put $\xi = \phi \times \zeta$, and consider any coarser filter δ with the same limit, $x \times y$. Let η be the projection of δ on Y, choose (H_n) in η as above, and let $D_n = X \times H_n$. For each $E \in \delta$, there are $F \in \phi$ and $G \in \zeta$ such that $E \supset F \times G$. Now $E \backslash D_n \supset F \times (G \backslash H_n)$, a non-void set without compact supersets, if n exceeds some n_0. In short, $E\backslash D_n$ is not sub-compact for all these n, that is, δ is not sc-complete.

4.7. Corollary. *Let Z be the Tychonov product of the spaces Z_j, for j in J. Then the point $z = (z_j)$ is* P/C *in Z in and only in the following cases:*

(i) *z_j is* P *in Z_j for all j, and the indiscrete filter $\{Z_j\}$ converges to z_j in Z_j, for almost all j,* OR

(ii) *z_j is* lc *in Z_j for all j, and Z_j is compact for almost all j,* OR

(iii) *z_i is* P/C *but neither* P *nor* lc *for some $i \in J$, and z_j is both* P *and* lc *in Z_j for all other j, and the indiscrete filter $\{Z_j\} \to z_j$ for almost all j.*

Since this follows directly from 4.6 itself, we omit the proof.

4.8. Definition. We call y a P/C-*productive point* of Y if $x \times y$ is P/C in $X \times Y$ whenever x is P/C in X, and we call Y itself P/C-*productive* if it consists wholly of P/C-productive points.

4.9. Corollary. *A point is* P/C-*productive iff it is both* P *and* lc. *Further,* P/C *productive* T_1 *spaces are discrete (since compact sets in* T_1 P-*spaces are finite).*

§5. Box-products.

Recall that a box product of T_2 topological spaces is compact iff its factors are all compact and almost all are singletons: see Knight [8]. The same is true for filter convergence spaces in general. We emphasise something not mentioned in §1: the box product of topological spaces is topological, even under the 'convergence' definition given in §4 above.

Major result. To our surprise though, we found something more general, namely, the box product of a sequence of locally compact spaces is cc-productive.

5.1. Theorem. *Let $y = (y_j)$ be a point in Y, the box product of a sequence of spaces Y_j. Suppose y_j is* lc *in Y_j, for all j. Then y is a* cc-*productive point, and in particular,* cc.

Proof. Take a space X, a cc-point x in X, a sequence ρ_m of covers of $X \times Y$ consisting of boxes

$$V \times W = V \times W_1 \times W_2 \times \dots$$

indexed as in 4.5, and filters σ and ν converging to x and y respectively. Next, choose compact sets H_j belonging to the projection ν_j of ν on Y_j, and put $K_m = H_1 \times \dots \times H_m$. Further, let Γ_m be the set of all filters ξ converging in Y with the same projection on $Y_{m+1} \times Y_{m+2} \times \dots$ as ν, and let p_m and q_m be the projections from Y and $X \times Y$ onto $Y_1 \times \dots \times Y_m$ and $X \times Y_1 \times \dots \times Y_m$ respectively.

Much as in 4.5, for each m and each filter ϕ converging in X, we find finite sets

$k(m, \phi)$ in Γ_m and b_m of filters converging in X, along with $M \in \sigma$ and $A(m, \phi) \in \phi$, such that

$$K_m \subset \bigcup \{p_m W(m, \phi, \xi) : \xi \in k(m, \phi)\},$$

$$A(m, \phi) = \bigcap \{V(m, \phi, \xi) : \xi \in k(m, \phi)\},$$

$$A(m, \phi) \times \subset \bigcup \{q_m U(m, \phi, \xi) : \xi \in k(m, \phi)\}, \text{ and} \tag{3}$$

$$M \subset \bigcup \{A(m, \phi) : \phi \in b_m\}. \tag{4}$$

Now put $N_1 = H_1$. For $j > 1$, let $N_j = G_j \cap H_j$, where

$$G_j = \bigcap \{W_j(m, \phi, \xi) : m < j, \phi \in b_m \text{ and } \xi \in k(m, \phi)\}, \tag{5}$$

an intersection of a finite number of sets. When these indices m, ϕ and ξ run through their possibilities, each $\xi \in k(m, \phi)$ 'has the same m-tail' as v, and in particular, $\xi_j = v_j$. Thus the sets $W_j(m, \phi, \xi)$ in (5) all belong to v_j. Consequently, $N_j \in v_j$.

Let $N = \prod N_j$. Then $M \times N \in \sigma \times v$. Furthermore, $M \times N < \rho_m$ for all m. To prove this, fix m and take $v \times w \in M \times N$. By (4), $v \in A(m, \phi)$, for some $\phi \in b_m$. Next, $w_j \in N_j \subset H_j$, for $1 \leqq j \leqq m$. Thus by (3),

$$v \times w_1 \times \ldots \times w_m \in q_m U(m, \phi, \xi), \tag{6}$$

for some $\xi \in k(m, \phi)$. Now let $j > m$: then by definition and by (5),

$$w_j \in N_j \subset G_j \subset W_j(m, \phi, \xi). \tag{7}$$

By (6) and (7), $v \times w \in U(m, \phi, \xi)$. In short,

$$M \times N \subset \bigcup \{U(m, \phi, \xi) : \phi \in b_m \text{ and } \xi \in k(m, \phi)\},$$

completing the proof that $x \times y$ is cc.

Application. In the context of 0.2, because scalar multiplication in topological vector spaces provides a very weak form of first countability, we wondered whether c = lu topological vector spaces had to be locally compact. They do not.

Let T be the set of all real sequences and S be the set of all real sequences eventually zero. As a countable box product of real lines, T boasts a complete Hausdorff c = lu group topology which becomes a c = lu locally convex vector topology, when restricted to the closed infinite dimensional sub-space S. Thus neither S nor T is locally compact.

In fact, as we show below, S and T are not even P/C. On this evidence, we suspect that P/C topological vector spaces must be locally compact.

5.2. Example. The topological vector space S is c = lu but not P/C.

Proof. Let 0_S stand for the zero vector in S. For all n, let D_n be the set of all sequences z in T with $|z_i| < 1/n$ for all i, and put $C_n = D_n \cap S$. By definition, each C_n is a neighbourhood of 0_S in S. We claim: the sequence C_n kills the P/C-property at 0_S in S.

To see why, consider a neighbourhood D of 0_S in T, and let $C = D \cap S$. Without loss of generality, we assume there is a decreasing sequence $q = (q_i)$ of positive real numbers, such that D is the set of all sequences z with $|z_i| \leqq q_i$ for all i. For each n and for any sequence z, let $t_n(z)$ be the truncated sequence, whose first n terms coincide with those of z, and are zero from then on. Choose $n > 1/q_1$. Then $t_m(q) \in C \backslash C_n$, for all $m > n$. Thus by 5.5, the set $C \backslash C_n$ is not sub-compact.

Reduced products. Tychonov's theorem characterises compact boxes in T_2 spaces, but it does not force compact sets to lie in a compact box! Though this must surely be known, if not folklore, we prove it below, as we cold not find it anywhere. Reduced products enter our proof: see van Douwen's survey [17] for their use in the 'para-compactness problem'. We begin with their 'convergence' definition, and then show its agreement with the usual 'topological' one.

Given a sequence Y_n of spaces, consider the equivalence $\sim$ on their box product Y under which $x \sim z$ iff $x_m = z_m$ for almost all m, and its quotient map s from Y to the factor-set $Y/\sim$. Let Z denote the resulting minimal quotient, usually known as the *reduced product* ∇Y_n.

As s sends any box product filter $\Box \phi_j$ in Y to an s-complete filter, this quotient is a P-space. Indeed, so are its pre-topological and topological modifications, by 2.8. Further, for $i = 1$ or 2, the space Y is T_i iff each factor is T_i, while the reduced product is T_i iff almost every Y_j is T_i. For each y in Y, let $E(y) = \{x : x \sim y\}$ and $E_n(y) = \{x : x_m = y_m \text{ for all } m > n\}$. Both sets are closed in Y, if Y is T_1, and in any case, $E(y) = \bigcup E_n(y)$.

Next, let $x \sim z$ and $\phi \to x$ in Y. Then $s(\phi) = s(\eta)$, for some $\eta \to z$. (To construct η, suppose x and z agree for $j > m$, let θ be the 'm-tail' of ϕ and put $\eta = \tau \times \theta$, where τ is the ultra-filter over $z_1 \times \ldots \times z_m$.) As a result, in our standard notation for neighbourhood filters,

(i) $\mathrm{nb}_Z(s(y)) = s(\mathrm{nb}_Y(y))$ for all $y \in Y$, and

(ii) if U is open in Y then $s(U)$ is open in ∇Y_n.

Together, these facts prove 5.3 below, and this in turn explains the good behaviour of the usual topological reduced product. See Kunen [9] for more details.

5.3. Proposition. *Let each Y_j be topological. Then the space Z constructed above is a topological* P*-space, which coincides with the usual topological reduced product ∇Y_n. Furthermore, s is open as well as continuous.*

Compacta in box products. Now take T_2 spaces Y_j, and consider a compact set L in Y. Being compact in the T_2 P-space ∇Y_n, its image $s(L)$ is finite. Hence

$L \subset \bigcup \{E(y) : y \in G\}$, for some finite subset G of L. In other words, L is the finite union of the 'one-legged' compact sets $L \cap E(y)$, for y in G. In fact, 'the legs are bare', as we explain in Theorem 5.5 below.

5.4. LEMMA. *Let L be a compact subset of $E(y)$. Then $L \subset E_n(y)$, for some n.*

PROOF. Assume the contrary. Then there is an infinite set J of integers, and for each n in J, a point y_n in L which differs from y in the n-th component but has the same n-tail: that is, $\mathrm{pr}_n(y_n) \neq \mathrm{pr}_n(y)$, and $\mathrm{pr}_m(y_n) = \mathrm{pr}_m(y)$ for $m > n$. Put $M = \{y_n : n \in J\}$. Then M is an infinite set, since its points y_n are all distinct.

We claim: for each z in $E(y)$, we can find an open neighbourhood U of z with $M \cap U$ finite. In the T_2 space Y, this makes M a closed, discrete and compact subset of L, and hence, finite.

To prove the claim, take z in $E(y)$. Then there is some n_z such that $\mathrm{pr}_n(z) = \mathrm{pr}_n(y)$ for all $n > n_z$. In particular, if $n_z < n \in J$ then $y_n \neq z$. Put

$$U = \{w : \mathrm{pr}_n(w) \neq \mathrm{pr}_n(y_n), \text{ for } n \text{ in } J \text{ with } n > n_z\},$$

Since Y is T_2, U forms an open box around z, and by construction, if y_l belongs to U then $l < n_z$. In other words, the claim is true.

5.5. THEOREM. *Let Y be the box product of a sequence Y_n of T_2 spaces, and let L be compact in Y. Then for some m, the projections K and F of L on $Y_1 \times \ldots \times Y_m$ and $Y_{m+1} \times \ldots$ are compact and finite respectively, and $L \subset K \times F$.*

P/C *in box products.* Together with 4.6, this allows us to characterize P/C in Hausdorff box products: we omit the proof.

5.6. PROPOSITION. *Let Z be the box product of the T_2 spaces Z_j, for $j = 1, 2, \ldots$ Then z is P/C in Z iff*

(i) *z_j is P in Z_j for all j,* OR

(ii) *z_j is lc in Z_j for all j, and for almost all j, the singleton z_j is open in Z_j,* OR

(iii) *z_i is P/C but neither P nor lc for some i, and z_j is both P and lc in Z_j for all other j, and the singleton z_j is open in Z_j for almost j.*

§6. Topological Aspects.

Our first results apply to a few (nearly topological) pre-topological limits and other spaces not covered by the general techniques of §2 and §3.

Then we save something from Schroder [14], theorems 5.1 and 5.2. The former exaggerates the influence of the set of non-compact points on the c = lu-property, while according to the latter, the c = lu-property resides wholly outside the compact sets. This idea led us to decompactification, a process which generates spaces known variously as anti-compact, compact-finite, or pseudo-finite. See Bankston [1] or Hutton and Reilly [6].

Two topological limits. We begin with a definition and a lemma. A point z in a space Z is *topological* if its neighbourhood filter $\mathrm{nbt}(z)$ in tZ converges to z in Z; we call it *strange* otherwise.

6.1. LEMMA, *Let K be a compact set of topological* cc-*points in a space Y, and let γ_n be a sequence of covers of Y. Then there is an open neighbourhood U of K, refining each γ_n weakly.*

We omit the proof. Without the 'topological' requirement, this result is false, as any strange pre-toplogical cc-point z shows, with $K = \{z\}$.

Now let Y be the pre-topological inductive limit of a sequence $\ldots X_k > X_{k+1} > \ldots$ of spaces, and let $Z = tY$. Suppose z is cc in almost all X_k. Then by 3.5, z is cc in Y (and in Z too, if z is topological in Y). But otherwise? Even .for compact Hausdorff topological spaces, we do not know.

To get some sort of answer, we assume roughly that (i) Y is cc and Z is T_3, (ii) 'not many' strange points lie near z, and (iii) 'neighbourhood-layers near z are σ-compact'. To be precise, see below.

6.2. PROPOSITION. *Let Y and Z be respectively, the pre-topological and topological limits of a sequence $\ldots X_k > X_{k+1} > \ldots$ of topological spaces. Suppose that for each sequence (C_n) in $\mathrm{nbt}(z)$ there is a 'smaller' sequence B_n of open sets in $\mathrm{nbt}(z)$ such that*

(i) *$C_n \supset A_n \supset B_n \supset A_{n+1}$ for all n, where A_n is the closure of B_n in Z,*
(ii) *each strange point in A_0 belongs to $\bigcap A_n$, and*
(iii) *the sets L_n are all compact, where $L_n = X_n \cap (A_n \backslash B_{n+2})$.*

Then z is cc in Z.

PROOF. Take covers γ_m of Z, choose C_m from $\gamma_m \cap \mathrm{nbt}(z)$, and hence obtain the sets A_m, B_m and L_m as above. Put $L = \bigcup L_m$ and $M = A \cup L$. Then $M \cap X_k$ is a neighbourhood of y in X_k, for all y in $A \cap X_k$, as B_k is Z-open and $M \cap X_k \supset B_k \cap X_k$. Thus $M \in \mathrm{nb}(y)$ for all $y \in A$.

Next, use 6.1 to insert a Z-open set W_k between $B_{k+1} \supset L_k$, with W_k refining each γ_m weakly. Now put $W = \bigcup W_k$ and $V = A \cup W$. Then $V \in \mathrm{nb}(y)$ for all $y \in A$, because $V \supset M$. Further, $V \in \mathrm{nb}(y)$ for all $y \in W$, because W is open. In short, V is open in both Y and Z. Finally, $V < \gamma_k$ for all k, as $V \subset B_k \cup W_1 \cup \ldots \cup W_{k+1}$.

A simple example illustrates this. Let H be the open right half complex plane together with the origin. For all k, consider the set X_k of all $x + iy$ in H such that $y = 0$ or $x > 1/k$, with its locally compact (Euclidean) topology. Their minimal inductive limit Y is pre-topological and locally compact, and 0 is its only strange point. To form a neighbourhood of 0 in $Z = tY$, take a positive real number r and a Euclidean open set U in H containing the half-open interval $(0, r]$: then $\{0\} \cup U$ is open in Z. In fact, these sets ('tear-drops') generate the neighbourhood filter of

0, and so Z is T_3. Because the other conditions needed for 6.2 clearly hold, 6.2 and 2.6 justify our claim below.

6.3. EXAMPLE. The topological inductive limit Z of the locally compact spaces X_k above is P/C.

A result like 6.2 holds for pre-topological spaces in general.

6.4. PROPOSITION. *Consider a pre-topological cc-space Y and its topological modification Z. Assume that*

(i) *Y is T_1 and strongly regular at z (that is, each $C \in \mathrm{nb}(z)$ contains a closed neighbourhood A of z),*

(ii) *z is P/C in Y,*

(iii) *z is isolated in Z from all (other) strange points, and*

(iv) *for each sequence C_n in $\mathrm{nb}(z)$ there is a smaller sequence A_n in $\mathrm{nb}(z)$ such that $\bigcap A_n$ is compact.*

Then z is cc in Z.

PROOF. Take an open set U over z, such that $U \backslash \{z\}$ consists entirely of topological points. Now let γ_n cover Z with open sets, choose C_n from the family $\gamma_n \cap \mathrm{nbt}(z)$, and pick closed sets A_n from $\mathrm{nb}(z)$ whose intersection A is compact, such that $C_0 \cap U \supset A_0$ and $C_n \supset A_n \supset A_{n+1}$ for all n. Further, take a closed set $B_z \in \mathrm{nb}(z)$ such that $B_z \subset A_0$ and $B_z \backslash A_n$ is sub-compact, for all n. Next, for each $y \in A \backslash \{z\}$, choose an open neighbourhood B_y of y, with $B_y < \gamma_k$ for all k. By compactness, a finite union B of these sets contains A, and in fact, $B \in \mathrm{nb}(y)$ for all $y \in A$. (Do not confuse this with 6.1: one needs B_z in general, and z need not be topological!)

Slice up B_z as follows: $S_n = B_z \cap (A_n \backslash A_{n+1})$. For each n, insert a compact set L_n between $A_n \supset S_n$ and an open set W_n between $C_n \supset L_n$, such that $W_n < \gamma_k$ for all k. As in 6.2, put $W = \bigcup W_n$ and $Q = B \cup W$.

By construction, $Q \in \mathrm{nb}(y)$ for all $y \in B_z$ while $Q \backslash B_z$ is clearly open. In short, Q is open. Furthermore, $Q < \gamma_k$ for all k, as $Q \subset B \cup C_k \cup W_1 \cup \ldots \cup W_{k+1}$.

Not quite by coinicidence, the space Y which appeared in the construction of example 6.3 illustrates the use of 6.4 as well.

Errors and corrections. Let us now turn to the ideas and errors in Schroder [14], theorems 5.1 and 5.2. The former claims: a completely regular topological space X is $\mathrm{c} = \mathrm{lu}$ iff the collection X_{nc} of all its points devoid of compact neighbourhoods forms a P-set in X. Butzmann and others gave examples showing the independence of these conditions.

On the one hand, as the rational field Q has no lc-points, Q_{nc} is trivially a P-set in Q, even though by 0.2, Q is not $\mathrm{c} = \mathrm{lu}$. On the other hand, in the $\mathrm{c} = \mathrm{lu}$-space V of 0.5, V_{nc} is not a P-set, since its only member, $\omega_0 \times \omega_1$, is not a P-point.

Nevertheless, as 6.5 shows, the situation of X_{nc} in X does help determine whether X has the cc-property.

6.5. PROPOSITION. *Let X be a topological space.*
(i) *Suppose X_{nc} is compact. Then X is* cc *iff X_{nc} is a* P/C*-set in X.*
(ii) *If X_{nc} is a union of compact* P/C*-sets, then X is* cc.

PROOF. To prove (ii), take a compact P/C-set K, and let γ_n cover X for all n. By compactness, for each n there is a neighbourhood U_n of K, with $U_n < \gamma_n$. By P/C, an open neighbourhood U of K and compact sets K_n exist, with $U \subset U_n \cup K_n$. Thus $U < \gamma_n$ for all n. Consequently, each point of K is cc. (Note: this does not make each point of K a P/C-point – counter-examples exist.)

As (ii) implies half of (i), suppose now that X is cc and X_{nc} is compact. Let κ denote the family of compact subsets of X, take open sets $U_n \supset X_{nc}$, form the covers $\{U_n\} \cup \kappa$ of X, and use 6.1 to find an open neighbourhood U of X_{nc} and compact sets K_n with $U \subset U_n \cup K_n$. Hence X_{nc} is a P/C-set, as desired.

§7. Decompactification.

All this led Schroder to see the c = lu-property in the non-compact convergent filters: let us give him better glasses.

7.1. DEFINITION. Given a space Z, we call a filter on Z *non-compact* if no compact set belongs to it and *anti-compact*, if the complement of every compact set belongs to it.

Anti-compact filters are non-compact, non-compact ultra-filters are anti-compact, and more generally, a filter is non-compact iff some finer ultra-filter is anti-compact.

7.2. DEFINITION. One calls a space *compact-finite* if all its compact subsets are finite. Given a space Z, we define its *decompactification* dZ as follows: $\phi \to z$ in dZ iff
(i) $\phi \to z$ in Z, and
(ii) $s(z, K) \in \phi$ for all compact K in Z, where $s(z, K) = \{z\} \cup (Z \backslash K)$.

Bankston [1] had good reason to call compact-finite spaces anti-compact, but that would cause confusion here. The rather irregular properties of decompactification appear below.

7.3. PROPOSITION. *For all spaces Z,*
(i) *$dZ > Z$ and $dZ = ddZ$,*
(ii) *dZ is compact-finite and T_1,*
(iii) *Z is a compact-finite and T_1 iff $Z = dZ$,*
(iv) *dZ is (pre-topological or) topological if Z is,*

(v) *z is* lc *in Z iff it is discrete in dZ, and*
(vi) *dA is a closed sub-space of dZ, if A is closed in Z.*

PROOF. Consider a space Z.

(i) By definition, $dZ > Z$. Consequently, $ddZ > dZ$. Conversely, any filter converging in dZ also converges in ddZ, because dZ has fewer compact sets to impose restrictions than does Z.

(ii) Let K be compact in dZ. Taken an ultra-filter ρ over K. Since $\rho \to z$ in dZ for some $z \in K$, the definition requires $s(z, K)$ to belong to ρ. Hence, either $\{z\} \in \rho$ or $Z \backslash K \in \rho$, but the latter contradicts $K \in \rho$. Thus $\rho = \hat{z}$. So K is finite, as it admits trivial ultra-filters only.

Now suppose $\hat{y}$ converges to z in dZ. Since the singleton y is compact, $s(z, \{y\}) \in \hat{y}$. Thus $y = z$. In short, dZ is T_1.

(iii) Parts (i) and (ii) do half the work. Conversely, let Z be compact-finite and T_1. Suppose $\phi \to z$ in Z. For each compact K, the co-finite set $s(z, K)$ is an open neighbourhood of z in Z, and as such, it belongs to ϕ. Consequently, $\phi \to z$ in dZ, as desired.

(iv) The pre-topological case being trivial, let Z be topological. For any open U, compact K and finite F in Z, put $Q(U, K, F) = U \backslash (K \backslash F)$. These sets form a base for a topology. Further, the neighbourhood filter of z in dZ is clearly based on the sets $Q(N, K, z)$, as N runs through the open neighbourhoods of z in Z. In short, the topology generated by these sets coincides with that of dZ.

(v) This is obvious.

(vi) Let $\zeta \to z$ in A as a sub-space of dZ, and let ϕ be the image of ζ in Z. Then $\phi \to z$ in Z and $\zeta \to z$ in A, as a sub-space of Z. Also, $s(z, K) = Q(Z, K, z) \in \phi$, for all compact K in Z. Hence $Q(A, K, z) \in \zeta$ for all compact K in A. In short, $\zeta \to z$ in dA.

Conversely, let A be closed and let $\zeta \to z$ in dA. Taking ϕ as above, we want ϕ to converge to z in dZ. First, $\phi \to z$ in Z. Next, take a compact set K in Z. Then $z \cup (A \backslash K) \in \zeta$, because $K \cap A$ is compact in Z. As a result, $z \cup (Z \backslash K) \in \phi$. All in all, $\phi \to z$ in Z, as desired.

One really does need T_1 in 7.3 (iii) above. For example, the set of natural numbers becomes a topological space M which is not T_1, when given neighbourhood filters as follows: $nb(m) = \hat{0} \cap \hat{m}$ for all m.

7.4. EXAMPLE. *The space M is locally compact and compact-finite, even though $dM \neq M$.*

Compact-finite cc-*spaces.*

Further, prompted by 7.3(v) and by the compact-finiteness of T_1 P-spaces, one might guess: maybe compact-finiteness converts cc into P. We refute this with the help of Martin's axiom, which provides an sf-complete ultra-filter ξ on the natural numbers and so guarantees non-trivial P-points in this Čech-Stone compactification. See Jech [7], theorem 57 and lemma 24.9. (We call ξ *sf-complete* if for each sequence (Q_n) in ξ, some Q in ξ makes all the sets $Q \backslash Q_n$ finite.)

One obtains a T_4 topological space S by giving the natural numbers the discrete topology except at 0, where the neighbourhood filter is $\hat{0} \cap \xi$.

7.5. EXAMPLE. *The space S is para-compact, anti-compact and* P/C, *but not* P.

Before leaving decompactification, we emphasise its peculiar behaviour in sub-spaces (which the proof of 7.3(vi) illustrates), under continuity, and in products (which we did not discuss).

Decompactification and cc. Take a space Z, and let Υ_{acu} and $\Upsilon_{\mathrm{acu}}(z)$ be the sets of anti-compact ultra-filters which converge in Z, and to z in Z respectively. Consider the following analogues of the cc-property:

(a) for every sequence (γ_n) of covers of Υ_{acu}, there is a cover γ of $\Upsilon_{\mathrm{acu}}(z)$ which refines each γ_n weakly,

(b) as above, using Υ_{ac}, the set of anti-compact convergent filters,

(c) as above, using Υ_{nc}, the set of non-compact convergent filters,

(d) z is cc in dZ, and

(e) z is *cc* in Z.

7.6. REMARK. Take a collection Ξ of filters with a coarsest member ϕ. Then the set of covers of Ξ is weakly countably directed iff ϕ is s-complete.

7.7. PROPOSITION. *In any space Z,*

(a) $\Leftrightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (e), *and* (c) $\Rightarrow$ (e),

(b) *and* (c) *are independent of one another, and each '$\Rightarrow$' given above is proper.*

In particular, though the cc-*property passes to a space from its decompactification, the decompactification of a* cc-*space need not be* cc.

(a) $\Leftrightarrow$ (b). This follows directly from 1.6.

(d) $\Rightarrow$ (e). Suppose z is cc in dZ, and let κ be the family of all compact subsets of Z. Take covers γ_n of Z. As they also cover dZ, some cover γ of z in dZ refines them all weakly. Now $\gamma \cup \kappa$ covers the ultra-filters converging to z in Z, and so $\mathfrak{p}\gamma + \kappa$ covers z in Z, by 1.6. Further, $\mathfrak{p}\gamma + \kappa$ refines each γ_n weakly, for again by 1.6, κ refines δ weakly as soon as δ covers the ultra-filters converging in Z. In all, z is cc in Z.

(b) $\Rightarrow$ (d) and (c) $\Rightarrow$ (e). These are proved similarly.

(e) $\not\Rightarrow$ (c) and (e) $\not\Rightarrow$ (d). Several counter-examples deal with these non-implications. The first, the cc-space V of 0.5, consists entirely of lc-points except for ω, the corner point $\omega_0 \times \omega_1$. Thus Υ_{ac} has a minimum, the filter α based on the open boxes $(\nu, \omega_0) \times (\lambda, \omega_1)$. Now α is obviously not s-complete: it is not even sf-complete.

Since the neighbourhood filter $\hat{\omega} \cap \alpha$ of ω in the topological space dV is not sf-complete, (d) does not hold. Further, the neighbourhood filter of ω in V is the coarsest member of Υ_{nc}: as it is not s-complete, (c) fails too, by 7.6.

(d) $\not\Rightarrow$ (b) and (d) $\not\Rightarrow$ (c). Assume Martin's axiom, and consider the compact-finite cc-space S of 7.5. Clearly, (d) holds but (c) does not, because the coarsest member $\hat{0} \cap \xi$ of Υ_{nc} is not s-complete. Similarly, as ξ is the coarsest member of Υ_{ac}, condition (b) fails.

(b) $\not\Rightarrow$ (c) and (e) $\not\Rightarrow$ (c). Take the P-space U of 0.6 and the ordinal space W_0, and let ω be the point in the topological quotient T of $U + W_0$ obtained by identifying ω_1 and ω_0. In T, Υ_{nc} and Υ_{ac} both have minima, the neighbourhood filter of ω in T and the filter based on the co-countable sets in $U \backslash \omega_1$ respectively. Because the latter is s-complete, (b) – and hence (e) – both hold. On the other hand, (c) does not, as the former is not s-complete.

(c) $\not\Rightarrow$ (d), (c) $\not\Rightarrow$ (b) and (e) $\not\Rightarrow$ (d). Take the c = lu-spaces V and L of 0.5 and 0.6, and let M be the topological quotient of $V + L$ obtained by identifying the points of $W_0 \times \omega_1$ in V with their counterparts in L. Here, the minimal members of Υ_{nc} are the neighbourhood filters in M of the points $v \times \omega_1$, for v in W_0. By a standard argument based on 'if $\chi_n < \omega_1$ for all n, then $\sup(\chi_n) < \omega_1$', the reader can easily verify (c) for the point $\omega_0 \times \omega_1$ in M.

On the other hand, 7.3(vi) makes dV (homemorphic to) a closed sub-space of dM, as V is closed in M. Now, as noted above, $\omega_0 \times \omega_1$ is not cc in dV. So by 3.1, $\omega_0 \times \omega_1$ is not cc in dM: in other words, (d) fails. This completes the list of non-implications, since (b) implies (d) and (c) implies (e).

Finally, we return to Schroder [14], theorem 5.2. This claims the equivalence for any space Z, of the conditions

(A) the set of all w-covers of Υ_{acu} is weakly countably directed,

(C) the set of all w-covers of Υ_{nc} is weakly countably directed, and

(E) Z is a c = lu-space.

The reader will see from 7.7 what really happens: (A) $\Rightarrow$ (E) and (C) $\Rightarrow$ (E), as claimed, but the rest is false, as these examples show.

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