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THE SMOOTH SURFACES ON CUBIC HYPERSURFACE IN P^4 WITH ISOLATED SINGULARITIES

ALF BJØRN AURE

§0. Introduction.

It is wellknown that a smooth surface on a quadric hypersurface in P^4 is either a complete intersection of the quadric and another hypersurface, or it is linked to a plane on the quadric. The purpose of this paper is to find a similar result for smooth surfaces on a cubic hypersurface in P^4. If the cubic hypersurface has only isolated singularities, then we are able to do so; a smooth surface on such a cubic is either a complete intersection of the cubic and another hypersurface, or it is linked to either a plane, a quadric surface, a cubic scroll, a Veronese surface or an elliptic quintic scroll on the cubic (Theorem 1.1). If the cubic hypersurface has a curve in its singular locus, then there are open questions. Those cubic hypersurfaces will not be treated here.

The proof of the theorem goes as follows: Since the cubic hypersurface has only isolated singularities, we get a smooth cubic Del Pezzo surface in P^3 by intersecting with a general hyperplane. We will show that for a smooth surface on the cubic hypersurface, the induced hyperplane section is a curve of "nearly maximal" genus on the Del Pezzo surface. Furthermore, this curve can be linked to a curve of degree 5 or less on the cubic surface, and then finally we show that this linkage can be lifted to a linkage of the surface and one of the surfaces mentioned above.

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§1. The results.

We work over an algebraically closed field of characteristic 0. Recall that the sectional genus of a surface in a projective space is the genus of a general hyperplane section.

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THEOREM 1.1. *Let S be a smooth surface of degree d and sectional genus π contained in a cubic hypersurface V in P^4 with only isolated singularities.*

Then $\pi = 1 + d/6\,(d-3) - \mu/6$, where $\mu = \#(\operatorname{Sing} V) \cap S$, and S is linked to a surface S' on the cubic with the following possibilities

S'	$d \bmod 3$	μ
$\emptyset$	0	0
P^2	2	4
Quadric (possibly singular)	1	4
Cubic scroll	0	6
Veronese surface	2	10
Elliptic quintic scroll	1	10

The surfaces S' in the table with $\mu \leqq 6$ are projectively Cohen-Macaulay; hence a surface S linked to such an S' has this property too. These surfaces are in the linkage class of the complete intersection surfaces in P^4 (see [PS]). An elliptic quintic scroll is linked to a Veronese surface, so we have the following:

COROLLARY 1.2. *A smooth surface on a cubic hypersurface in P^4 with isolated singularities is either projectively Cohen-Macaulay or in the linkage class of a Veronese surface.*

If we have a linkage $S \cup S' = V \cap V_t$ as in the theorem where V_t is a hypersurface of degree t, then the exact sequence of linkage is (see [PS])

(1.3) $$0 \to \omega_{S'}(2-t) \to \mathcal{O}_{S\cup S'} \to \mathcal{O}_S \to 0,$$

where $\omega_{S'}$ is the dualizing sheaf of S'. From this exact sequence we can calculate the cohomology groups $H^i(\mathcal{O}_S(n)), i = 0, 1, 2, n \in \mathbf{Z}$, and the Euler-Poincaré characteristic $\chi = \chi(\mathcal{O}_S)$. We omit listing these formulae, but we can mention one application:

COROLLARY 1.4. *An irregular smooth surface on a cubic hypersurface with isolated singularities in P^4 is necessarily an elliptic quintic scroll.*

PROOF. With the notation as above, interchange the roles of S and S' in (1.3) and twist with $(t-2)$ to get

$$0 \to \omega_S \to \mathcal{O}_{S\cup S'}(t-2) \to \mathcal{O}_{S'}(t-2) \to 0$$

The irregularity of S is $h^1(\mathcal{O}_S) = h^1(\omega_S) =$ the dimension of coker $(H^0(\mathcal{O}_{S\cup S'}(t-2)) \to H^0(\mathcal{O}_{S'}(t-2))$. The hypersurfaces of degree $t-2$ cut out a complete linear series for our possible S', unless S' is a Veronese surface and $t = 3$; but then S is an elliptic quintic scroll.

REMARK 1.5. That all possibilities listed in the theorem may occur is the case for the Segre cubic primal; a cubic hypersurface with 10 ordinary double points as singular locus. It can be realized as the image of a rational map $\mathsf{P}^3 \to \mathsf{P}^4$ defined by the quadrics through 5 fixed points in P^3. The 10 nodes are the images of the 10 lines joining the pairs among the 5 fixed points. The image of a plane P in P^3 is either a Veronese surface, a cubic scroll, a quadric surface, or a plane according to whether P contains 0, 1, 2, or 3 of the fixed 5 points. The image of a quintic surface in P^3 with triple points in the 5 fixed points can be shown to be an elliptic quintic scroll. Hence the Segre cubic primal contains all possible S' of the theorem. Since an S' is cut out by cubics, one gets examples with all possible d and μ of the theorem by linkage.

§2. Proof of the theorem.

LEMMA 2.1. *Let S and V be as in Theorem* 1.1, *then*

(2.2) $$\pi = 1 + d/6(d-3) - \mu/6 \text{ where } \mu = \#(\operatorname{Sing} V) \cap S.$$

If $d \equiv 0 \pmod 3$, then $\mu \in \{0, 6, 12\}$, otherwise $\mu \in \{4, 10, 16\}$.

PROOF. Let I denote the ideal sheaf of S in P^4, and let F be a homogeneous polynomial of degree 3 defining V. Then multiplication with F defines a section of a twist of the conormal bundle

$$0 \to \mathcal{O}_S \xrightarrow{F} I/I^2(3).$$

This section vanishes precisely where the Jacobi ideal of F restricted to S vanishes; hence (as a definition when counted with multiplicity) $\mu = c_2(I/I^2(3)) = \#(\operatorname{Sing} V) \cap S$, where c_2 denotes the second Chern class. Let N be the normal bundle of S in P^4, then (see [H, p 434]) $c_2(N) = d^2$, and for the first Chern class $c_1(N) = 5H + K$, where H is the class of a hyperplane section and K the canonical divisor. This gives

$$\mu = c_2(I/I^2(3)) = c_2(N(-3)) = d^2 + (5H + K)(-3H) + 9d.$$

By adjunction $H \cdot K = 2\pi - 2 - d$, and (2.2) follows by solving the equation above for π.

Since V has only isolated singularities, $\mu \leqq 2^4 = 16$ by Bezout's theorem used on four (general) partial derivatives of F. The possible values of μ in the lemma then follow since π is an integer.

Let us next finish the cases $\mu = 0, 4, 16$:

The following result is a consequence of a theorem of Gruson and Peskine together with a result of Roth (see [A2, Prop. 1.7]).

Proposition 2.3. *Let S be a smooth surface in $\mathbf{P}^4$ of degree $d > 6$ and sectional genus π not contained in a quadric hypersurface. Then $\pi \leqq \pi_3(d) = 1 + [d/6(d-3)]$. If $\pi = \pi_3(d)$, then S is linked to a surface of degree $r \leqq 2$, where $d + r \equiv 0 \pmod 3$, by a cubic and a hypersurface of degree $(d+r)/3$.*

For a surface S on V with $\mu = 0$ or 4 we have $\pi = \pi_3(d)$ by Lemma 2.1. If $d \leqq 6$ and $\pi = \pi_3(d)$, then S is either a plane, a quadric surface, a cubic or a quartic Del Pezzo surface, a Castelnuovo surface, or a complete intersection of type $(2, 3)$ (by the classification of surfaces of low degree in $\mathbf{P}^4$ [S.R., p. 218]). These surfaces are contained in a $\mathbf{P}^3$ or a quadric; the residual surface on V has degree 2 or less. If $d > 6$, then S is not contained in a quadric and Prop. 2.3 applies (if S is contained in more than on cubic ($d + r = 9$), then we can choose one of the cubics in Prop. 2.3 to be V).

Lemma 2.4. *The case $\mu = 16$ is impossible.*

Proof. If $\mu = 16$, then $(\operatorname{Sing} V) \cap S$ is a complete intersection scheme defined by four general partial derivatives of the form defining V. The fifth partial derivative vanishes too on this scheme, so by Noether's theorem it must be a linear combination of the four others, and V is a cone. It is straightforward to check that a surface on a cubic cone has π_3 as sectional genus (consider a general hyperplane section through the vertex and use [H, p. 374), hence $\mu = 0$ or 4; contradiction.

Remark 2.5. The case $\mu = 12$ does not occur in the theorem, and we will later exclude this possibility. A cubic hypersurface cannot have 12 distinct double points (consider the degree of the dual hypersurface), but the number μ is counted with multiplicity. For instance the singular locus of the cubic $V' = \{x_0x_1x_2 + x_3^3 + x_4^3 = 0\}$ is a finite scheme of length 12, hence V' could a priori contain a surface with $\mu = 12$.

Instead of classifying and then examine all different cubics with isolated singularities, we choose a more uniform approach:

Let S_3 be a general hyperplane section of V, and let C denote the induced hyperplane section of the surface S on V. The surface S_3 is a smooth cubic Del Pezzo surface in $\mathbf{P}^3$ with Picard group $\operatorname{Pic} S_3 \cong \mathbf{Z}^7$. A smooth curve of degree d and genus g will be denoted by C_d^g. A divisor D on S_3 is said to be of type $(\delta; m_1, \ldots, m_6)$ if there exists a morphism $S_3 \to \mathbf{P}^2$ blowing down six skew lines E_i in S_3, $i = 1, \ldots, 6$, such that $D = \delta l - \Sigma m_i E_i$ in $\operatorname{Pic} S_3$. Here l denotes the pullback of a line in $\mathbf{P}^2$, and $\{l, E_1, \ldots, E_6\}$ is a basis of $\operatorname{Pic} S_3$. Let H_3 denote a hyperplane section of S_3; it is of type $(3; 1, 1, 1, 1, 1, 1)$. The canonical class $K = -H_3$.

Proposition 2.6. *Let C be a smooth curve on S_3 of degree d and genus $\pi = 1 + d/6(d-3) - \mu/6$ with ideal sheaf $I_C = I_{C/S_3}$. If the values of d and μ are as* *listed in the table below, then C is linked to a smooth curve D on S_3 by a surface of degree t, such that we have one of the following possibilities*

$d \bmod 3$	μ	D	type of D	$h^0(\mathcal{O}_{S_3}(D))$	$h^0(\mathcal{O}_{S_3}(D-H_3))$	$h^1(I_C(n))$
0	6	C_3^0	$(1;0,0,0,0,0,0)$	3	0	0
2	10	C_4^0	$(2;1,1,0,0,0,0)$	4	0	$\delta_{n,t-2}$
1	10	C_5^1	$(3;1,1,1,1,0,0)$	6	1	$\delta_{n,t-2}$
0	12	C_6^2	$(4;2,1,1,1,1,0)$	8	2	$\delta_{n,t-2}+\delta_{n,t-3}$

PROOF. We will omit the details since the proof is mainly elementary but tedious. For a given curve C of degree d on S_3 there exists a so called adequate basis of $\operatorname{Pic} S_3$ w.r.t. C (see [G.P]), i.e. in this basis C is of type $(\delta; m_1, \ldots, m_6)$, where i) $\delta \geqq m_1+m_2+m_3$, ii) $m_1 \geqq m_2 \geqq \ldots \geqq m_6$, iii) $r := \delta - \mathrm{m}_1 \leqq 2/3\,d$, iv) $d = C \cdot H_3 = 3\delta - \Sigma m_i$.
By adjunction, the genus π of C is $1 + 1/2\,(C\cdot C - C\cdot H_3)$ leading to

$$\pi = 1 + 1/2((r-1)d - 3/4r^2 - \sum_{i=2}^{6}(r/2 - m_i)^2). \tag{2.7}$$

Because of iii), write $r = 1/3(2d-b) - a$, where $a \geqq 0$ and $0 \leqq b \leqq 2$, $b + d \equiv 0 \pmod 3$. Comparing (2.7) and $\pi = 1 + d/6(d-3) - \mu/6$ one finds

$$\mu/6 = 1/24(3a+b)^2 + 1/2\sum_{i=2}^{6}(r/2 - m_i)^2. \tag{2.8}$$

Using i), ii), iii), and iv), the solutions of (2.8) for the given μ and b are:
$\mu = 6$ and $b = 0$:
$a = 1$; $m_1 = \ldots = m_6 = 1/2(r-1)$; $\delta = 1/2(3r-1)$.
Let $t = 1/2(r+3)$, then $d+3 = 3t$ and $C + D' = tH_3$, where $D' = (5;2,2,2,2,2,2)$ in the adequate basis of $\operatorname{Pic} S_3$.
$\mu = 10$ and $b = 1$:
$a = 1$; $m_1 = \ldots m_4 = r/2$, $m_5 = m_6 = r/2 - 1$; $\delta = 3/2r$.
Let $t = r/2 + 2$, then $d + 4 = 3t$ and $C + D' = tH_3$, where $D' = (6;2,2,2,2,3,3)$.
$\mu = 10$ and $b = 2$:
$a = 1$; $m_1 = m_2 = 1/2(r+1)$, $m_3 = \ldots = m_6 = 1/2(r-1)$; $\delta = 1/2(3r+1)$.
Let $t = 1/2(r+5)$, then $d+5 = 3t$ and $C + D' = tH_3$, where $D' = (7;2,2,3,3,3,3)$.
$\mu = 12$ and $b = 0$:
$a = 2$; $m_1 = r/2 + 1$, $m_2 = \ldots = m_5 = r/2$, $m_6 = r/2 - 1$; $\delta = 3/2r + 1$.
Let $t = r/2 + 3$, then $d + 6 = 3t$ and $C + D' = tH_3$, where $D' = (8;2,3,3,3,3,4)$.
In the cases above C is linked to a curve D' of the indicated type. Let $G_j = 2l - \sum_{i \neq 7-j} E_i$, $j = 1, \ldots, 6$, then the G_j's are six skew lines contracted under

a morphism $S_3 \to \mathbf{P}^2$. The pullback of a line for this morphism is of type $(5; 2, 2, 2, 2, 2, 2)$. Using this basis for $\operatorname{Pic} S_3$, D' gets the type of D as written in the table.

The general curve in the linear system $|D|$ is clearly smooth and of the indicated degree and genus.

It remains to fill in the three rightmost columns of the table. We have $h^1(\mathcal{O}_{S_3}(n)) = 0$, for $n \in \mathbf{Z}$, and $K = -H_3$. From the exact sequence

$$(*) \qquad 0 \to \mathcal{O}_{S_3} \to \mathcal{O}_{S_3}(D) \to \mathcal{O}_D(D) \to 0,$$

we find $h^0(\mathcal{O}_{S_3}(D)) = D \cdot H_3 + p(D)$, where $p(D) =$ genus of D, by using Riemann-Roch on D. Hence the first column follows. Tensoring $(*)$ with $\mathcal{O}_{S_3}(-H_3)$, we find by adjunction $h^0(\mathcal{O}_{S_3}(D - H_3)) = p(D)$.

For the last column we have $C + D = tH_3$ in $\operatorname{Pic} S_3$; hence $h^1(I_C(n)) = h^1(\mathcal{O}_{S_3}(D + (n - t)H_3)) = h^1(I_D(t - 1 - n))$ by Serre duality. The latter cohomology group is easily studied by considering the exact sequences

$$0 \to I_D(r) \to \mathcal{O}_{S_3}(r) \to \mathcal{O}_D(r) \to 0,$$

and

$$0 \to \mathcal{O}_{S_3}(-D + rH_3) \to \mathcal{O}_{S_3}(-D + (r + 1)H_3) \to \mathcal{O}_{H_3}(-D + (r + 1)H_3) \to 0.$$

We skip this calculation.

Now we can finish the proof of the theorem for the remaining cases with $\mu \in \{6, 10, 12\}$. A curve C (resp. D), as in Prop. 2.6 when $\mu = 6$ or 10, is the hyperplane section of a surface S (resp. S'), as in the theorem, because of Lemma 2.1. We must show that we can lift the linkage of C and D to a linkage of S and an S'. Furthermore, we must exclude the possibility $\mu = 12$. Since the proofs for the various values of μ are rather similar, we will treat the case $\mu = 10$, $d \equiv 1 \pmod 3$ in detail and then just sketch what differs for the other values of μ and d.

The case $\mu = 10$, $d \equiv 1 \pmod 3$:

Let $I_S = I_{S/V}$ and as before, S_3 is a general hyperplane section of V. Let C denote the induced hyperplane section of S and $I_C = I_{C/S_3}$. By Prop. 2.6, we have $C + D = tH_3$ and $D = C_5^1$. Consider the exact sequence

$$(2.9) \qquad 0 \to I_S(n - 1) \to I_S(n) \to I_C(n) \to 0.$$

Since $h^1(I_S(m)) = 0$ when $m \ll 0$, and $h^1(I_C(n)) = \delta_{n,t-2}$ (prop. 2.6), it follows by induction on n that

$$(2.10) \qquad h^1(I_S(t - 2)) \leqq 1.$$

For $n = t - 1$ we get from (2.9) the long exact sequence

$$0 \to H^0(I_S(t - 1)) \to H^0(I_C(t - 1)) \to H^1(I_S(t - 2)) \to H^1(I_S(t - 1)) \to 0.$$

We have that $H^0(I_C(t-1)) \simeq H^0(\mathcal{O}_{S_3}(E_5+E_6))$ has rank 1, where E_5+E_6 is the union of two skew lines being of type $(0;0,0,0,0,-1,-1)$ by Prop. 2.6. But then $H^0(I_S(t-1))=0$; otherwise S is linked to the union of two planes intersecting in one point and having E_5+E_6 as a hyperplane section. This is impossible because the union of the two planes is not Cohen-Macaulay in the point of intersection. This contradicts the fact that S is a Cohen-Macaulay scheme; a property preserved by linkage ([P.S., Prop. 1.3]). So (2.10) and the long exact sequence above imply $H^1(I_S(t-1))=0$; hence there is a surjection $H^0(I_S(t)) \to H^0(I_C(t))$. We can then lift the linkage of C and $D=C_5^1$ to a linkage of S and a quintic surface S_5 having D as a hyperplane section. Since D is smooth, S_5 is reduced, irreducible and has only isolated singularities (if any). By Serre's criterion for normality ("R_1+S_2"), S_5 is normal (the S_2 property comes from linkage with S [P.S, Prop. 1.3]). If S_5 is smooth, then it is an elliptic quintic scroll by the classification of surfaces of degree 5, and we are through.

Assume that S_5 is singular. First of all, S_5 is not a cone over D because such a cone is not Cohen-Macaulay in the vertex (D is not projectively Cohen-Macaulay).

Let $\tilde{S} \to S_5$ be a minimal desingularization, i.e. no -1 curves are contracted. Let $|L|$ be the linear system on $\tilde{S}$ defining this morphism. Since S_5 is normal and not a cone, Severi's theorem [M, p. 72] applies: $h^0(\mathcal{O}_{\tilde{S}}(L))=5$. Furthermore, $L^2=5$ and by adjunction $L\cdot K=-5$, so $\tilde{S}$ is ruled [B, p. 112]. Let $\chi=\chi(\mathcal{O}_{\tilde{S}})$, then $\tilde{S}$ is ruled over a curve of genus $g=1-\chi$. The elliptic curve D dominates this curve so $g \leqq 1$. By Riemann-Roch $\chi(\mathcal{O}_{\tilde{S}}(L)=5-h^1(\mathcal{O}_{\tilde{S}}(L))=\chi+5$; hence $g=1$. By Hurwitz' formula, D is a section and S_5 is ruled in lines, because otherwise we would have ramification points. Since L intersects a line in the ruling once, the minimality of the desingularization implies that a section of $\tilde{S}$ is contracted. But then S_5 is a cone; contradiction.

The proof in the case $\mu=10, d\equiv 2 \pmod 3$ is similar. For $\mu=6, d\equiv 0 \pmod 3$ the lifting of the linkage is without difficulty since the twisted cubic curve $D=C_3^0$ is projectively Cohen-Macaulay. To avoid linkage with a cubic cone use $h^0(I_C(t))=h^0(\mathcal{O}_{S_3}(D))=3$: By the proof of Lemma 2.4, V itself is not a cone so it contains only a finite number of such cones (the vertex is in a singular point of V, so the cone is contained in the tangent cone of a singular point).

The case $\mu=12, d\equiv 0 \pmod 3$: There are no smooth surfaces of degree 6 with sectional genus 2, and we will show that a normal surface S_6 of degree 6 in $\mathbf{P}^4$ with sectional genus 2 is a cone. The proof is analogous to the treatice of the elliptic quintic scroll: Let $\tilde{S} \to S_6$ be a minimal desingularization given by a linear system $|L|$. As before, $\tilde{S}$ is ruled, and we must show that $\tilde{S}$ is ruled in lines, i.e. $\tilde{S}$ is ruled over a genus $g=2$ curve. By Riemann-Roch $\chi(\mathcal{O}_{\tilde{S}}(L)=5-h^1(\mathcal{O}_{\tilde{S}}(L))=1-g+5$; hence $g \geqq 1$ and for K the canonical divisor, $\chi(\mathcal{O}_{\tilde{S}}(L+K)=h^0(\mathcal{O}_{\tilde{S}}(L+K))-h^1(\mathcal{O}_{\tilde{S}}(L+K))=2-g$. If $g=1$, then $h^0(\mathcal{O}_{\tilde{S}}(L+K))>0$. But this contradicts an easy modification of a result of A. Sommese [So, Lemma 2.3.3]: Since the morphism given by $|L|$ does not contract any -1 curve and $h^0(\mathcal{O}_{\tilde{S}}(L + K)) > 0$, we have $0 \leqq (L + K)^2 = L^2 + 2L \cdot K + K^2 = 6 - 8 + K^2 < 0$ (since $K^2 \leqq 8(1 - g) = 0$); contradiction. Hence $g = 2$ and S_6 is a cone.

One shows that a smooth surface S on V with $\mu = 12$ and $d \equiv 0 \pmod 3$ is linked to such a cone; this is impossible sine the cone is not Cohen-Macaulay in the vertex, and therefore this case is excluded.

REFERENCES

[A1] A. B., Aure, *On surfaces in projective* 4-*space*, Thesis Oslo 1987.

[A2] A. B. Aure, *The smooth surfaces in* $\mathbf{P}^4$ *without apparent triple points*, Duke Math. J. 57 (1988), 423–430.

[B] A. Beauville, *Surfaces algébriques complexes*, Astérisque 54, Soc. Math. France, 1978.

[GP] L. Gruson, C. Peskine, *Genre des courbes de l'espace projectif* (II), Ann. Sci. Ecole Norm. Sup. 15. (1982), 401–418.

[H] R. Hartshorne, *Algebraic Geometry*, Graduate texts in Math., Vol 52, Springer Verlag, 1977.

[M] B. Moishezon, *Complex Surfaces and Connected Sums of Complex Projective Planes*, Springer Lecture Notes 603, 1977.

[PS] C. Peskine, L. Szpiro, *Liaison des variétés algébriques* I, Invent. Math. 26 (1974), 271–302.

[So] A. J. Sommese, *Hyperplane sections of projective surfaces* I. *The adjunction mapping*, Duke Math. J. 46 (1979), 377–401.

[SR] J. G. Semple, L. Roth, *Introduction to Algebraic Geometry*, Clarendon Press, Oxford, 1949.

INSTITUTT FOR MATEMATISKE FAG
NORGES TEKNISKE HØGSKOLE
N-7034 TRONDHEIM
NORWAY