ON MODULES INDUCED OR COINDUCED FROM HOPF SUBALGEBRAS

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Let $A$ be a Hopf algebra over a commutative ring $k$, $B \subset A$ a Hopf subalgebra, and $W$ a left $A$-module. Koppinen and Neuvonen [2] showed that $W$ is induced from $B$, that is $W \cong A \otimes_B V$ for a left $B$-module $V$, if and only if $W$ admits a system of imprimitivity based on $B$; their result assumes however that the antipode of $A$ is bijective and that $A$ as a right $B$-module is a finitely generated and projective generator. Essentially the same result holds for coinduced modules, i.e. modules of the form $W \cong \text{Hom}_B(A, V)$. The rather strong assumptions in [2] were made in order to apply the Morita theorems, and Koppinen and Neuvonen asked whether these assumptions can be weakened ([2], Remark). The present paper gives proofs for the above results which do not use Morita theory, and which only assume $A$ to be finitely generated and projective over $B$. In the induced case a more general result is given which only needs $A$ to be flat over $B$; this is closely related to [1] and [4].

In the following $A$ denotes a Hopf algebra over a commutative ring $k$, and $B \subset A$ a Hopf subalgebra. The antipode and counit are denoted by $\lambda$ and $\varepsilon$, respectively, and the coproduct by $\delta$. "$A$-module" will mean left $A$-module.

1. Induced Modules.

Let $F$ denote the $k$-algebra considered in [2]; as a $k$-module $F$ consists of all right $B$-linear maps $f: A \to k$, i.e. $f(ab) = f(a)\varepsilon(b)$ for $a \in A, b \in B$, and the product is given by

$$(f \cdot f')(a) = \sum f(a_{(2)})f'(a_{(1)}), \quad f, f' \in F.$$  

$F$ is an $A$-module with $(af)(a') = f(\lambda(a)a')$ for $a, a' \in A$. By definition, an $A$-module $W$ admits a system of imprimitivity based on $B$ if it is a left $F$-module satisfying

$$(1) \quad a(fw) = \sum (a_{(1)}f)(a_{(2)}w), \quad a \in A, f \in F, w \in W.$$  

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In the following let $C = A \otimes_B k$. This is naturally an $A$-module coalgebra with coproduct $C \to C \otimes C$, $a \otimes 1 \mapsto \sum (a_{(1)} \otimes 1) \otimes (a_{(2)} \otimes 1)$, $a \in A$. (We note that $C \cong A/AB^+$ in the notation of [4], and that $C$ represents the kernel of $Sp(A) \to Sp(B)$ if $A$ is commutative, [5], p. 14). Let $\mathcal{C}_A$ be the category of $A$-modules $W$ which are supplied with a left $C$-coaction $\alpha: W \to C \otimes W$ such that

\[(2) \quad \alpha(aw) = \delta(a)\alpha(w), \quad a \in A, w \in W.\]

A morphism in $\mathcal{C}_A$ is a map which is both $A$-linear and $C$-colinear.

**Lemma 1.1.** Assume $A_B$ is finitely generated and projective. Then an $A$-module $W$ admits a system of imprimitivity based on $B$ if and only if $W$ is an object of $\mathcal{C}_A$.

**Proof.** First note that $F$ is essentially the opposite of the dual algebra $C^*$ of $C$. For

\[\zeta: \text{Hom}_k(C, k) \to F, \quad \zeta(g)(a) = g(a \otimes 1),\]

is an anti-isomorphism of $k$-algebras with $\zeta^{-1}(f)(a \otimes 1) = f(a)$; $\zeta$ is an algebra antimorphism since the product of $F$ is defined by the transpose coproduct of $A$. Since $A_B$ is finitely generated and projective, $C = A \otimes_B k$ is so over $k$. Therefore, $W$ is a left $F$-module if and only if $W$ is a left $C$-comodule, the actions being determined by each other through the formula

\[f \cdot w = \sum \langle \zeta^{-1}(f), w_{(1)} \rangle w_{(0)}, \quad f \in F, w \in W.\]

Since $C \otimes W \cong \text{Hom}_k(F, W)$, one sees that (1) is equivalent to

\[(3) \quad (1 \otimes a)\alpha(w) = \sum (\lambda(a_{(1)}) \otimes 1)\alpha(a_{(2)}w), \quad a \in A, w \in W.\]

The latter is evidently satisfied if (2) holds. Conversely, applying (3) with $a$ replaced by $a_{(2)}$ one obtains $\delta(a)\alpha(w) = \sum (a_{(1)}\lambda(a_{(2)}) \otimes 1)\alpha(a_{(3)}w) = \alpha(aw)$. Hence (2) is equivalent to (1), and this completes the proof.

For any left $B$-module $V$, $W = A \otimes_B V$ is naturally an object of $\mathcal{C}_A$ with coaction $W \to C \otimes W$, $a \otimes v \mapsto \sum (a_{(1)} \otimes 1) \otimes (a_{(2)} \otimes v)$. Conversely, we want to show that any $W \in \mathcal{C}_A$ is induced from $B$ if $A_B$ is flat. This is closely related to [1], Thm. 2.11, and [4], Thm. 2, and essentially the same proof as in [1] can be employed. It is based on the following lemma.

**Lemma 1.2.** For any $A$-module $X$ the map

\[\eta_X: A \otimes_B X \to C \otimes X, \quad a \otimes x \mapsto \sum (a_{(1)} \otimes 1) \otimes a_{(2)}x,\]

is an isomorphism.

**Proof.** For $b \in B$ we have $\eta_X(ab \otimes x) = \sum (a_{(1)} \otimes \varepsilon(b_{(1)})) \otimes (a_{(2)}b_{(2)}x) = \eta_X(a \otimes bx)$; hence $\eta_X$ is well-defined, and $(a \otimes 1) \otimes x \mapsto \sum a_{(1)} \otimes \lambda(a_{(2)})x$ gives a (well-defined) inverse.
THEOREM 1.3. Let $A$ be a Hopf $k$-algebra, $B \subset A$ a Hopf subalgebra, and $C = A \otimes_B k$. Assume $A$ is flat as a right $B$-module. Then an $A$-module $W$ is induced from $B$ if and only if $W$ is an object of $\mathcal{A}M$ (i.e. a left $C$-comodule satisfying (2)).

PROOF. Let $\alpha: W \to C \otimes W$ be the coaction of $W$. Regard $C \otimes W$ as an $A$-module by $a \cdot (c \otimes w) = \delta(a)(c \otimes w)$. Then both $\alpha$ and $i: W \to C \otimes W, w \mapsto (1 \otimes 1) \otimes w$, are $B$-linear. Set $W_0 = \{w \in W | \alpha(w) = i(w)\}$. Since $A$ is $B$-flat, the sequence

$$
A \otimes_B W_0 \to A \otimes_B W \xrightarrow{1 \otimes \alpha} A \otimes_B (C \otimes W) \xrightarrow{1 \otimes i}
$$

is exact. Consider $\mu_W: A \otimes_B W_0 \to W, a \otimes w \mapsto aw$, which is a morphism in $\mathcal{A}M$. It is easy to see that $(\mu_W, \eta_W, \eta_C \otimes W)$ transforms (4) into the sequence

$$
W \xrightarrow{\alpha} C \otimes W \xrightarrow{1 \otimes \alpha} C \otimes C \otimes W.
$$

But this sequence is exact for any comodule. Hence Lemma 1.2 implies that $\mu_W$ is an isomorphism.

Let $b\mathcal{M}$ denote the category of left $B$-modules.

COROLLARY 1.4. (cf. [4], Thm. 2). Assume $A_B$ is faithfully flat. Then the functor $b\mathcal{M} \to \mathcal{C}M, V \mapsto A \otimes_B V$, is an equivalence.

PROOF. For $V \in b\mathcal{M}$ consider the $B$-linear map $v: V \to (A \otimes_B V)_0, v \mapsto 1 \otimes v$. Since $\mu(1 \otimes v): A \otimes_B V \to A \otimes_B V$ is the identity, we have that $1 \otimes v$, hence $v$, is an isomorphism. It follows that $W \mapsto W_0$ is a quasi-inverse for $V \mapsto A \otimes_B V$.

REMARK 1.5. For $B = k$ theorem 1*3 gives the descent theorem for Hopf modules [3], Thm. 4.1.1. It should be noted however that the proof given in [3] works also for $A$ not $k$-flat (and $k$ not a field).

2. Coinduced Modules.

In the following we work with the $k$-algebra $E = \text{Hom}_B(A, k)$ of all left $B$-linear maps $f: A \to k$. The product is defined by

$$(f \cdot f')(a) = \sum f(a) f'(a) , \quad f, f' \in E.$$

$E$ is an $A$-module algebra with $(af)(a') = f(a'a)$. We shall consider the category $\mathcal{A}M_E$ of left $A$-modules and right $E$-modules $W$ satisfying

$$
a(wf) = \sum (a(1)w)(a(2)f), \quad a \in A, w \in W, f \in E.
$$
EXAMPLE. Let $W = \text{Hom}_B(A, V)$ for a left $B$-module $V$. For $g \in W$ and $f \in E$ define $gf \in W$ by $(gf)(a) = \sum g(a(1))f(a(2))$, $a \in A$. Then $W$ is an object of $\mathcal{M}_E$ with natural $A$-action $(ag)(a') = g(a' a)$.

REMARK 2.1. The condition for an $A$-module $W$ to be an object of $\mathcal{M}_E$ is slightly different from that of admitting a system of imprimitivity. There is no difference if $A$ is cocommutative, for then $E$ is commutative and $f \mapsto f \lambda$ gives an algebra isomorphism $F \cong E$. There appears however to be a gap at the end of the proof in [2] for the coinduced case. The proof provides an action on $W$ by the right $B$-endomorphisms of $A$, but for $W \cong \text{Hom}_B(A, V)$ a (right) action by the left $B$-endomorphisms of $A$ is needed.

Let $W \in \mathcal{M}_E$. We define a left $B$-module $W_0$ by the exact sequence

\begin{equation}
W \otimes E \xrightarrow{m} W \xrightarrow{p} W_0
\end{equation}

where $m(w \otimes f) = wf$, and $m'(w \otimes f) = wf(1)$. Note that $m$ and $m'$ are $B$-linear if we regard $W \otimes E$ as a $B$-module by $b \cdot (w \otimes f) = \delta(b)(w \otimes f)$. We want to show that the left $A$- and right $E$-linear map

\begin{equation}
\mu_W: W \rightarrow \text{Hom}_B(A, W_0), \quad \mu_W(w)(a) = p(aw),
\end{equation}

is an isomorphism if $BA$ is finitely generated and projective. The proof is in some sense dual to that in section 1, and uses the following lemma (cf. the lemma in [2]).

LEMMA 2.2. Let $X$ be an $A$-module, and suppose $BA$ is finitely generated and projective. Then

\begin{equation}
\delta_X: X \otimes E \rightarrow \text{Hom}_B(A, X), \quad \delta_X(x \otimes f)(a) = \sum a(1)f(a(2))x,
\end{equation}

is an isomorphism.

PROOF. Let $^*X = X$ with left $B$-action $b \cdot x = \varepsilon(b)x$. Then

\begin{equation}
\beta: \text{Hom}_B(A, ^*X) \rightarrow \text{Hom}_B(A, X), \quad \beta(\varphi)(a) = \sum a(1)\varphi(a(2)),
\end{equation}

is an isomorphism with $\beta^{-1}(\psi)(a) = \sum \lambda(a(1))\psi(a(2))$ for $\psi \in \text{Hom}_B(A, X)$. Set $^*\delta = \beta^{-1}\delta$. Then for $f \in E$, $a \in A$, and $x \in X$

\begin{equation}
^*\delta(x \otimes f)(a) = \sum \lambda(a(1))\delta(x \otimes f)(a(2)) = \sum \lambda(a(1)a(2)f(a(3))x = f(a)x.
\end{equation}

Now choose a projective coordinate system $f_i \in \text{Hom}_B(A, B)$, $a_i \in A$, $1 \leq i \leq n$; then $\text{Hom}_B(A, ^*X) \rightarrow X \otimes E$, $\varphi \mapsto \sum \varphi(a_i) \otimes \varepsilon \circ f_i$, is an inverse for $^*\delta$ as follows from $\sum f_i(a)a_i = a$ for $a \in A$.

THEOREM 2.3. Let $A$ be a Hopf $k$-algebra, $B \subset A$ a Hopf subalgebra, $E = \text{Hom}_B(A, k)$, and assume that $A$ is finitely generated and projective as a left
B-module. Then an A-module W is coinduced from B if and only if W is an object of $A \mathcal{M}_E$ (i.e. a right E-module satisfying (5)).

**Proof.** For any right E-module W there is a canonical exact sequence

$$W \otimes E \otimes E \longrightarrow W \otimes E \longrightarrow W$$

defined by the action of E on W. Suppose $W \in A \mathcal{M}_E$ and consider $W \otimes E$ as a left A-module by $a \cdot (w \otimes f) = \delta(a)(w \otimes f)$. Then $(\iota_{W \otimes E}, \iota_W, \mu_W)$, with $\mu_W$ defined in (7), transforms (8) into the exact sequence obtained from (6) by applying $\text{Hom}_B(A, \_)$, $\_$. Observe that

$$\iota_W(w \otimes f \otimes f')(a) = \sum a_{(1)}(a_{(3)})(w \otimes f) = \sum f'(a_{(3)})(a_{(1)}w \otimes a_{(2)}f),$$

and $p(wf) = p(wf(1))$. In particular, $(\mu_W m)(w \otimes f)(a) = p(a(wf)) = p(\sum (a_{(1)}w)(a_{(2)}f)) = p(\sum a_{(1)}w(\otimes f)(a_{(2)})) = p \iota_W(w \otimes f)(a)$. It follows therefore from lemma 2.2 that $\mu_W$ is an isomorphism.

**Corollary 2.4.** Suppose $pA$ is finitely generated and projective. Then the functor $A \mathcal{M}_E \rightarrow p \mathcal{M}$, $W \mapsto W_0$, is an equivalence if and only if $B \subset A$ is a left B-direct summand of A.

**Proof.** First observe that $V \mapsto \text{Hom}_B(A, V)$ is a right adjoint for $W \mapsto W_0$, the adjunction morphisms being $\mu$, and $\nu$: $\text{Hom}_B(A, V)_0 \rightarrow V, p(g) \mapsto g(1)$, for $g \in \text{Hom}_B(A, V)$. Furthermore, the composite

$$\text{Hom}_B(A, V) \xrightarrow{\mu} \text{Hom}_B(A, \text{Hom}_B(A, V)_0) \xrightarrow{\text{Hom}(A, \_)} \text{Hom}_B(A, V)$$

is the identity, since $p(ag) = (ag)(1) = g(a)$ for $a \in A$. Hence $\text{Hom}(A, \_)$ is bijective. Now, if $A = B \oplus X$, we may conclude that $\nu$ is bijective by decomposing $\text{Hom}(A, \_)$ = $\text{Hom}(B, \_)$ $\oplus$ $\text{Hom}(X, \_)$, Conversely, suppose that $\nu$ is bijective for $V = B$. Then there exists $g \in \text{Hom}_B(A, B)$ with $g(1) = 1$, and therefore $A = B \oplus \text{Ker}(g)$. □

**Remark 2.5.** $B \subset A$ is a left B-direct summand of A iff $A$ is a left B-generator; for suppose there exists a B-epimorphism $A^{(t)} \rightarrow B$. Pick $u \in A^{(t)}$ such that $u \mapsto 1$. Then $g$: $A \rightarrow Au \rightarrow B$ is a B-epimorphism with $g(1) = 1$, and hence with $A = B \oplus \text{Ker}(g)$.

**Remark 2.6.** For $B = k$ the assumption of thm. 2.3 can not be omitted. This can be seen as follows. Let X be an A-module. Then $W = X \otimes E$ is naturally a right E-module and an object of $A \mathcal{M}_E$ with A-action defined by $\delta$. It is not difficult to see that

$$W \otimes E \xrightarrow{m} W \xrightarrow{p} X$$

is exact where $p(x \otimes f) = f(1)x$ for $x \in X$, and $f \in E$. Hence in this case $W_0 = X$. 
Furthermore, for $\mu_w : X \otimes E \to \text{Hom}_B(A, X)$ we obtain

$$\mu_w(x \otimes f)(a) = p(a \cdot (x \otimes f)) = p(\delta(a)(x \otimes f)) = \sum a_{(1)} f(a_{(2)}) x.$$ 

Thus $\mu_w = \delta_x$. Suppose now that $\mu_w$ is bijective for $X = A$. Then also $\epsilon \delta = \beta^{-1} \delta$ is an isomorphism, and in case $B = k$ this means that $A$ is finitely generated and projective over $k$.

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