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INVOLUTORY ANTIAUTOMORPHISMS OF VON NEUMANN AND C*-ALGEBRAS

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1. Introduction.

In this paper we study the relationship between involutory antiautomorphisms of C*- and von Neumann algebras on one hand and Jordan algebras on the other. If A is either a C*- algebra or a von Neumann algebra with an involutory antiautomorphism α (i.e. a *-preserving antiautomorphism of order two), we will denote by A^α the self-adjoint part of its fixed point algebra. We will find conditions under which the following statements are true.

1. A^α generates A.
2. The Jordan structure of A^α determines the conjugacy class of α.
3. If $\phi : A^\alpha \to A^\beta$ is a Jordan isomorphism, then ϕ can be extended to a *-automorphism of A implementing a conjugacy between α and β.

(Two antiautomorphisms α and β are said to be conjugate if there exists a *-automorphism ϕ of A such that $\phi \circ \alpha \circ \phi^{-1} = \beta$. The set of involutory antiautomorphisms conjugate to α is called its conjugacy class.) Our results generalise previous results by Størmer and Hanche-Olsen, [2] and [7]. We also prove partial analogues in the C*-algebra case of results by Størmer [7] concerning closeness of antiautomorphisms.

2. Antiautomorphisms of von Neumann and C*-algebras.

For every JW-algebra P (i.e. a weakly closed Jordan subalgebra of the selfadjoint part of some von Neumann algebra) there exists a unique von Neumann algebra $W^*(P)$ called the universal von Neumann algebra of P and a Jordan isomorphism ψ: $P \to W^*(P)_{sa}$ with the following property: If M is a von Neumann algebra and ϕ: $P \to M_{sa}$ is a Jordan homomorphism, there exists a unique *-homomorphism $\hat{\phi}$: $W^*(P) \to M$ such that $\hat{\phi} \circ \psi = \phi$. Moreover, there exists a canonical (involutory) antiautomorphism Φ of $W^*(P)$ which is uniquely

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determined by the property that $\psi(P) \subset \mathrm{W}^*(P)_\Phi$ [2]. Reference [2] also defines JW-factors as well as JW-algebras of types I_n, II and III analogously with von Neumann algebras.

2.1. LEMMA. *Let P be a* JW*-algebra of the form* $\mathrm{C}(X, Q)$ *where X is a compact Hausdorff space and Q is a* JW*-factor of finite dimension. Let $N = \mathrm{W}^*(Q)$ with canonical antiautomorphism Φ, then* $\mathrm{W}^*(P) = \mathrm{C}(X, N)$ *with inclusion map induced by the inclusion map of Q in N and canonical antiautomorphism Φ_P given by $\Phi_P(f)(x) = \Phi(f(x))$.*

PROOF. We may identify $\mathrm{C}(X, N)$ with $\mathrm{C}_\mathrm{C}(X) \otimes N$ via an isomorphism that takes $f \times a$ to $f \otimes a$ for $f \in \mathrm{C}_\mathrm{C}(X)$ and $a \in N$. Since $\mathrm{C}_\mathrm{R}(X)$ generates $\mathrm{C}_\mathrm{C}(X)$ and Q generates N, it follows that P generates $\mathrm{C}(X, N)$. Let $\phi\colon P \to M_{sa}$ be a normal Jordan homomorphism into the self-adjoint part of a von Neumann algebra M. Then ϕ restricts to a homomorphism ϕ_0 on the constant functions in P, which may be extended to a C*-homomorphism $\hat{\phi}_0\colon 1 \otimes N \to M$ by the universal property of N relative to Q. We define $\hat{\phi}\colon \mathrm{C}_\mathrm{C}(X) \otimes N \to M$ by $\hat{\phi}(f \otimes a) = (\phi(\mathrm{Re}\, f) + i\phi(\mathrm{Im}\, f)) \times \hat{\phi}_0(a)$. Then clearly $\hat{\phi}$ is a normal C*-homomorphism extending ϕ. By the uniqueness of the universal von Neumann algebra, it follows that $\mathrm{W}^*(P) = \mathrm{C}(X, N)$. The antiautomorphism Φ_P is seen to leave P pointwise invariant, thus Φ_P is the canonical antiautomorphism.

A Jordan subalgebra A of an associative algebra B is said to be reversible in B if all elements of the form $a_1 a_2 \ldots a_n + a_n a_{n-1} \ldots a_1$ with $a_i \in A$ for all i belong to A. A representation of a Jordan algebra into an associative algebra is said to be reversible if its image is a reversible subalgebra. Note that the class of JW-algebras of type I_2 can be further subdivided into types $I_{2,n}$ corresponding to the spin factors V_n of dimension $n + 1$, generated by n anticommuting symmetries.

2.2. THEOREM. *Let M be a von Neumann algebra and α an involutory anti-automorphism of M. Let M^α be the self-adjoint part of the fixed point algebra of M. Then,*

(a) *M^α has no part of type $\mathrm{I}_{2,n}$, $n \in \{4, 6, 7, 8, \ldots\} \cup \{\infty\}$.*

Suppose in addition that M^α generates M, then

(b) *If M^α has no part of type $\mathrm{I}_{2,5}$, there exists an isomorphism $\gamma\colon M \to \mathrm{W}^*(M^\alpha)$ leaving M^α pointwise fixed such that $\alpha = \gamma^{-1} \circ \Phi \circ \gamma$, where Φ is the canonical antiautomorphism of* $\mathrm{W}^*(M^\alpha)$.

(c) *If M^α is of type $\mathrm{I}_{2,5}$, then M is of type I_4 and α is central on M. If s_1, s_2, s_3, s_4 are anti-commuting symmetries in M^α, $s_5 = s_1 s_2 s_3 s_4$ and $\psi\colon M^\alpha \to M \otimes M$ is given by*

(1) $\psi(s_i) = s_i \oplus s_i, i = 1, 2, 3, 4.$

(2) $\psi(s_5) = s_5 \oplus -s_5.$

(3) $\psi(z) = z \oplus z$, *z in the center Z of M^α,*

then ψ is a Jordan isomorphism, and with the imbedding ψ, $M \oplus M$ is the universal von Neumann algebra $W^*(M^\alpha)$. The canonical antiautomorphism is given by $\alpha \oplus \alpha$.

PROOF. (a). Suppose $n \in \{4, 6, 7, 8, \ldots\} \cup \{\infty\}$, and let π be a factor representation of the $I_{2,n}$ part of M^α onto a JW-factor P. Let z be the central support projection of π, then $zM^\alpha \cong P$, thus, P is of type $I_{2,n}$. The map $\pi(x) \to zx$ is then a reversible representation of P, which is impossible by [2, 6.2.5] Thus, M^α has no part of type $I_{2,n}$.

(b). Let $M^\alpha = P_1 \oplus P_2 \oplus P_3$ where P_2 is the type $I_{2,2}$-part, and P_3 is type $I_{2,3}$-part of M^α. By [2, Theorem 6.3.13], P_2 is of the form $C(X, V_2)$ with X a compact Hausdorff space and V_2 the three-dimensional spin-factor. By [2, Section 6.2.5], V_2 is reversible in *every* representation, so in particular V_2 is reversible in $W^*(V_2)$. From this and lemma 2.1 it follows that P_2 is reversible in $W^*(P_2)$. *If* Φ_2 is the canonical antiautomorphism of $W^*(P_2)$, then $P_2 = W^*(P_2)^{\Phi_2}$. Indeed, choose $z \in W^*(P_2)^{\Phi_2}$, then z is a limit of sums of elements of the form $x = a_1 a_2 \ldots a_n$ and $y = ib_1 b_2 \ldots b_m$, with $a_i, b_j \in P_2$. Define the linear map Λ: $W^*(P_2) \to W^*(P_2)^{\Phi_2}$ by $\Lambda(w) = 1/4(w + \alpha(w) + w^* + \alpha(w^*))$, then $\Lambda(z) = z$, $\Lambda(x) \in P_2$ because of the reversibility of P_2, while $\Lambda(y) = 0$. Thus, z is a limit of elements from P_2, so it follows that $z \in P_2$. Similarly $P_3 = W^*(P_3)^{\Phi_3}$ where Φ_3 is the canonical antiautomorphism of $W^*(P_3)$. Clearly, $W^*(M^\alpha) = W^*(P_1) \oplus W^*(P_2) \oplus W^*(P_3)$ with canonical antiautomorphism $\Phi = \Phi_1 \oplus \Phi_2 \oplus \Phi_3$, where Φ_1 is the canonical antiautomorphism of $W^*(P_1)$. By hypothesis, M^α has no part of type $I_{2,5}$, so by (a), P_1 has no part of type I_2. Thus, by [2, Theorem 7.3.3] and the above argument, $M^\alpha = W^*(M^\alpha)^\Phi$. Now, let ϕ: $W^*(M^\alpha) \to M$ be a normal C*-homomorphism satisfying $\phi(x) = x, x \in M^\alpha$ (that is, ϕ is the canonical extension of the inclusion map). Let e be the central projection in $W^*(M^\alpha)$ such that $eW^*(M^\alpha)$ is the kernel of ϕ. By the uniqueness of ϕ, $\phi = \alpha \circ \phi \circ \Phi$, so that $\phi \circ \Phi = \alpha \circ \phi$. It follows that $\Phi(e) \in \ker \phi$ so that $\Phi(e)$ is a sub-projection of e. Since Φ has period two, it follows that $\Phi(e) = e$. Thus $e \in M^\alpha$ which implies $e = 0$. Thus ϕ is an isomorphism (it is *onto* since M^α generates M). Putting $\gamma = \phi^{-1}$, this yields the desired result.

(c). Note that the anti-commutation relations between the elements s_i, $i = 1, 2, 3, 4$, show that $s_5 = 1/2(s_1 s_2 s_3 s_4 + s_4 s_3 s_2 s_1)$, so that $s_5 \in M^\alpha$. Also it is seen that s_5 is a symmetry anti-commuting with s_i, $t = 1, 2, 3, 4$, so that s_1, s_2, s_3, s_4 and s_5 form a spin system. Thus if P is the Jordan algebra generated by the elements s_i, then, by [2, 6.3.9], the JW-algebra ZP generated by P and the center Z of M^α equals M^α. $M^\alpha = zP$, and the map ψ is a well defined Jordan isomorphism. Suppose first that $M^\alpha \cong V_5$, the six-dimensional spin factor. Then by [2, 7.1.12], $W^*(M^\alpha) = M_1 \oplus M_2$ where M_i is isomorphic to $M_4(\mathrm{C})$, $i = 1, 2$. We want to prove that the canonical extension $\hat{\psi}$ of ψ is an isomorphism. We have $\psi(s_1)\psi(s_2)\psi(s_3)\psi(s_4) + \psi(s_4)\psi(s_3)\psi(s_2)\psi(s_1) + 2\psi(s_5) = 4s_5 \oplus 0$, so that $1 \oplus 0 = s_5^2 \oplus 0$ lies in the C*-algebra generated by $\psi(M^\alpha)$. Clearly, also $0 \oplus 1$ lies in this algebra. Since M^α generates M, it follows that $\psi(M^\alpha)$ generates $M \oplus M$.

Thus $\hat{\psi}$ is surjective so that $\hat{\psi}$ must be an isomorphism. Thus $M \oplus M$ with imbedding ψ is the universal von Neumann algebra if M^α is a factor. Note that $1/2(\psi(s_1)\psi(s_2)\psi(s_3)\psi(s_4) + \psi(s_4)\psi(s_3)\psi(s_2)\psi(s_1)) = s_5 \oplus s_5 \notin \psi(M^\alpha)$, so $\psi(M^\alpha)$ is not reversible in $M \oplus M$. From this it follows that if $\phi\colon V_5 \to N_{sa}$ is a Jordan isomorphism into the self-adjoint part of a von Neumann algebra N such that $\phi(V_5)$ is a reversible sub-algebra of N_{sa} generating N, then the surjective homomorphism $\hat{\phi}\colon \mathrm{W}^*(V_5) \to N$ can not be an isomorphism since otherwise $V_5 = \hat{\phi}^{-1} \circ \phi(V_5)$ would be a reversible sub-algebra of $\mathrm{W}^*(V_5)$. It follows that $N \cong \mathrm{M}_4(\mathrm{C})$, since N is a homomorphic but *not* an isomorphic image of $\mathrm{M}_4(\mathrm{C}) \oplus \mathrm{M}_4'(\mathrm{C})$. Now, in the general case, let N be the C*-algebra generated in M by P. Then P is a reversible sub-algebra of N. To see this, note that if x is a symmetric product of n of the symmetries s_i, $i = 1, 2, \ldots, 5$, then, by permuting the order of the symmetries in the product and cancelling all terms of the form $s_i^2 = 1$, we may assume that $n \leqq 5$. Note also that if σ is a permutation of the set $\{1,2,3,4,5\}$, then $s_{\sigma(1)}s_{\sigma(2)}s_{\sigma(3)}s_{\sigma(4)}s_{\sigma(5)} = \pm 1$. Thus, it suffices to prove $1/2(s_{\sigma(1)}s_{\sigma(2)}s_{\sigma(3)}s_{\sigma(4)} + s_{\sigma(4)}s_{\sigma(3)}s_{\sigma(2)}s_{\sigma(1)}) \in P$ for any permutation σ of $\{1,2,\ldots,5\}$. This is obvious if $\sigma(i) \neq 5$ for each i; if $\sigma(i) = 5$ for some i, we replace s_5 by $s_1s_2s_3s_4$ in the first summand, and by $s_4s_3s_2s_1$ in the second summand, and use cancellation again. Thus P is reversible in N as asserted. Thus by the same proof that was used in part (b), (using the map Λ), $P = N^\alpha$. Thus, using the factor case proved above, $N \oplus N$ with the imbedding $\psi|_P$ is the universal von Neumann algebra of P. Obviously, $Z + iZ$ commutes with N, so by lemma 2.1, $\mathrm{W}^*(M^\alpha) = \mathrm{W}^*(ZP) = (Z + iZ) \otimes N \otimes (Z + iZ) \otimes N$ with the imbedding ψ. Since $ZP = M^\alpha$ generates M, it is clear $(Z + iZ)N = M$. Since $N \cong \mathrm{M}_4(\mathrm{C})$, it follows that M is of type I_4. Since α is the identity on Z, and $Z + iZ$ is the center of M, it follows that α is central on M. Finally $\alpha \oplus \alpha$ is an involutory antiautomorphism of $M \oplus M$ leaving $\psi(M^\alpha)$ pointwise invariant. This completes the proof.

2.3. Corollary. *Let M be a von Neumann algebra, and α and β involutory antiautomorphisms of M. Suppose $\phi\colon M^\alpha \to M^\beta$ is a Jordan isomorphism, and that both M^α and M^β generate M, and have no parts of type $\mathrm{I}_{2,5}$. Then ϕ extends to an automorphism $\hat{\phi}$ of M such that $\beta = \hat{\phi} \circ \alpha \circ \hat{\phi}^{-1}$.*

Proof. By part (b) of the preceding theorem, M is the universal von Neumann algebra of M^α. By [2, 4.5.6], ϕ is normal, so ϕ has an extension to a normal C*-homomorphism $\hat{\phi}\colon M \to M$. Similarly ϕ^{-1} has an extension to a normal C*-homomorphism $\hat{\phi}^{-1}\colon M \to M$. Clearly, $\hat{\phi}^{-1} \circ \hat{\phi}$ restricts to the identity on M^α, and since M^α generates M it follows that $\hat{\phi}^{-1} \circ \hat{\phi}$ equals the identity on M. The same argument with M^α replaced by M^β shows that $\hat{\phi} \circ \hat{\phi}^{-1}$ equals the identity on M. It follows that $\hat{\phi}$ is an automorphism of M. Clearly $\hat{\phi}$ and $\beta \circ \hat{\phi} \circ \alpha$ have the same restrictions to M^α, hence they are equal on M. This implies the conjugacy of α and β via $\hat{\phi}$.

2.4. EXAMPLES. The following examples (a) and (b) show that the Jordan structure of the fixed algebra does not determine the conjugacy class of the antiautomorphism unless special assumptions are made for the abelian part of the fixed poin algebra.

Example (c) shows that the assumption that M^α and M^β have no parts of type $\mathrm{I}_{2,s}$ is necessary in the above corollary.

We denote by t_n the transposition map on $\mathrm{M}_n(\mathrm{C})$ and by q the quaternionic flip on $\mathrm{M}_2(\mathrm{C})$ defined by

$$q\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(a) Put $A = \mathrm{C}([-1,1])$. Define α and β by

$$\alpha(f)(x) = f(x), \qquad x \in [-1,1].$$

$$\beta(f)(x) = f(-x), \quad x \in [-1,1].$$

Then, clearly, $A^\alpha = C_{\mathrm{R}}([-1,1])$, whereas $A^\beta \cong \mathrm{C}_{\mathrm{R}}([0,1])$, so that A^α and A^β are isomorphic via the map $\phi(f)(x) = f(1/2(x+1))$. But α and β are not conjugate since $\psi \circ \alpha \circ \psi^{-1} = \alpha$ for each automorphism ψ of A.

Clearly the symbol C could be replaced by the symbol L^∞ in this argument to give a corresponding von Neumann algebra example.

(b) Put $A = \{f \in L^\infty([0,3], \mathrm{M}_2(\mathrm{C})) : f(x) \in \mathrm{C}, 0 \leqq x \leqq 1\}$. Define α by

$$\alpha(f)(x) = \begin{cases} f(x), & 0 \leqq x \leqq 1 \\ t_2(f(x)), & 1 \leqq x \leqq 3. \end{cases}$$

Define β by

$$\beta(f)(x) = \begin{cases} f(x), & 0 \leqq x \leqq 1 \\ q(f(x)), & 1 \leqq x \leqq 2 \\ t_2(f(x)), & 2 \leqq x \leqq 3. \end{cases}$$

Then $A^\alpha \cong L^\infty_{\mathrm{R}}([0,1]) \oplus L^\infty([1,3], \mathrm{H}_2(\mathrm{R}))$, $A^\beta \cong L^\infty_{\mathrm{R}}([0,2]) \oplus L^\infty([2,3], \mathrm{H}_2(\mathrm{R}))$, where $\mathrm{H}_2(\mathrm{R})$ denotes the symmetric real 2 by 2 matrices. Thus $A^\alpha \cong A^\beta$, but, since α does not restrict to a quaternionic flip anywhere, $\alpha \not\sim \beta$.

Compare with [7, Theorem 3.3] and note that the isomorphism between A^α and A^β is not central in the examples (In example (a) the involution β is not central either).

(c) Put $M = \mathrm{M}_4(\mathrm{C}) \otimes \mathrm{M}_2(\mathrm{C})$ and $\alpha = \beta = q \otimes t_2$. It is shown in [3] that M^α and M^β are isomorphic to V_5. Let $s_1, s_2\, s_3$ and s_4 be anti-commuting symmetries in M^α and put $s_5 = s_1 s_2 s_3 s_4$. Then the map $\phi\colon M^\alpha \to M^\beta$ defined by $\phi(s_i) = s_i, i = 1,2,3,4$, $\phi(s_5) = -s_5$ is a Jordan isomorphism of M^α onto M^β which has no extension to a C*-homomorphism on M.

2.5. DEFINITION. Suppose A and A' are reversible Jordan sub-algebras of associative algebras B and B' respectively. A Jordan homomorphism $\phi: A \to A'$ is said to be *reversible* if $\phi(x_1x_2\ldots x_n + x_nx_{n-1}\ldots x_1) = \phi(x_1)\phi(x_2)\ldots\phi(x_n) + \phi(x_n)\phi(x_{n-1})\ldots\phi(x_1), x_1, x_2, \ldots, x_n \in A$. Note that if α is an involutory anti-automorphism of M such that M^α has no type $I_{2,5}$ part, and if $\phi: M^\alpha \to N_{sa}$ is a normal Jordan homomorphism into the self-adjoint part of some von Neumann algebra generated by M^α, then $K = W^*(M^\alpha)$, so that ϕ can be extended to a C*-homomorphism $\hat{\phi}: K \to N$.

2.6. THEOREM. *Let α and β be involutory antiautomorphisms of a von Neumann algebra M with fixed point algebras M^α and M^β respectively. Suppose $\phi: M^\alpha \to M^\beta$ is a reversible Jordan isomorphism, and suppose that M^α and M^β generate M.*

Then ϕ can be extended to an automorphism $\hat{\phi}$ of M such that $\beta = \hat{\phi}\circ\alpha\circ\hat{\phi}^{-1}$.

PROOF. Let w_2 be a central projection in M such that $M^\alpha w_2$ is the type $I_{2,5}$ part of M^α, and put $w_1 = 1 - w_2$. Let ϕ_1 be the restriction of ϕ to $M^\alpha w_1$. Then ϕ_1 is a Jordan isomorphism *onto* $M^\beta\phi(w_1)$. Since clearly, $M^\alpha w_1$ and $M^\beta\phi(w_1)$ generate Mw_1 and $M\phi(w_1)$ respectively, it follows, by the same argument as in the proof of corollary 2.3, that ϕ_1 extends to a C*-isomorphism $\hat{\phi}_1: Mw_1 \to M\phi(w_1)$ such that $\beta(x) = \hat{\phi}_1\circ\alpha\circ\hat{\phi}_1^{-1}(x), x \in M\phi(w_1)$. Now choose anti-commuting symmetries s_1, s_2, s_3, s_4 and $s_5 = 1/2(s_1s_2s_3s_4 + s_4s_3s_2s_1)$ in $M^\alpha w_2$ and define the Jordan isomorphism $\psi: M^\alpha w_2 \to Mw_2 \oplus Mw_2$ as in part (c) of theorem 2.2. Let ϕ_2 be the restriction of ϕ to $M^\alpha w_2$. Then, by theorem 2.2, part (c), there exists a C*-homomorphism $\hat{\phi}_2: Mw_2 \oplus Mw_2 \to M\phi(w_2)$ such that $\hat{\phi}_2\circ\psi = \phi_2$. Then we have

$$\begin{aligned}\hat{\phi}_2(0\oplus s_5) &= 1/4\hat{\phi}_2(\psi(s_1)\psi(s_2)\psi(s_3)\psi(s_4) + \psi(s_4)\psi(s_3)\psi(s_2)\psi(s_1)) - 1/2\hat{\phi}_2\circ\psi(s_5)\\ &= 1/4(\phi_2(s_1)\phi_2(s_2)\phi_2(s_3)\phi_2(s_4) + \phi_2(s_4)\phi_2(s_3)\phi_2(s_2)\phi_2(s_1))\\ &\quad - 1/2\phi_2(s_5)\\ &= 1/2\phi_2(s_5) - 1/2\phi_2(s_5) = 0\end{aligned}$$

The last step is justified by the reversibility of ϕ. Thus $\hat{\phi}_2(0\oplus 1) = \hat{\phi}_2(0\oplus s_5^2) = 0$, hence ϕ_2 extends to a C*homomorphism $\tilde{\phi}_2: Mw_2 \to M\phi(w_2)$. Again arguing as in the proof of corollary 2.3 we conclude that $\tilde{\phi}_2$ is an isomorphism and $\beta(x) = \hat{\phi}_2\circ\alpha\circ\tilde{\phi}_2^{-1}(x), x \in M\phi(w_2)$. Define the automorphism $\tilde{\phi}: M \to M$ by $\tilde{\phi}(x) = \hat{\phi}_1(xw_1) + \hat{\phi}_2(xw_2)$. Then $\tilde{\phi}$ is the desired automorphism. Thus, the assertion follows. This completes the proof.

2.7. COROLLARY. *Let α and β be involutory antiautomorphisms of a C*-algebra A. Suppose that A^α and A^β generate A. Suppose that there exists a reversible Jordan isomorphism ϕ from A^α onto A^β. Then ϕ can be extended to an automorphism of A satisfying $\phi\circ\alpha\circ\phi^{-1} = \beta$.*

PROOF. We extend α and β by ultra-weak continuity to involutory antiautomorphisms of A^{**}. Then clearly the fixed point algebras are the ultra-weak closures $\overline{A^\alpha}$ and $\overline{A^\beta}$ respectively. Similarly ϕ can be extended to an isomorphism from $\overline{A^\alpha}$ onto $\overline{A^\beta}$. Since ϕ is ultra weakly continuous, it follows, by induction, that ϕ is reversible on $\overline{A^\alpha}$. Thus, ϕ can be extended to an automorphism of A^{**} satisfying $\beta = \phi \circ \alpha \circ \phi^{-1}$ by theorem 2.6 (denoting the extensions to A^{**} by the same symbols as the original maps). Since A^α generates A, it follows that ϕ restricts to an automorphism of A.

Examples (a) and (b) of section 2.4 also show that the fixed point algebra does not always generate the original *-algebra. As the next theorem shows, this is due to the presence of a type I_1 part in the fixed point algebra.

2.8. THEOREM. *Let M be a von Neumann algebra and α an involutory antiautomorphism of M. Let M^α be the self-adjoint part of the fixed point algebra of α. If M^α has no part of type I_1, then M^α generates M.*

PROOF. Let z be the central projection in M such that Mz is the non-type I part of M. Then, clearly, Mz is invariant under α. By [5, appendix 3] there exist central projections w_1, w_2 such that $z = w_1 + w_2 + \alpha(w_2)$ with the center of Mw_1 pointwise fixed under α (see also [1, prop. 2.7]). Then by [7, Proposition 3.1., and lemma 3.2], $Mw_1 = \mathrm{W}^*(M^\alpha w_1)$ and, by [2, Theorem 7.4.7], $M(w_2 + \alpha(w_2)) = \mathrm{W}^*(M^\alpha(w_2 + \alpha(w_2)))$. $(M\alpha(w_2) \cong (Mw_2)^\circ$ via the map α, and $M^\alpha(w_2 + \alpha(w_2)) \cong (Mw_2)_{sa}$ via the map $x \to x + \alpha(x))$ Thus, $Mz = \mathrm{W}^*(M^\alpha z)$, and in particular $M^\alpha z$ generates Mz. If z' is a central projection in M such that Mz' is the type I_n part of M, $n = 2, 3, \ldots$ or $n = \infty$, then z' can be split as an orthogonal sum $z' = w_1 + w_2 + \alpha(w_2)$ in the same way as the projection z above. For the non-central part of $\alpha_{|Mz'}$ the proof is then the same as above. For the central part we refer to the classification of involutions of type I algebras given in [7, proposition 2.6] together with the fact that in each of these cases the fixed point algebra generates the full algebra (see e.g. the remark after corollary 4.6 of [3]). This completes the proof.

2.9. COROLLARY. *Let A be a C*-algebra with an involutory α, such that the fixed point algebra A^α has no abelian representations. Then A^α generates A. If A^α has no representations of type $\mathrm{I}_{2,5}$, $A \cong \mathrm{CU}^*(A^\alpha)$, where $\mathrm{CU}^*(A^\alpha)$ denotes the universal C*-algebra of A^α.*

PROOF. Let B be the C*-algebra generated by A^α in A. Then α restricts to an involutory antiautomorphism of B and $B^\alpha = A^\alpha$. Let $r\colon A^* \to B^*$ be the restriction map. Then r is surjective by the Hahn-Banach theorem. Clearly, the map $r^*\colon B^{**} \to A^{**}$ restricts to the identity map from B^α into A^α. By ultra-weak continuity r^* restricts to a reversible isomorphism from $\overline{B^\alpha}$ onto $\overline{A^\alpha}$ (the ultra-weak closures are with respect to B^{**} and A^{**} respectively). Since $A^\alpha = B^\alpha$ has no abelian representations, neither $\overline{A^\alpha}$ nor $\overline{B^\alpha}$ have parts of type I_1, so, by theorem 2.8, they generate A^{**} and B^{**} respectively. Clearly r^* is the unique extension of $r^*|_{\overline{B^\alpha}}$ to an ultra-weak–ultra continuous C*-homomorphism $B^{**} \to A^{**}$, so, by theorem 2.6, r^* is an isomorphism. In particular, r^* is surjective so that r is injective. It follows from the Hahn-Banach theorem that $B = A$. The last claim follows from theorems 2.2 and 4.4 of [3].

2.10. THEOREM. *Suppose α and β are involutory antiautomorphisms of the von Neumann algebra M and that $M^\alpha = M^\beta$, then $\alpha = \beta$.*

PROOF. Let z be the central projection in M such that $M^\alpha z$ is the abelian part of M^α. Let $w = 1 - z$. Then, by theorem 2.8, $M^\alpha w = M^\beta w$ both generate Mw so that α and β have common restrictions to Mw. Thus we may assume that M^α is abelian. By [5, appendix 3] and [7, lemma 2.3 and proposition 2.6] there exist central projections z_1, z_2 and z_3 in M such that $z_1, z_2, \alpha(z_2)$ and z_3 are orthogonal, $z_1 + z_2 + \alpha(z_2) + z_3 = 1$, α restricts to the identity on Mz_1, and α is of the form $\iota \oplus q$ on Mz_3, which is homogeneous of type I_2. A similar decomposition with respect to β obviously gives the same projections z_1 and z_3. Then $z_2\beta(z_2)$ is a subprojection of z_2 in $M^\beta = M^\alpha$, which implies that $\beta(z_2)$ is orthogonal to z_2. Thus, if $x \in Mz_2$ then, $\beta(x) \in M\alpha(z_2)$ so that $(x + \alpha(x)) - (x + \beta(x)) = \alpha(x) - \beta(x) \in M\alpha(z_2) \cap M^\alpha = \{0\}$. It follows that $\alpha(x) = \beta(x)$ so that α and β coincide on Mz_2, hence also on $M\alpha(z_2)$. Obviously, α and β coincide on Mz_1. Also note that the antiautomorphism q does not depend on the choice of matrix units for $M_2(C)$ since $q(x) = x^{-1} \det x$ and hence α and β coincide on Mz_3. This completes the proof.

REMARK. This result also holds for C*-algebras. Indeed, if A is a C*-algebra with involutory antiautomorphisms α and β, and if $A^\alpha = A^\beta$, the α and β have extensions (also called α and β) to A^{**}, with fixed point algebras $\overline{A^\alpha}$ and $\overline{A^\beta}$ respectively. Hence we can apply theorem 2.10.

2.11. THEOREM. *Let A be a* C**-algebra, α: $A \to A$ an involutory antiautomorphism and suppose u is a unitary in A with a square root v in the* C**-algebra* $C^*(u)$ *generated by u. Then*

(a) *If $\alpha(u) = u$, then $\beta = \alpha \operatorname{Ad} u$ is an involutory antiautomorphism conjugate to α via* $\operatorname{Ad} v$.

(b) *If β: $A \to A$ is an involutory antiautomorphism with $\|\alpha - \beta\| < 2$ and $\alpha\beta =$* $\operatorname{Ad} u$, *then $\beta \sim \alpha$ via* $\operatorname{Ad} v$.

PROOF. It is trivial to check that $\alpha \operatorname{Ad} u$ is an antiautomorphism of period 2. Let v be a square root of u. Since $v \in C^*(u)$, $\alpha(v) = v$. So $\operatorname{Ad} v^* \circ \alpha \circ \operatorname{Ad} v(x) = v^*\alpha(vxv^*)v = (v^*)^2\alpha(x)v^2 = u^*\alpha(x)u = \alpha(uxu^*) = \alpha \operatorname{Ad} u(x)$. Denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the extensions of α and β to A^{**}. Note that $\tilde{\alpha} - \tilde{\beta} = (\alpha - \beta)^{**}$ and $\|\tilde{\alpha} - \tilde{\beta}\| = \|(\alpha - \beta)^{**}\| = \|\alpha - \beta\| < 2$. See for instance [2, Lemma 1.21]. By the proof of [7, Theorem 4.2], $\alpha(u) = \tilde{\alpha}(u) = u$, so $\alpha \sim \beta$ by part (a).

2.12. THEOREM. *Suppose A is a simple C*-algebra, α and β are involutory antiautomorphisms of A with $\|\alpha - \beta\| < 2$, then $\alpha \sim \beta$.*

PROOF. Since $\|\alpha - \beta\| < 2$, we have $\|\alpha\beta - \iota\| < 2$. By [4], α and β lie on a norm continuous one-parameter group of automorphisms of A, and since A is simple $\alpha\beta = \operatorname{Ad} e^{ih}$ for some $h \in A_{sa}$ by [6, Corollary 4.1.14]. Also, by [4], the spectrum of $u = e^{ih}$ is not all of the unit circle, so we can choose h such that $ih = \log u \in C^*(u)$. Define $v = e^{1/2ih}$. Then $v^2 = u$ and $v \in C^*(u)$. The assertion then follows from proposition 2.11.

2.13. EXAMPLE. The following example shows that we need not have $\alpha \sim \alpha \operatorname{Ad} u$ even if $\alpha(u) = u$. Let $A = C(\mathrm{T}, M_2(\mathrm{C}))$ where T is the unit circle in C. Put $\alpha(f)(\theta) = t_2(f(\theta))$ and put $u(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}$. Then $\alpha(u) = u$, so $\beta = \alpha \operatorname{Ad} u$ is an involutory antiautomorphism.

Choose an arbitrary unitary matrix $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^\beta = \{x \in A : \beta(x) = x^*,\}$. We have

$$\beta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(\theta) = t_2 \circ (\operatorname{Ad} u(\theta))\left(\begin{bmatrix} a(\theta) & b(\theta) \\ c(\theta) & d(\theta) \end{bmatrix}\right)$$
$$= \begin{bmatrix} a(\theta) & c(\theta)e^{-i\theta} \\ b(\theta)e^{i\theta} & d(\theta) \end{bmatrix}$$
$$= \begin{bmatrix} \bar{a}(\theta) & \bar{c}(\theta) \\ \bar{b}(\theta) & \bar{d}(\theta) \end{bmatrix}$$

If $c(\theta) \neq 0$ for every $\theta \in \mathrm{T}$, we get $\bar{c}(\theta)^2/|c(\theta)|^2 = e^{-i\theta}$ so that $(c(\theta)/|c(\theta)|)^2 = e^{-i\theta}$. Since the function $\theta \to e^{-i\theta}$ defined on T has no continuous square root, this is impossible. Thus we must have $c(\theta) = 0$ for some $\theta \in \mathrm{T}$. Since v is unitary we also have $b(\theta) = 0$. It follows that R^β has the following property. There exists projections p_1, p_2 such that for each unitary $f \in R^\beta$ there exists θ such that $f(\theta)$ is a linear combination of $p_1(\theta)$ and $p_2(\theta)$. This property is clearly preserved under isomorphisms, and, since $R^\alpha = C(\mathrm{T}, M_2(\mathrm{R}))$ does not have this property, we conclude that $R^\alpha \not\cong R^\beta$, so that $\alpha \not\sim \beta$. Thus the existence of a square root of u is necessary in proposition 2.11, part (a).

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