MATH. SCAND. 67 (1990), 56–60

ON A SPANNED TAUTOLOGICAL BUNDLE

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Abstract.

In this article we show that if X is a complex projective three dimensional cubic del Pezzo fibre space, then the spannedness of ζ^n for some $n > 0$, where ζ is an associated tautological bundle, implies that ζ itself is spanned by global sections.

Introduction.

In [D] we studied a class of threefolds X, whose hyperplane section S are elliptic surfaces of non-negative Kodaira dimension. These threefolds turned out to be del Pezzo fiber spaces over some smooth curve Y, belonging to the class of threefolds of Kodaira dimension $\kappa(X) = -\infty$. In this article we study a particular class of del Pezzo fiber spaces, namely the fiber space of cubic surfaces in $\mathbf{P}^3$. These surfaces have been studied extensively because of the famous 27 lines on them and their beautiful symmetries. In [D] we showed that the fiber space of cubic surfaces can be relatively embedded into a $\mathbf{P}^3$-bundle over Y. In this article we prove that, in order to show that the tautological bundle ζ of this $\mathbf{P}^3$-bundle over Y is spanned by global sections, it is sufficient to know that some high power of ζ is spanned by global sections.

Notation and background material.

(0.0) Throughout this paper X is an irreducible complex projective threefold. X contains a very ample divisor S, an elliptic surface of non-negative Kodaira dimension. Moreover, X is a del Pezzo fibering (see 0.7) whose general fiber is a cubic surface. The fibering map is denoted by p.

(0.1) Let L be a line bundle over X. We say that L is *big* if $c_1(L)^n > 0$. We say that L is *nef* if $c_1(L)\cdot[C] \geqq 0$, for all effective curves C on X. We say that L is semi-ample if there exists an $m > 0$, such that $Bs\,|mL|$, the base locus of $|mL|$, is empty.

(0.2) Let $X, L)$ be a polarized manifold. A *reduction of* (X, L) *is a polarized manifold* (X', L') such that:

Received March 28, 1988; in revised form August 3, 1989.

a) there exists a morphism π: $X \to X'$ expressing X' as X with a finite F in X' set blown up.

b) $L = \pi^*(L') \otimes [\pi^{-1}(F)]^{-1}$ or equivalently $K_X \otimes L^{n-1} = \pi^*(K_{X'} \otimes L'^{n-1})$

(0.3) We use $\pi_{(i)}$ instead of $R^i\pi_*$, for higher direct image sheaves.

(0.4) We use the multiplicative and the additive notation interchangeably in the tensor powers of line bundles, i.e. nL is the same as $L^{\otimes n}$ (or L^n).

(0.5) We denote the sheaf of holomorphic functions on X, by $\mathcal{O}_X$ or $\mathcal{O}$.

(0.6) A smooth surface F is said to be a *del Pezzo surface* if $-K_F$ is ample. If F is a smooth cubic del Pezzo surface then it is well known that $-kF$ is *very ample*. A quick reference for del Pezzo surfaces is [H; p. 400–401].

(0.7) A del Pezzo fiber space consists of a smooth threefold X, a smooth curve Y and a surjective morphism p: $X \to Y$ whose *generic* fiber $p^{-1}(y) = X_y$ is a del Pezzo surface.

(0.8) Let p: $X \to Y$ be a surjective morphism, where X is a smooth complex projective manifold and Y is a smooth curve. Let $\mathcal{L}$ be a holomorphic line bundle on X. Suppose also that for all $y \in Y$, $\mathcal{L}$ restricted to $p^{-1}(\Delta_y)$ is very ample, where Δ_y is a neighborhood of y, then $\mathcal{L}$ is said to be *locally very ample* with respect to p.

Main results.

(1.0) THEOREM. *Given (X, S) as in* (0.0), *assume moreover that X is not a holomorphic* P^1*-bundle over a smooth surface $\tilde{S}$ with smooth S in $|L|$ as meromorphic sections, then there exists a pair (X', S') which is a reduction of (X, S) in the sense of* (0.2), *where $\pi(S) = S'$ is a minimal model of S.*

PROOF. See [D; (0.6)].

(1.1) REMARKS a) We denote the del Pezzo fibering of this map, also by p: $X' \to Y$, where Y is a smooth curve. See [D; (0.7)].

b) Also by [D; (0.7)] we see that $K_{X'} + L' = p^*(M)$ for some line bundle M on Y, with $\deg(M) > 0$.

(1.2) THEOREM. *Let (X', S') and p be as in* (1.1), *then there exists a morphism φ, where φ: $X' \to \mathrm{P}(p_*(-K_{X'}))$, such that φ is a relative embedding with the following commutative diagram of morphisms*:

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d] & \mathrm{P}(p_*(-K_{X'})) \arrow[dl, "\pi"] \\ Y & \end{tikzcd}$$

where $\mathrm{P}(p_*(-K_{X'}))$ is a P^3*-bundle over Y.*

PROOF. See [D; (2.2.1)].

This brings us to the main result of this article.

(1.3) THEOREM. *Let ζ denote the tautological bundle of* $\mathsf{P}(p_*(-K_{X'}))$. *If for some* $n > 0$, ζ^n *is spanned by global ssections, then* ζ *itself must be spanned.*

PROOF. Let F denote a general fiber of p. Since φ is an embedding on fibers, we can identify F with the image under φ, and so F is contained in the fiber, P^3, of π, and so $\zeta^n \cong (-K_{X'|F})^n = (K_F)^{-n}$. Let η be the map associated with $(K_{X'})^{-n}$. Since $K_{X'}$ is not trivial, the image cannot be zero-dimensional. Now by the adjunction formula $-K_F \cong (-K_{X'} - F)_{|F}$, F being a fiber $F \cdot F = 0$ in X'. Hence by (0.6), $-K_{X'|F}$ is very ample, and so the image under η is at least two dimensional.

Claim 1: Suppose the image under η is two dimensional. The image of X' under η is a smooth cubic surface, and the general fiber of η is an elliptic curve.

PROOF (of claim 1). Since by above $-K_{X'|F} (\cong -K_F)$ is very ample, $\eta(X')$ has to be a cubic surface, if the image is two dimensional. Moreover, if E denotes the general fiber of η then $(-nK_{X'})_{|E}$ is trivial, i.e. by adjunction, $-nK_E$ is trivial. Hence K_E itself is trivial, and so E is an elliptic curve. This proves the claim. Hence $\eta(X') = F$, where F *is a smooth cubic surface.*

Claim 2: $X' \cong Y \times F$.

PROOF (of claim 2). Let q: $X' \to Y \times F$, be given by $q(x) = (p(x), \eta(x))$. Since S is very ample in X, it follows by [D; (1.6.1)] that L' is locally *very ample* with respect to p (see (0.8)). By (1.1) (b) η is a local embedding on every fiber of p. Hence it follows that q is bijective. Moreover q is birational, since the general fibers of p are cubic surfaces; and $Y \times F$ is normal. Hence by [H; 5.1 p. 410], it follows that q^{-1} is also a bijective map. Hence q must be an isomorphism, thereby proving the claim.

Hence $-K_{X'} \cong (-K_F, -K_Y)$. Since the image under η is two-dimensional, it follows that $(-K_Y)^n$ must be trivial. Hence K_Y is trivial and Y is an elliptic curve. Hence $-K_{X'}$ is spanned. Now we repeat the argument as above for $\mathsf{P}(p_*(-K_{X'}))$, and look at $\eta(\mathsf{P}(p_*(-K_{X'})))$, and prove that $\eta(\mathsf{P}(p_*(-K_{X'}))) \cong \mathsf{P}^3$, whence as in a similar situation before, $\mathsf{P}(p_*(-K_{X'})) \cong Y \times \mathsf{P}^3$, with Y an elliptic curve, Hence ζ must be spanned.

Since ζ is the tautological bundle associated with $\mathsf{P}(p_*(-K_X))$, in order to show the completeness of ζ^r for $r \gg 0$, it suffices to show that $H^1(X, \mathcal{O}(-\mathrm{r}\mathrm{K}_X - \mathrm{F})) = 0$.

Assume that there exists some $n > 0$, such that $|-nK_X|$ is base point free. Since L is ample, then for $M \in |-nK_X|$ it follows that $L_{|M}$ is ample. But $L \cdot M = -nK_X \cdot L$. Hence $-K_X \cdot L = -(K_X + L) \cdot L + L \cdot L$ is ample, i.e. $L_S - K_S$ is ample, where $[S] = L$. Now choose $m > 0$, *such that* L^m is very ample. We observe that if F denotes the general fiber of p in (0.7), then $L \cdot F = E$ is the general fiber of p_S. Let $\mathscr{H} \in |(r + 1)L|$, were $r \gg 0$. Now we observe that $-(r + 1)(K_X + L) + K_X - F = -rK_X - F - (r + 1)L$, and consider the following short exact sequence:

$$0 \to -(r+1)(K_X+L)+K_X-F \to -rK_X-F \to$$
$$[-r(r+1)K_X \cdot L-(r+1)E]_{|\mathscr{H}} \to 0$$

Now by [H: III, p. 232, ex. 5.7] and a well know result of Serre (see [H: III, 5.2]) we can choose r large enough so that $H^1(\mathscr{H}, \mathscr{O}_{\mathscr{H}}(-r(r+1)K_X \cdot L-(r+1)E))=0$.

Hence by Serre duality and the Leray spectral sequence: $H^1(X, \mathscr{O}(-(r+1)(K_X+L)+K_X-F))=H^2(X, \mathscr{O}((r+1)(K_X+L)+F))=H^2(\mathrm{Y}, (\mathrm{r}+1)\mathscr{L}+[P])$, where by [D; 0.10] $p^*\mathscr{L}=K_X+L$ and $p^{-1}(P)=F$. Since Y is a curve $H^2(Y, (r+1)L+[P])=0$. Hence from the associated long exact sequence to the short exact sequence, $H^1(X, \mathscr{O}(-rK_X-F))=0$.

Claim 3: Suppose the image under η is three-dimensional. Then the base curve is P^1.

PROOF (of claim 3). Since the image under η is three-dimensional, $-K_{X'}$ is nef and big. Hence by the Kodaira-Ramanujan-Kawamata-Viehweg (KRKV) theorem [K] or [V], it follows that $H^1(X', \mathscr{O}_{X'})=0$. Hence by [D; (1.4.1)], and the Leray spectral sequence, it follows that $H^1(X', \mathscr{O}_{X'}) \cong H^1(Y, \mathscr{O}_Y)$. Hence $H^1(Y, \mathscr{O}_Y)=0$, and since Y is smooth, $Y \cong \mathsf{P}^1$. This proves the claim.

Since for a general cubic surface F, $(K_F)^{-1} \cong \mathscr{O}_{\mathsf{P}^3}(1)$, $\pi_*(\zeta)$ is locally free of rank 4 over P^1, we have $\pi_*(\zeta)=\mathscr{O}(\mathrm{a}) \oplus \mathscr{O}(b) \oplus \mathscr{O}(c) \oplus \mathscr{O}(d)$ for some a, b, c and d in Z. Hence it suffices to show that a, b, c and d are each non-negative. Using the KRKV theorem on $-2K_{X'}$, we see that $H^1(X', -K_{X'})=0$. Now we know that for $i>0$, by a standard result, $\pi_{(i)}(\zeta)=0$. Hence by the Leray spectral sequence $H^1(\mathsf{P}(\mathrm{p}_*(-\mathrm{K}_{X'})), \zeta) \cong H^1(Y, \pi_*(\zeta) \cong H^1(X', -K_{X'})=0$. In particular, we get $H^1(Y, \mathscr{O}(\mathrm{a}))=0$, hence $a \geqq -1$ and similarly each of b, c, and d is $\geqq -1$. Similarly, on considering $-3K_{X'}$, we see that $H^1(X', -2K_{X'})=0$. Hence on noting that $\pi_{(i)}(\zeta^2)=0$, we get $H^1(\mathsf{P}(p_*(-K_{X'})), \zeta^2) \cong H^1(Y, \pi_*(\zeta^2)) \cong H^1(X', -2K_{X'})=0$. Hence $2a \geqq -1$, or $a \geqq 0$ and similarly each of b, c and d is $\geqq 0$. Since $\pi_*(\zeta)=\mathscr{O}(a) \oplus \mathscr{O}(b) \oplus \mathscr{O}(c) \oplus \mathscr{O}(d)$, it follows that $\pi_*(\zeta)$ is spanned, and hence that ζ itself is spanned.

ACKNOWLEDGEMENTS. I would like to thank A. Sommese for bringing this problem to my attention, and for the many helpful discussions. I would also like to thank the referee for the many useful comments in improving the style of presentation.

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