ON A SPANNED TAUTOLOGICAL BUNDLE

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Abstract.

In this article we show that if $X$ is a complex projective three dimensional cubic del Pezzo fibre space, then the spannedness of $\zeta^n$ for some $n > 0$, where $\zeta$ is an associated tautological bundle, implies that $\zeta$ itself is spanned by global sections.

Introduction.

In [D] we studied a class of threefolds $X$, whose hyperplane section $S$ are elliptic surfaces of non-negative Kodaira dimension. These threefolds turned out to be del Pezzo fiber spaces over some smooth curve $Y$, belonging to the class of threefolds of Kodaira dimension $\kappa(X) = -\infty$. In this article we study a particular class of del Pezzo fiber spaces, namely the fiber space of cubic surfaces in $\mathbb{P}^3$. These surfaces have been studied extensively because of the famous 27 lines on them and their beautiful symmetries. In [D] we showed that the fiber space of cubic surfaces can be relatively embedded into a $\mathbb{P}^3$-bundle over $Y$. In this article we prove that, in order to show that the tautological bundle $\zeta$ of this $\mathbb{P}^3$-bundle over $Y$ is spanned by global sections, it is sufficient to know that some high power of $\zeta$ is spanned by global sections.

Notation and background material.

(0.0) Throughout this paper $X$ is an irreducible complex projective threefold. $X$ contains a very ample divisor $S$, an elliptic surface of non-negative Kodaira dimension. Moreover, $X$ is a del Pezzo fibering (see 0.7) whose general fiber is a cubic surface. The fibering map is denoted by $p$.

(0.1) Let $L$ be a line bundle over $X$. We say that $L$ is big if $c_1(L)^n > 0$. We say that $L$ is nef if $c_1(L) \cdot [C] \geq 0$, for all effective curves $C$ on $X$. We say that $L$ is semi-ample if there exists an $m > 0$, such that $Bs|mL|$, the base locus of $|mL|$, is empty.

(0.2) Let $X, L$ be a polarized manifold. A reduction of $(X, L)$ is a polarized manifold $(X', L')$ such that:

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a) there exists a morphism \( \pi: X \to X' \) expressing \( X' \) as \( X \) with a finite \( F \) in \( X' \) set blown up.

b) \( L = \pi^*(L) \otimes [\pi^{-1}(F)]^{-1} \) or equivalently \( K_X \otimes L^{n-1} = \pi^*(K_{X'} \otimes L'^{n-1}) \)

(0.3) We use \( \pi_{(0)} \) instead of \( R^i\pi_* \), for higher direct image sheaves.

(0.4) We use the multiplicative and the additive notation interchangeably in the tensor powers of line bundles, i.e. \( nL \) is the same as \( L^{\otimes n} \) (or \( L^n \)).

(0.5) We denote the sheaf of holomorphic functions on \( X \), by \( O_X \) or \( O \).

(0.6) A smooth surface \( F \) is said to be a del Pezzo surface if \( -K_F \) is ample. If \( F \) is a smooth cubic del Pezzo surface then it is well known that \( -kF \) is very ample.

A quick reference for del Pezzo surfaces is [H; p. 400–401].

(0.7) A del Pezzo fiber space consists of a smooth threefold \( X \), a smooth curve \( Y \) and a surjective morphism \( p: X \to Y \) whose generic fiber \( p^{-1}(y) = X_y \) is a del Pezzo surface.

(0.8) Let \( p: X \to Y \) be a surjective morphism, where \( X \) is a smooth complex projective manifold and \( Y \) is a smooth curve. Let \( L \) be a holomorphic line bundle on \( X \). Suppose also that for all \( y \in Y \), \( L \) restricted to \( p^{-1}(\Delta_y) \) is very ample, where \( \Delta_y \) is a neighborhood of \( y \), then \( L \) is said to be locally very ample with respect to \( p \).

**Main results.**

(1.0) **Theorem.** Given \((X, S)\) as in (0.0), assume moreover that \( X \) is not a holomorphic \( P^1 \)-bundle over a smooth surface \( \tilde{S} \) with smooth \( S \) in \(|L|\) as meromorphic sections, then there exists a pair \((X', S')\) which is a reduction of \((X, S)\) in the sense of (0.2), where \( \pi(S) = S' \) is a minimal model of \( S \).

**Proof.** See [D; (0.6)].

(1.1) **Remarks**

a) We denote the del Pezzo fibering of this map, also by \( p: X' \to Y \), where \( Y \) is a smooth curve. See [D; (0.7)].

b) Also by [D; (0.7)] we see that \( K_{X'} + L' = p^*(M) \) for some line bundle \( M \) on \( Y \), with \( \deg(M) > 0 \).

(1.2) **Theorem.** Let \((X', S')\) and \( p \) be as in (1.1), then there exists a morphism \( \varphi \), where \( \varphi: X' \to P(p_*(-K_{X'})) \), such that \( \varphi \) is a relative embedding with the following commutative diagram of morphisms:

\[
\begin{array}{ccc}
X' & \xrightarrow{\varphi} & P(p_*(-K_{X'})) \\
\downarrow & & \downarrow \pi \\
Y & & \\
\end{array}
\]

where \( P(p_*(-K_{X'})) \) is a \( P^3 \)-bundle over \( Y \).

**Proof.** See [D; (2.2.1)].

This brings us to the main result of this article.
(1.3) **Theorem.** Let $\zeta$ denote the tautological bundle of $P(p_*(-K_X))$. If for some $n > 0$, $\zeta^n$ is spanned by global sections, then $\zeta$ itself must be spanned.

**Proof.** Let $F$ denote a general fiber of $p$. Since $\varphi$ is an embedding on fibers, we can identify $F$ with the image under $\varphi$, and so $F$ is contained in the fiber, $P^3$, of $\pi$, and so $\zeta^n \cong ((-K_{X'}/F))^n = (K_F)^{-n}$. Let $\eta$ be the map associated with $(K_X)^{-n}$. Since $K_{X'}$ is not trivial, the image cannot be zero-dimensional. Now by the adjunction formula $-K_F \cong ((-K_{X'} - F)_{|F}$, $F$ being a fiber $F \cdot F = 0$ in $X'$. Hence by (0.6), $-K_{X'}/F$ is very ample, and so the image under $\eta$ is at least two dimensional.

**Claim 1:** Suppose the image under $\eta$ is two dimensional. The image of $X'$ under $\eta$ is a smooth cubic surface, and the general fiber of $\eta$ is an elliptic curve.

**Proof (of claim 1).** Since by above $-K_{X'}/F (\cong -K_F)$ is very ample, $\eta(X')$ has to be a cubic surface, if the image is two dimensional. Moreover, if $E$ denotes the general fiber of $\eta$ then $(-nK_{X'})_{|E}$ is trivial, i.e. by adjunction, $-nK_E$ is trivial. Hence $K_E$ itself is trivial, and so $E$ is an elliptic curve. This proves the claim. Hence $\eta(X') = F$, where $F$ is a smooth cubic surface.

**Claim 2:** $X' \cong Y \times F$.

**Proof (of claim 2).** Let $q: X' \to Y \times F$, be given by $q(x) = (p(x), \eta(x))$. Since $S$ is very ample in $X$, it follows by [D; (1.6.1)] that $L'$ is locally very ample with respect to $p$ (see (0.8)). By (1.1) (b) $\eta$ is a local embedding on every fiber of $p$. Hence it follows that $q$ is bijective. Moreover $q$ is birational, since the general fibers of $p$ are cubic surfaces; and $Y \times F$ is normal. Hence by [H; 5.1 p. 410], it follows that $q^{-1}$ is also a bijective map. Hence $q$ must be an isomorphism, thereby proving the claim.

Hence $-K_{X'} \cong (-K_F, -K_Y)$. Since the image under $\eta$ is two-dimensional, it follows that $(-K_Y)^n$ must be trivial. Hence $K_Y$ is trivial and $Y$ is an elliptic curve. Hence $-K_{X'}$ is spanned. Now we repeat the argument as above for $P(p_*(-K_X))$, and look at $\eta(P(p_*(-K_{X'})))$, and prove that $\eta(P(p_*(-K_{X'}))) \cong P^3$, whence as in a similar situation before, $P(p_*(-K_{X'})) \cong Y \times P^3$, with $Y$ an elliptic curve, Hence $\zeta$ must be spanned.

Since $\zeta$ is the tautological bundle associated with $P(p_*(-K_X))$, in order to show the completeness of $\zeta$ for $r \gg 0$, it suffices to show that $H^1(X, \mathcal{O}(-rK_X - F)) = 0$.

Assume that there exists some $n > 0$, such that $| -nK_X |$ is base point free. Since $L$ is ample, then for $M \in | -nK_X |$ it follows that $L_{|M}$ is ample. But $L \cdot M = -nK_X \cdot L$. Hence $-K_X \cdot L = -(K_X + L) \cdot L + L \cdot L$ is ample, i.e. $L_{|S} - K_S$ is ample, where $[S] = L$. Now choose $m > 0$, such that $L^n$ is very ample.

We observe that if $F$ denotes the general fiber of $p$ in (0.7), then $L \cdot F = E$ is the general fiber of $p_S$. Let $\mathcal{M} \in |(r + 1)L|$, were $r \gg 0$. Now we observe that $-(r + 1)(K_X + L) + K_X - F = -rK_X - F - (r + 1)L$, and consider the following short exact sequence:
$0 \to -(r + 1)(K_X + L) + K_X - F \to -rK_X - F \to$
\[\left[ -r(r + 1)K_X \cdot L - (r + 1)E \right]_r \to 0\]

Now by [H: III, p. 232, ex. 5.7] and a well known result of Serre (see [H: III, 5.2])
we can choose $r$ large enough so that $H^1(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(-r(r + 1)K_X \cdot L - (r + 1)E)) = 0$.

Hence by Serre duality and the Leray spectral sequence:

\[H^1(X, \mathcal{O}(-(r + 1)(K_X + L) + K_X - F)) = H^2(X, \mathcal{O}((r + 1)(K_X + L) + F)) = H^2(Y, (r + 1)L + [P]),\]
where by [D; 0.10] $p^*L = K_X + L$ and $p^{-1}(P) = F$.
Since $Y$ is a curve $H^2(Y, (r + 1)L + [P]) = 0$. Hence from the associated long exact sequence to the short exact sequence, $H^1(X, \mathcal{O}(-rK_X - F)) = 0$.

Claim 3: Suppose the image under $\eta$ is three-dimensional. Then the base curve is $P^1$.

Proof (of claim 3). Since the image under $\eta$ is three-dimensional, $-K_X$ is nef and big. Hence by the Kodaira-Ramanujan-Kawamata-Viehweg (KRKV) theorem [K] or [V], it follows that $H^1(X', \mathcal{O}_{X'}) = 0$. Hence by [D; (1.4.1)], and the Leray spectral sequence, it follows that $H^1(X', \mathcal{O}_{X'}) \cong H^1(Y, \mathcal{O}_Y)$. Hence $H^1(Y, \mathcal{O}_Y) = 0$, and since $Y$ is smooth, $Y \cong P^1$. This proves the claim.

Since for a general cubic surface $F, (K_F)^{-1} \cong \mathcal{O}_F(1), \pi_*(\zeta)$ is locally free of rank 4 over $P^1$, we have $\pi_*(\zeta) = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \oplus \mathcal{O}(d)$ for some $a, b, c$ and $d$ in $\mathbb{Z}$. Hence it suffices to show that $a, b, c$ and $d$ are each non-negative. Using the KRKV theorem on $-2K_X$, we see that $H^1(X', -K_X) = 0$. Now we know that for $i > 0$, by a standard result, $\pi(i)(\zeta) = 0$. Hence by the Leray spectral sequence $H^1(\mathcal{P}(\pi_*(-K_X)), \zeta) \cong H^1(Y, \pi_*(\zeta)) \cong H^1(X', -K_X) = 0$. In particular, we get $H^1(Y, \mathcal{O}(a)) = 0$, hence $a \geq -1$ and similarly each of $b, c,$ and $d$ is $\geq -1$. Similarly, on considering $-3K_X$, we see that $H^1(X', -2K_X) = 0$. Hence on noting that $\pi(i)(\zeta^2) = 0$, we get $H^1(\mathcal{P}(\pi_*(-K_X)), \zeta^2) \cong H^1(Y, \pi_*(\zeta^2)) \cong H^1(X', -2K_X) = 0$. Hence $2a \geq -1$, or $a \geq 0$ and similarly each of $b, c$ and $d$ is $\geq 0$. Since $\pi_*(\zeta) = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \oplus \mathcal{O}(d)$, it follows that $\pi_*(\zeta)$ is spanned, and hence that $\zeta$ itself is spanned.

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References


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