CANCELLATION OF ABELIAN GROUPS OF FINITE RANK 
MODULO ELEMENTARY EQUIVALENCE

FRANCIS OGER

Introduction.

Following R. Hirshon, we say that a group $A$ may be cancelled in direct products if, for any groups $G, H$, $A \times G \cong A \times H$ implies $G \cong H$. Results, examples and references concerning the cancellation properties of groups can be found in [5] and other papers of R. Hirshon. Finite groups and many other familiar groups may be cancelled. On the other hand, [5] gives some examples of non abelian groups $G, H$ which satisfy $Z \times G \cong Z \times H$ without being isomorphic.

According to [7], if $G$ and $H$ are groups such that $Z \times G \cong Z \times H$, then $G$ and $H$ are elementarily equivalent. By [9], the converse is true for finitely generated finite-by-nilpotent groups. On the other hand, [8, Proposition, p. 1042] gives an example of two polycyclic abelian-by-finite groups $G, H$ which are elementarily equivalent while $Z \times G$ and $Z \times H$ are not isomorphic. According to [4], the result from [7] remains true if we replace $Z$ by any subgroup of $Q^n$ for an integer $n \in \mathbb{N}$.

In the present paper, we give conditions on the abelian groups $A, B$ which imply that, for any groups $G, H$, if $A \times G$ and $B \times H$ are isomorphic, then $G$ and $H$ are elementarily equivalent. In particular, we prove the following result:

Let $A$ be an abelian group such that, for each prime number $p$ and for each subgroup $S$ of $A$, $S/pS$ is finite. If $G$ and $H$ are groups such that $A \times G$ and $A \times H$ are isomorphic, then $G$ and $H$ are elementarily equivalent.

The condition on $A$ is satisfied by any abelian group which is an homomorphic image of a subgroup of $Q^n$ for an integer $n \in \mathbb{N}$. In particular, our result generalizes [7] and [4] (see Examples 2 and 3 below). On the other hand, we have $M^{(\omega)} \times \{1\} \cong M^{(\omega)} \times M$ for each group $M$, while $\{1\}$ and $M$ are elementarily equivalent if and only if $M$ is trivial.

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Definitions and main theorem.

For each integer \( n \geq 1 \), we consider \( \mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z} \). For each prime number \( p \), we denote by \( \hat{\mathbb{Z}}_p \) the \( p \)-adic completion of \( \mathbb{Z} \) and we write \( \mathbb{Z}(p^{\infty}) = \{a/p^k \mid a \in \mathbb{Z} \text{ and } k \in \mathbb{N}\}/\mathbb{Z} \).

If \( S \) is a subset of a group \( M \), we denote by \( \langle S \rangle \) the subgroup of \( M \) which is generated by \( S \). If \( G \) is a subgroup of a group \( M \), we say that a subgroup \( A \) of \( M \) is a supplementary of \( G \) in \( M \) if we have \( A \cap G = \{1\}, [A, G] = \{1\} \) and \( \langle A, G \rangle = M \). Similarly, if \( R \) is a ring and if \( G \) is a submodule of an \( R \)-module \( M \), we say that a submodule \( A \) of \( M \) is a supplementary of \( G \) in \( M \) if we have \( A \cap G = \{0\} \) and \( A + G = M \).

Now, let us consider an abelian group \( M \) with additive notation. We denote by \( t(M) \) the torsion subgroup of \( M \) and \( d(M) \) the subgroup which consists of the elements which are divisible by each integer \( k \geq 1 \). For each prime number \( p \), we write \( M[p] = \{x \in M \mid px = 0\} \); we denote by \( t_p(M) \) the subgroup which consists of the elements \( x \in M \) which satisfy \( p^k x = 0 \) for an integer \( k \geq 1 \) and \( d_p(M) \) the subgroup which consists of the elements which are divisible by each integer \( k \) which is not divisible by \( p \).

For each prime number \( p \) and for each integer \( k \), we consider the following invariants, according to the notations of [2]:

\[
T_f(p; M) = \inf_{h \in \mathbb{N}} \dim (p^h M/p^{h+1} M) \text{ if finite, } \infty \text{ otherwise;}
\]
\[
D(p; M) = \inf_{h \in \mathbb{N}} \dim ((p^h M)[p]) \text{ if finite, } \infty \text{ otherwise;}
\]
\[
U(p, k; M) = \dim (((p^h M)[p])/(p^{h+1} M)[p])) \text{ if finite, } \infty \text{ otherwise.}
\]

For each prime number \( p \), \( \dim (p^h M/p^{h+1} M) \) and \( \dim ((p^h M)[p]) \) are monotonically decreasing functions of \( h \). The invariants \( T_f(p; M), D(p; M) \) and \( U(p, k; M) \) are first-order definable.

Now, we investigate the abelian groups \( M \) which satisfy the following property:

(P) For each subgroup \( S \) of \( M \) and for each prime number \( p \), \( S/pS \) is finite. This condition is equivalent to saying that the torsion-free rank \( r_0(M) \) and all the \( p \)-torsion ranks \( r_p(M) \) are finite, or equivalently, that the injective hull of \( M \) has only finitely many copies of \( \mathbb{Q} \) and of \( \mathbb{Z}(p^{\infty}) \) for each prime number \( p \) (the torsion-free rank and the \( p \)-torsion ranks are defined in [3, §16]; the injective hull is defined in [3, §24]). Finite direct products, subgroups and homomorphic images of abelian groups which satisfy (P) also satisfy (P).

If \( M \) satisfies (P), then, for each subgroup \( S \) of \( M \) and for each integer \( n \geq 1 \), \( S/nS \) is finite. The invariants \( T_f(p; M), D(p; M) \) and \( U(p, k; M) \) are finite, as well as \( \dim (M[p]) = D(p; M) + \sum_{k \in \mathbb{N}} U(p, k; M) \). The subgroup \( d(M) \) is divisible: for each
$x \in d(M)$ and for each integer $n \geq 1$, there are finitely many elements $y \in M$ such that $ny = x$; as $x$ is divisible by $nk$ for each integer $k$, one of these elements is divisible by $k$! for infinitely many integers $k$ and therefore belongs to $d(M)$. We can show in a similar way that $nd_p(M) = d_p(M)$ for each prime number $p$ and for each integer $n$ which is not divisible by $p$. The divisible torsion-free group $d(M)/(d(M) \cap t(M))$ is isomorphic to the additive structure of a vector space over $Q$. We denote by $Q(M)$ the finite dimension of this vector space. The invariant $Q$ is not first-order definable.

We consider the following relation between abelian groups which satisfy (P):

(R) $A$ and $B$ are elementarily equivalent and satisfy $Q(A) = Q(B)$.

It follows from [2, Theorem 2.2 and Theorem 2.6] that $A$ and $B$ satisfy (R) if and only if they satisfy $Q(A) = Q(B)$, $T(p; A) = T(p; B)$, $D(p; A) = D(p; B)$ and $U(p, k; A) = U(p, k; B)$ for each prime number $p$ and for each integer $k \in N$.

Now, we state the main theorem:

**Theorem.** Let $A, B$ be abelian groups such that, for each prime number $p$ and for each subgroup $S$ of $A$ (respectively $B$), $S/pS$ is finite. Let us suppose that $A$ and $B$ are elementarily equivalent and that the divisible torsion-free groups $d(A)/(d(A) \cap t(A))$ and $d(B)/(d(B) \cap t(B))$ have the same dimension over $Q$. If $G$ and $H$ are groups such that $A \times G$ and $B \times H$ are isomorphic, then $G$ and $H$ are elementarily equivalent.

**Remark 1.** We do not suppose that $G$ and $H$ are abelian.

**Remark 2.** We must suppose that $A$ and $B$ satisfy (R) and not only that they are elementarily equivalent. For instance, $A = Q$ and $B = Q \times Q$ are elementarily equivalent by [2, Theorem 2.6] and do not satisfy $Q(A) = Q(B)$; we have $A \times G \cong B \times H$ for $G = Q$ and $H = \{1\}$, but $G$ and $H$ are not elementarily equivalent.

**Example 1.** In [6], B. Jonsson gives an example of a subgroup $A$ of $Q$ and two nonisomorphic subgroups $G, H$ of $Q^2$ such that $A \times G \cong A \times H$.

**Example 2.** If $G$ and $H$ are abelian groups, then $Z \times G \cong Z \times H$ implies $G \cong H$. On the other hand, [5] gives some examples of nonisomorphic finitely generated nilpotent groups $G, H$ which satisfy $Z \times G \cong Z \times H$. By [7], if $G$ and $H$ are groups such that $Z \times G \cong Z \times H$, then $G$ and $H$ are elementarily equivalent. This result is generalized by the theorem above.

**Example 3.** If $M$ is a homomorphic image of a subgroup of $Q^n$ for an integer $n \in N$, then $M$ satisfies (P). In order to prove this result, it suffices to show that $Q$ satisfies (P). As a matter of fact, for each subgroup $S$ of $Q$ and for each prime number $p$, we have $|S/pS| \leq p$ since any finitely generated subgroup of $S$ is generated by one element.

So, the theorem above can be applied for $A, B$ homomorphic images of subgroups of $Q^n$. In particular, it generalizes [4].
Example 4. The abelian group $M = \bigoplus_{p \text{ prime}} \mathbb{Z}(p)^d$ is not a homomorphic image of a subgroup of $\mathbb{Q}^n$ for an integer $n \in \mathbb{N}$. Anyhow, $M$ satisfies (P) since we have $S = \bigoplus_{p \text{ prime}} (S \cap \mathbb{Z}(p)^d)$ for each subgroup $S$ of $M$.

Proof of the theorem.

If $I(M)$ is any of the invariants $U(p, k; M)$, $Tf(p; M)$, $D(p; M)$ and $Q(M)$ defined above, then we have $I(M \times N) = I(M) + I(N)$ for any abelian groups $M, N$ which satisfy (P). Moreover, $I(A)$ and $I(B)$ are finite since $A$ and $B$ satisfy (P) and we have $I(A) = I(B)$ since $A$ and $B$ satisfy (R). If $G$ and $H$ are abelian and satisfy (P), it follows that $I(G) = I(M) - I(A) = I(M) - I(B) = I(H)$. So, $G$ and $H$ satisfy (R) if they satisfy (P). This particular case of the theorem will be used in the proof of lemma 1 below.

We shall prove the following result, which is clearly equivalent to our theorem:

If $M$ is a group, if $A, B, G, H$ are subgroups of $M$, if $A$ and $B$ are abelian and satisfy (P) and (R), if $M = \langle A, G \rangle = \langle B, H \rangle$ and if $A \cap G = [A, G] = B \cap H = [B, H] = \{1\}$, then $G$ and $H$ are elementarily equivalent.

We write $S = \langle A, B \rangle$; in $S$, $A$ is a supplementary of $S \cap G$ and $B$ is a supplementary of $S \cap H$; however, we must not suppose that $S \cap G$ and $S \cap H$ have a common supplementary in $S$, or that $S \cap G \cap H$ has supplementaries in $S \cap G$ and $S \cap H$, or, even, that $S \cap G \cap H$ is a pure subgroup of $S$; on the other hand, we have the following result:

Lemma 1. $S \cap G$ and $S \cap H$ satisfy (P) and (R); moreover, $(S \cap G)/(S \cap G \cap H)$ and $(S \cap H)/(S \cap G \cap H)$ satisfy (P) and (R).

Proof. The groups $S \cap G$ and $S \cap H$ satisfy (P) since they are respectively isomorphic to $B/(A \cap B)$ and $A/(A \cap B)$. The groups $(S \cap G)/(S \cap G \cap H)$ and $(S \cap H)/(S \cap G \cap H)$ also satisfy (P) since they are images of $S \cap G$ and $S \cap H$ respectively.

The groups $S \cap G$ and $S \cap H$ satisfy (R) since their supplementaries $A, B$ satisfy (R). Let us consider $T = \langle S \cap G, S \cap H \rangle$; $T \cap A$ and $T \cap B$ satisfy (R) since, in $T$, $T \cap A$ is a supplementary of $S \cap G$ and $T \cap B$ is a supplementary of $S \cap H$. The groups $(S \cap G)/(S \cap G \cap H)$ and $(S \cap H)/(S \cap G \cap H)$ satisfy (R) since they are respectively isomorphic to $T/(S \cap H) \cong T \cap B$ and $T/(S \cap G) \cong T \cap A$.

We are going to prove that, for each countably incomplete ultrafilter $U$, we have $G^U \cong H^U$; $G$ and $H$ will be elementarily equivalent according to [1, Corollary 4.1.10].

By [1, Theorem 6.1.1], if $U$ is a countably incomplete ultrafilter and if $K$ is a structure associated with a countable language, then $K^U$ is $\omega_1$-saturated. In
particular, if $K$ is an abelian group which satisfies (P), it follows from [2, Theorem 1.11] that we have the following decomposition of $K^U$:

$$
(D) \quad K^U \cong \left[ \prod_{p \text{ prime}} \left( 2^{T_f(p;K)}_p \oplus \left( \bigoplus_{n \geq 1} \mathbb{Z}(p^n)^{U(p,n-1;K)} \right) \right) \right] \oplus \left( \bigoplus_{p \text{ prime}} \mathbb{Z}(p^{\infty})^{D(p;K)} \right) \oplus \mathbb{Q}^{(d)}
$$

with $d = 0$ if $K$ is finite and $d \geq \omega_1$ if $K$ is infinite. As a matter of fact, if $K$ is infinite, we have $d = |K^U| = |N^U|$.

In order to prove the last point, we consider, for each countably incomplete ultrafilter $U$ over a set $I$, a sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint subsets of $I$ which do not belong to $U$ and such that $\bigcup_{n \in \mathbb{N}} A_n = I$. As $K$ is a group of unbounded order, there exists, for each integer $n \geq 1$, an element $x_n \in K$ such that $k x_n \neq 0$ for each $k \in \{1, \ldots, (n!)^3\}$. We define an injection from $[0, 1[ \subset \mathbb{R}$ to $d(\mathbb{K}^U)/(d(\mathbb{K}^U) \cap t(\mathbb{K}^U))$ as follows: For each $u \in [0, 1[$, we consider the sequence of integers $(u_n)_{n \in \mathbb{N}}$ such that $u_n \leq n! \cdot u < u_n + 1$ for each integer $n$, and the element $y(u) \in \mathbb{G}^U$ which admits the system of representatives $(y_i(u))_{i \in I}$ in $G^U$ with $y_i(u) = u_n \cdot x_n$ for each $n \in \mathbb{N}$ and each $i \in A_n$; the element $y(u)$ is divisible in $K^U$ since, for each integer $n$, the elements $y_i(u)$ for $i \in \bigcup_{m \geq n} A_m$ are divisible by $n!$. For any elements $u \neq v$ in $[0, 1[$, we have $n! \cdot (y_i(u) - y_i(v)) \neq 0$ for each integer $n$ such that $n! \cdot |u - v| \geq 1$ and for each $i \in \bigcup_{m \geq n} A_m$, it follows that $y(u) - y(v)$ does not belong to $t(\mathbb{K}^U)$. Consequently, we have $|d(\mathbb{K}^U)/(d(\mathbb{K}^U) \cap t(\mathbb{K}^U))| \geq 2^{\omega}$, which implies $d \geq 2^{\omega}$, and therefore $d = |K^U|$ since $K^U$ satisfies (D) with

$$
\left[ \prod_{p \text{ prime}} \left( 2^{T_f(p;K)}_p \oplus \left( \bigoplus_{n \geq 1} \mathbb{Z}(p^n)^{U(p,n-1;K)} \right) \right) \right] \oplus \left( \bigoplus_{p \text{ prime}} \mathbb{Z}(p^{\infty})^{D(p;K)} \right) \leq 2^{\omega}.
$$

It follows that, if $U$ is a countably incomplete ultrafilter and if $K$, $L$ are abelian groups which satisfy (P) and (R), then $K^U$ and $L^U$ are isomorphic.

It suffices to show that $G^U \cap \langle A^U, B^U \rangle$ and $H^U \cap \langle A^U, B^U \rangle$ have a common supplementary $R$ in $\langle A^U, B^U \rangle$; then, $R$ is also a supplementary of $G^U$ and $H^U$ in $M^U$ and we have $G^U \cong M^U/R \cong H^U$.

We write $S' = S^U = \langle A^U, B^U \rangle$, $A' = A^U$, $B' = B^U$, $G' = (G \cap S)^U = G^U \cap \langle A^U, B^U \rangle$ and $H' = (H \cap S)^U = H^U \cap \langle A^U, B^U \rangle$. In $S'$, $A'$ is a supplementary of $G'$ and $B'$ is a supplementary of $H'$. According to lemma 1 and the definition of $U$, $A'$ is isomorphic to $B'$, $G'$ is isomorphic to $H'$ and $G'/(G' \cap H')$ is isomorphic to $H'/(G' \cap H')$. We must show that $G'$ and $H'$ have a common supplementary in $S'$.

From now on, we only have to consider subgroups of the abelian group $S'$. So, we use the additive notation instead of the multiplicative notation. We write $S_1 = t(d(S')) = t(S') \cap d(S')$ and $S_2 = d(S')$; we define similarly $A_1, A_2, B_1, B_2$ in $B'$, $G_1, G_2$ in $G'$ and $H_1, H_2$ in $H'$. 
We have \(d(S') \cap A' = d(A')\) since \(A'\) has a supplementary in \(S'\). Clearly, we also have \(t(S') \cap A' = t(A')\). It follows that we have \(S_1 \cap A' = A_1\) and \(S_2 \cap A' = A_2\). Similar equalities hold for \(B', G'\) and \(H'\).

According to [3, Theorem 21.2], \(S_1\) has a supplementary in \(S_2\) and \(S_2\) has a supplementary in \(S'\). In the three following sections, we are going to show that:

1) \(G_1\) and \(H_1\) have a common supplementary in \(S_1\);
2) \(\langle G_2, S_1 \rangle / S_1\) and \(\langle H_2, S_1 \rangle / S_1\) have a common supplementary in \(S_2 / S_1\);
3) \(\langle G', S_2 \rangle / S_2\) and \(\langle H', S_2 \rangle / S_2\) have a common supplementary in \(S' / S_2\).

This result implies that \(G'\) and \(H'\) have a common supplementary in \(S'\) according to the following lemma:

**Lemma 2.** Let \(S\) be a group, let \(G, H\) be subgroups of \(S\) and let \(M\) be a subgroup of \(S\) which has a supplementary in \(S\). If \(G \cap M\) and \(H \cap M\) have a common supplementary in \(M\) and if \(\langle G, M \rangle / M\) and \(\langle H, M \rangle / M\) have a common supplementary in \(S / M\), then \(G\) and \(H\) have a common supplementary in \(S\).

**Proof of Lemma 2.** Let \(N\) be a supplementary of \(M\) in \(S\), let \(A\) be a supplementary of \(G \cap M\) and \(H \cap M\) in \(M\) and let \(B\) be a supplementary of \(\langle G, M \rangle / M\) and \(\langle H, M \rangle / M\) in \(S / M\); let us denote by \(C\) the subgroup of \(N\) which consists of the representatives of elements of \(B\) in \(N\).

As the subgroup \(A\) is a supplementary of \(G \cap M\) in \(M\), it is also a supplementary of \(G\) in \(\langle G, M \rangle\). Moreover, \(C\) is a supplementary of \(\langle G, M \rangle\) in \(S\). So, \(\langle A, C \rangle\) is a supplementary of \(G\) in \(S\). We prove in a similar way that \(\langle A, C \rangle\) is a supplementary of \(H\) in \(S\).

\(G_1\) and \(H_1\) have a common supplementary in \(S_1\).

We see from the decomposition (D) that \(S_1\) is isomorphic to \(\bigoplus_{p\text{ prime}} \mathbb{Z}(p^\infty)^{\mathbb{D}(p;S)}\). So, \(G_1\) and \(H_1\) have a common supplementary in \(S_1\) according to the second of the two following lemmas:

**Lemma 3.** Let \(K\) be a field, let \(S\) be a vector space over \(K\) and let \(G, H\) be two subspaces of \(S\). If \(G/\langle G \cap H \rangle\) and \(H/\langle G \cap H \rangle\) are isomorphic, and in particular if \(G\) and \(H\) have the same finite dimension over \(K\), then \(G\) and \(H\) have a common supplementary in \(S\).

**Proof.** Let \(M\) and \(N\) be supernumeraries of \(G \cap H\) in \(G\) and \(H\) respectively; let \(f\) be an isomorphism from \(M\) to \(N\). Then \(A = \{x + f(x) | x \in M\}\) is a common supplementary of \(G\) and \(H\) in \(G + H\). If \(B\) is a supplementary of \(G + H\) in \(S\), then \(A + B\) is a common supplementary of \(G\) and \(H\) in \(S\).

**Lemma 4.** Let \(S\) be a torsion group such that, for each prime number \(p\), \(\{x \in S | px = 0\}\) is finite. Let \(G, H\) be isomorphic subgroups of \(S\) which have supernumeraries in \(S\). Then \(G\) and \(H\) have a common supplementary in \(S\).
PROOF. We have $S = \bigoplus_{p \text{ prime}} t_p(S)$, $G = \bigoplus_{p \text{ prime}} t_p(G)$, $H = \bigoplus_{p \text{ prime}} t_p(H)$ and, for each prime number $p$, $t_p(S) \cap G = t_p(G)$ and $t_p(S) \cap H = t_p(H)$. So, it suffices to show that, for each prime number $p$, $t_p(G)$ and $t_p(H)$, which are isomorphic and have supplementaries in $t_p(S)$, have a common supplementary in $t_p(S)$. Consequently, we can suppose for the remainder of the proof that $S$ is a $p$-torsion group for a prime number $p$.

For each integer $i$, we write $S(i) = \{x \in p^iS \mid px = 0\}$, $G(i) = \{x \in p^iG \mid px = 0\}$ and $H(i) = \{x \in p^iH \mid px = 0\}$; we have $S(i) \cap G = G(i)$ and $S(i) \cap H = H(i)$ since $G$ and $H$ have supplementaries in $S$. We have $S(j) \subseteq S(i)$ for any integers $i < j$. As $S(0) = \{x \in S \mid px = 0\}$ is finite, there exists an integer $n$ such that $S(n) = \bigcap_{i \in \mathbb{N}} S(i)$.

$G(n)$ and $H(n)$ are isomorphic since $G$ and $H$ are isomorphic. So, according to lemma 3, $G(n)$ and $H(n)$ have a common supplementary in $S(n)$. We consider a basis $B$ of this supplementary. We define by induction on $k \geq 1$ a set $B(k)$ which consists of elements of $S$ which are divisible by $p^k$ for each integer $i \geq 1$ as follows: we write $B(1) = B$; for each integer $k \geq 1$ and for each $x \in B(k)$, there are finitely many elements $y \in S$ such that $py = x$; one of these elements is necessarily divisible by $p^k$ for each integer $i \geq 1$ since $x$ is divisible by $p^i$ for each integer $i \geq 1$; we define $B(k+1)$ from $B(k)$ by choosing for each $x \in B(k)$ an element $y \in S$ which is divisible by $p^i$ for each integer $i \geq 1$ and satisfies $py = x$.

For each integer $i \in \{1, \ldots, n\}$, $G(i-1)/G(i)$ and $H(i-1)/H(i)$ are isomorphic since $G$ and $H$ are isomorphic. So, according to lemma 3, $G(i-1)/G(i)$ and $H(i-1)/H(i)$ have a common supplementary in $S(i-1)/S(i)$. We denote by $C(i)$ a system of representatives in $S(i-1)$ of the elements of a basis of this supplementary. We choose for each $x \in C(i)$ an element $y \in S$ such that $p^{i-1}y = x$ and we denote by $D(i)$ the set which consists of these elements $y$.

We are going to prove that the subgroup $K$ of $S$ which is generated by

$$\left( \bigcup_{k \geq 1} B(k) \right) \cup \left( \bigcup_{1 \leq i \leq n} D(i) \right)$$

is a supplementary of $G$ and $H$ in $S$. As a matter of fact, we shall only give the proof for $G$, because the other proof is similar.

It follows from the definition of $K$ that, for each integer $i \in \mathbb{N}$, $K(i) = \{x \in p^iK \mid px = 0\}$ is a supplementary of $G(i)$ in $S(i)$ generated by $B \cup \left( \bigcup_{j > i} C(j) \right)$.

We show by induction on $i \geq 1$ that $K_i = \{x \in K \mid p^ix = 0\}$ is a supplementary of $G_i = \{x \in G \mid p^ix = 0\}$ in $S_i = \{x \in S \mid p^ix = 0\}$. This result is clear for $i = 1$ since we have $K_1 = K(0)$, $G_1 = G(0)$ and $H_1 = H(0)$. Now, we suppose that it is true for some integer $i \geq 1$ and we prove that it is also true for $i + 1$. For each $x \in S_{i+1}$, as $p^ix$ is an element of $S(i) = \langle G(i), K(i) \rangle$, there are two elements $y \in G_{i+1}$ and $z \in K_{i+1}$ such that $p^ix = p^iy + p^iz$. The element $u = x - (y + z)$, which satisfies $p^iu = 0$, belongs to $S_i = \langle G_i, K_i \rangle$ and $x$ belongs to $\langle G_{i+1}, K_{i+1} \rangle$. For
each \( x \in G_{i+1} \cap K_{i+1} \), as \( p^i x \) belongs to \( G_1 \cap K_1 = G(0) \cap K(0) \), we have \( p^i x = 0 \) and \( x \) belongs to \( G_1 \cap K_1 = \{0\} \).

\( \langle G_2, S_1 \rangle / S_1 \) and \( \langle H_2, S_1 \rangle / S_1 \) have a common supplementary in \( S_2 / S_1 \).

We see from the decomposition (D) that \( S_2 \) is isomorphic to

\[
\bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{D(p;S)} \bigoplus \mathbb{Q}^{(\delta)} \text{ with } \delta = 0 \text{ if } S \text{ is finite and } \delta = \lvert K^U \rvert = \lvert N^U \rvert \text{ if } S \text{ is infinite.}
\]

So, \( S_2 / S_1 \) is isomorphic to the additive structure of a vector space over \( \mathbb{Q} \). The subgroups \( \langle G_2, S_1 \rangle / S_1 \) and \( \langle H_2, S_1 \rangle / S_1 \) are subspaces of \( S_2 / S_1 \) since they have supplementaries. So, \( \langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1 \) is also a subspace of \( S_2 / S_1 \).

According to lemma 3, in order to prove that \( \langle G_2, S_1 \rangle / S_1 \) and \( \langle H_2, S_1 \rangle / S_1 \) have a common supplementary in \( S_2 / S_1 \), it suffices to show that the groups

\( \langle \langle G_2, S_1 \rangle / S_1 \rangle / \langle \langle G_2, S_1 \rangle / S_1 \rangle \cap \langle H_2, S_1 \rangle / S_1 \rangle \) and

\( \langle \langle H_2, S_1 \rangle / S_1 \rangle / \langle \langle G_2, S_1 \rangle / S_1 \rangle \cap \langle H_2, S_1 \rangle / S_1 \rangle \) are isomorphic. As a matter of fact, we are going to prove that these groups are respectively isomorphic to \( d(G'/(G' \cap H'))/\langle d(G'/(G' \cap H')) \rangle \) and \( d(H'/(G' \cap H'))/\langle d(H'/(G' \cap H')) \rangle \), which implies that they are isomorphic since \( G'/(G' \cap H') \) and \( H'/(G' \cap H') \) are isomorphic.

We prove this result only for \( \langle \langle G_2, S_1 \rangle / S_1 \rangle / \langle \langle G_2, S_1 \rangle / S_1 \rangle \cap \langle H_2, S_1 \rangle / S_1 \rangle \) since the proof for \( \langle \langle H_2, S_1 \rangle / S_1 \rangle / \langle \langle G_2, S_1 \rangle / S_1 \rangle \cap \langle H_2, S_1 \rangle / S_1 \rangle \) is similar. We have

\[
\langle \langle G_2, S_1 \rangle / S_1 \rangle / \langle \langle G_2, S_1 \rangle / S_1 \rangle \cap \langle H_2, S_1 \rangle / S_1 \rangle \cong \langle G_2, S_1 \rangle / \langle G_2 \cap H_1 \rangle \cong G_2 / (G_2 \cap H_1) = d(G') / d(G') \cap d(H'), t(d(S'))
\]

We denote by \( f \) the canonical surjection from \( G' \) to \( G'/(G' \cap H') \) and we show that \( f \) induces an isomorphism from \( d(G') / d(G') \cap d(H') \) to

\( d(G') / (G' \cap H') \). We observe that \( f(d(G')) = d(G') \). If \( w \) is an element of \( d(G') \) and if \( x \) is a representative of \( w \) in \( G' \), then, for each integer \( n \geq 1 \), there are two elements \( y, z \in G' \) such that \( nz = y \) and \( x - y \in G' \cap H' \). So, the set which consists of the formulas \( \varphi_n(u) = (x - u \in G' \cap H') \wedge (\exists v)(u = nv) \) for \( n \in \mathbb{N}^* \) is consistent, and therefore satisfied, in the \( \omega_1 \)-saturated structure \( G' \cap H' = (G \cap S, G \cap H \cap S)^U \). If \( y \in G' \) satisfies \( \varphi_n \) for each \( n \in \mathbb{N}^* \), then we have \( y \in d(G') \) and \( f(y) = w \).

Now, it suffices to prove that

\[
d(G') \cap f^{-1}(\langle d(G'/(G' \cap H')) \rangle) = d(G') \cap \langle d(H'), t(d(S')) \rangle.
\]

We observe that \( f(d(G') \cap \langle d(H'), t(d(S')) \rangle) \) is contained in \( t(d(G'/(G' \cap H'))) \) since, for each \( x \in \langle d(H'), t(d(S')) \rangle \), there exists an integer \( k \geq 1 \) such that \( kx \) belongs to \( d(H') \), which implies \( kf(x) = f(kx) = 0 \).
Then, we show that an element \( x \in d(G') \) such that \( f(x) \in t(d(G'/(G' \cap H'))) \) necessarily belongs to \( \langle d(H'), t(d(S')) \rangle \). If \( k \geq 1 \) is an integer such that \( kf(x) = 0 \), then \( kx \) belongs to \( G' \cap H' \). As \( x \) is divisible in \( G' \), \( kx \) is divisible in \( S' \), and therefore divisible in \( H' \) since \( H' \) has a supplementary in \( S' \). In \( H' \), there are only finitely many elements \( y \) which satisfy \( ky = kx \), and one of these elements is necessarily divisible. So, there exists an element \( y \in d(H') \) such that \( kx = ky \). The element \( x - y \), which satisfies \( k(x - y) = 0 \), belongs to \( t(d(S')) \) since \( x \) and \( y \) respectively belong to \( d(G') \) and \( d(H') \).

\[
\langle G', S_2 \rangle / S_2 \quad \text{and} \quad \langle H', S_2 \rangle / S_2 \quad \text{have a common supplementary in} \quad S'/S_2.
\]

According to (D) we have \( S'/S_2 \cong \prod_{\text{prime}} (\mathbb{Z}/p^{\nu}; S) \oplus \mathbb{Z}(p^{n}U(p,n-1;S)) \), and therefore \( S'/S_2 \cong \prod_{\text{prime}} d_p(S'/S_2) \); this property is also true for \( G'/G_2 \), \( H'/H_2 \), \( A'/A_2 \) and \( B'/B_2 \). Moreover, we have \( d_p(S'/S_2) \cap (G'/G_2) = d_p(G'/G_2) \) for each prime number \( p \), since \( G'/G_2 \) has a supplementary in \( S'/S_2 \); similar equalities hold for \( H'/H_2 \), \( A'/A_2 \) and \( B'/B_2 \). So, in order to prove that \( G'/G_2 \) and \( H'/H_2 \) have a common supplementary in \( S'/S_2 \), it suffices to show that, for each prime number \( p \), \( d_p(G'/G_2) \) and \( d_p(H'/H_2) \), which respectively are supplementaries of \( d_p(A'/A_2) \) and \( d_p(B'/B_2) \) in \( d_p(S'/S_2) \), have a common supplementary in \( d_p(S'/S_2) \).

For each prime number \( p \), \( t(d_p(S'/S_2)) \) has a supplementary in \( d_p(S'/S_2) \) according to (D). Moreover, we have \( t(d_p(S'/S_2)) \cap d_p(G'/G_2) = t(d_p(G'/G_2)) \) and similar equalities hold for \( H'/H_2 \), \( A'/A_2 \) and \( B'/B_2 \). So, by lemma 2, in order to prove that \( d_p(G'/G_2) \) and \( d_p(H'/H_2) \) have a common supplementary in \( d_p(S'/S_2) \), it suffices to show that:

1) \( t(d_p(G'/G_2)) \) and \( t(d_p(H'/H_2)) \) have a common supplementary in \( t(d_p(S'/S_2)) \); 2) \( d_p(G'/G_2)/t(d_p(G'/G_2)) \) and \( d_p(H'/H_2)/t(d_p(H'/H_2)) \) have a common supplementary in \( d_p(S'/S_2)/t(d_p(S'/S_2)) \).

As \( G' \) and \( H' \) are isomorphic, \( t(d_p(G'/G_2)) \) and \( t(d_p(H'/H_2)) \) are isomorphic. Moreover, \( t(d_p(S'/S_2)) \cong \mathbb{Z}(p^{n}U(p,n-1;S)) \) is a finite \( p \)-torsion group since \( \sum_{n \geq 1} U(p,n-1;S) \) is finite. So, 1) follows from lemma 4.

We have \( d_p(S'/S_2)/t(d_p(S'/S_2)) \cong \mathbb{Z}_p^{\text{TF}(p;S)} \). The subgroups \( d_p(G'/G_2)/t(d_p(G'/G_2)) \) and \( d_p(H'/H_2)/t(d_p(H'/H_2)) \) are closed for the \( p \)-adic topology since they have supplementaries in \( d_p(S'/S_2)/t(d_p(S'/S_2)) \). They are isomorphic since \( G' \) and \( H' \) are isomorphic. So, 2) is a consequence of the following lemma:

**Lemma 5.** Let \( S \) be a free \( \mathbb{Z}_p \)-module of finite dimension and let \( G, H \) be two isomorphic submodules which have supplementaries in \( S \). Then \( G \) and \( H \) have a common supplementary in \( S \).
PROOF. \( \hat{Z}_p \) is a principal ideal domain. The invertible elements of \( \hat{Z}_p \) are the elements which are not divisible by \( p \). The ideals of \( \hat{Z}_p \) are the subgroups \( p^k \hat{Z}_p \) for \( k \in \mathbb{N}^* \).

According to lemma 3, \( G/pG \) and \( H/pH \) have a common supplementary \( M \) in \( S/pS \). We consider a basis \( \{ x_1, \ldots, x_m \} \) of \( M \) and, for each \( i \in \{1, \ldots, m\} \), a representative \( y_i \) of \( x_i \) in \( S \). We denote by \( N \) the submodule of \( S \) which is generated by \( \{y_1, \ldots, y_m\} \). We are going to prove that \( N \) is a supplementary of \( G \) in \( S \). We can show in a similar way that \( N \) is a supplementary of \( H \).

In order to prove that \( N \cap G = \{0\} \), we consider an element \( y = a_1 y_1 + \ldots + a_m y_m \), with \( a_1, \ldots, a_m \in \hat{Z}_p \), which belongs to \( G \), we denote by \( k \) the largest integer such that \( a_1, \ldots, a_m \) belong to \( p^k \hat{Z}_p \) and we write \( a_i = p^k b_i \) for each \( i \in \{1, \ldots, m\} \). As the element \( y = p^k (b_1 y_1 + \ldots + b_m y_m) \) belongs to \( G \), the element \( b_1 y_1 + \ldots + b_m y_m \) also belongs to \( G \), whence a contradiction since at least one of the elements \( b_1, \ldots, b_m \) does not belong to \( p \hat{Z}_p \).

In order to prove that \( S \) is generated by \( G \) and \( N \), it suffices to show that \( S \) is generated by \( G \), \( N \) and \( p^k S \) for each integer \( k \geq 1 \). We consider the smallest integer \( k \geq 1 \) such that \( S \) is not generated by \( G \), \( N \) and \( p^k S \). For each \( y \in S \), there exist elements \( u \in G \), \( v \in N \) and \( z \in S \) such that \( y = u + v + p^k z \); according to the definition of \( N \) there are also elements \( u' \in G \), \( v' \in N \) and \( z' \in S \) such that \( z = u' + v' + p^k z' \); it follows that \( y = (u + p^{k-1} u') + (v + p^{k-1} v') + p^k z' \) with \( u + p^{k-1} u' \in G \) and \( v + p^{k-1} v' \in N \), contrary to the definition of \( k \).

REFERENCES

4. F. Haug, Cancellation and elementary equivalence for torsion-free abelian groups of finite rank, Colloquium on Model Theory, Oberwolfach, West Germany, January 1988.

U.A. 753, DÉPARTEMENT DE MATHEMATIQUES
UNIVERSITÉ PARIS VII
2 PLACE JUSSIEU
75 251 PARIS CÉDEX 05
FRANCE