MATH. SCAND. 66 (1990), 275–290

THE CONVEX FLOATING BODY

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Summary.

We define the convex floating body of a convex body in $\mathbb{R}^n$ and investigate its properties related to the Gauss-Kronecker curvature and the affine surface area. We use a distributional version of Blaschke's rolling theorem.

Preliminaries.

For two vectors $x, y \in \mathbb{R}^n$ we define

$$[x, y] = \{tx + (1-t)y \mid 0 \leqq t \leqq 1\}$$

More generally, the convex hull of two sets A and B is denoted by $[A, B]$.

By $d(z, A)$ we denote the distance of a point z to a set A.

The Euclidean ball in $\mathbb{R}^n$ with center x and radius ρ is $B_2^n(x, \rho)$. Let H be a hyperplane in $\mathbb{R}^n$. The two closed halfspaces determined by H are denoted by H^+ and H^-.

μ is usually the surface measure on the boundary ∂K of a convex body K, the restriction of the $n-1$ dimensional Hausdorff measure to ∂K [Fe].

The affine surface area of a convex body K in $\mathbb{R}^n$ with sufficiently smooth boundary is given by

$$\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \, d\mu$$

where μ is the surface measure on ∂K and $\kappa(x)$ the Gauss-Kronecker curvature. It was verified by Blascke [B] that this expression equals

$$\lim_{\delta \to 0} c_n \frac{\operatorname{vol}_n(K) - \operatorname{vol}_n(K_{[\delta]})}{\delta^{\frac{2}{n+1}}}$$

* supported by NSF Grant DMS 86-02395

Received May 20, 1989.

in the case $n = 3$. Here $K_{[\delta]}$ denotes the floating body introduced by Dupin [D] in 1822. $K_{[\delta]}$ is a floating body of K if each supporting hyperplane of $K_{[\delta]}$ cuts off a set of volume δ from K. If ∂K is sufficiently smooth, then for sufficiently small $\delta > 0$ $K_{[\delta]}$ exists. Leichtweiß [L1] showed that the above expressions are also equal for $n > 3$. Using this Leichtweiß [L2] extended the definition of affine surface area to all convex bodies.

Lutwak [Lu] extended the affine surface area at the same time to all convex bodies in a quite different approach.

Petty gave a definition for the surface area of a certain class of convex bodies [P].

We extend here Leichtweiß's results. We define here the convex floating body K_δ as the intersection of all halfspaces whose hyperplanes cut off a set of volume δ from K. As long as the floating body is convex it coincides with the convex floating body. Let $x \in \partial K$ and $\Delta(x, \delta)$ the height of a slice of volume δ cut off by a hyperplane orthogonal to the normal $N(x)$. Almost all the paper is devoted to prove that

$$\lim_{\delta \to 0} c_n \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = \int_{\partial K} \lim_{\delta \to 0} c_n \frac{\Delta(x, \delta)}{\delta^{\frac{2}{n+1}}} \, d\mu(x)$$

where $c_n = 2\left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))}{n+1}\right)^{\frac{2}{n+1}}$ and $B_2^n(0,1)$ the Euclidean ball of radius 1. The limit under the integral sign equals the $(n+1)$th root of the classical Gauss-Kronecker curvature where ∂K is of the class C^2 and almost everywhere the $(n+1)$th root of the generalized Gauss-Kronecker curvature [Sch]. Both expressions could be used for the definition of the affine surface area for all convex bodies.

The convex floating body.

Let K be a convex body in $\mathbb{R}^n$. The convex floating body K_δ of K is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume δ from the set K. Let A be the set of all (ξ, t), $\xi \in \mathbb{R}^n$, $t \in \mathbb{R}$, so that $\mathrm{vol}_n\{x \in K \mid \langle x, \xi \rangle \geqq t\} = \delta$. Then we have

$$K_\delta = \bigcap_{(\xi,t)\in A} \{x \in \mathbb{R}^n \mid \langle x, \xi \rangle \leqq t\}$$

For $x \in \partial K$, with a unique outer normal $N(x)$, $\|N(x)\|_2 = 1$, we denote the width of the slice of volume δ as $\Delta(x, \delta)$ where

$$\mathrm{vol}_n\{y \in K \mid \langle y, N(x) \rangle \geqq \langle x, N(x) \rangle - \Delta(x, \delta)\} = \delta$$

Also, we require the width of a slice defined by a hyperplane H orthogonal *to* $N(x)$

$$\Delta(x, H) = \langle x, N(x)\rangle - \langle y, N(x)\rangle \quad \text{for } y \in H.$$

We put

$$c_n = 2\left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))}{n+1}\right)^{\frac{2}{n+1}} \qquad n = 2, 3, \ldots$$

THEOREM 1. *Let K be a convex body in* $\mathbb{R}^n$ *and* K_δ, $0 < \delta < \frac{1}{2}\cdot \mathrm{vol}_n(K)$, *its convex floating body. Then we have*

$$\lim_{\delta\to 0} c_n \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = \int_{\partial K} \lim_{\delta\to 0} c_n \frac{\Delta(x,\delta)}{\delta^{\frac{2}{n+1}}}\, d\mu(x)$$

where μ is the surface measure on ∂K. The limit under the integral exists almost everywhere and can be evaluated as in (12), (13).

If K has a C^2 boundary the limit under the integral the $(n + 1)$th root of the Gauss-Kronecker curvature [L2]. By the Hahn-Banach theorem the convex floating body is the same as the floating body [L1] as long as the floating body is convex.

COROLLARY 2. *Let K be a convex body in R^n such that the surface measure of the extreme points of K is zero. Then we have*

$$\lim_{\delta\to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = 0$$

LEMMA 3. *Let K_1 and K_2 be convex bodies in* $\mathbb{R}^n$ *such that* 0 *is an interior point of K_2 and $K_2 \subset K_1$. Then we have*

$$\mathrm{vol}_n(K_1) - \mathrm{vol}_n(K_2) = \frac{1}{n}\int_{\partial K_1} \langle x, N(x)\rangle \left(1 - \left|\frac{\|x'\|_2}{\|x\|_2}\right|^n\right) d\mu(x)$$

where $x \in \partial K_1$, $x' \in \partial K_2$ and $x' \in [0, x]$.

The following is a quantitative version of Blaschke's rolling theorem. It is very close to a result of McMullen [McM].

LEMMA 4. *Let K be a convex body in* $\mathbb{R}^n$ *that contains the Euclidean ball of radius* 1 *and with* 0 *as center and for every $x \in \partial K$ let $r(x)$ be the radius of the Euclidean sphere that is contained in K and that contains x. Then we have*

$$\mathrm{vol}_{n-1}\{x \in \partial K \mid r(x) \geqq t\} \geqq (1-t)^{n-1}\,\mathrm{vol}_{n-1}(\partial K)$$

The inequality is optimal.

LEMMA 5. *We have for all α, $0 < \alpha < 1$,*

$$\int_{\partial K} r(x)^{-\alpha}\, d\mu(x) < \infty$$

where μ is the surface measure on ∂K.

Suppose that 0 is an interior point of the convex floating body K_δ of K. If $x \in \partial K$ then x_δ is the unique element with $x_\delta \in \partial K_\delta$ and $x_\delta \in [0, x]$.

LEMMA 6. *Suppose that 0 is an interior point of the convex body K. There is $\delta_0 > 0$ so that we have for all x with $r(x) > 0$ and all δ with $0 < \delta < \delta_0$*

$$0 \leqq \frac{1}{n} \delta^{-2/(n+1)} \langle x, N(x) \rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right) \leqq C r(x)^{-\frac{n-1}{n+1}}$$

where C does not depend on x and δ.

LEMMA 7. *Suppose that 0 is an interior point of the convex body K. Then we have for almost all $x \in \partial K$*

$$\lim_{\delta \to 0} \frac{1}{n} \delta^{-\frac{2}{n+1}} \langle x, N(x) \rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right) = \lim_{\delta \to 0} \frac{\Delta(x, \delta)}{\delta^{2/(n+1)}} \tag{1}$$

The right hand limit exists almost everywhere [L2, p. 450].

PROOF OF THEOREM 1. We may assume that 0 is an interior point of K_δ. By Lemma 3 we have

$$\frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = \frac{1}{n} \int_{\partial K} \delta^{-\frac{2}{n+1}} \langle x, N(x) \rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right) d\mu(x).$$

By Lemma 5 and 6 the functions under the integral signs are uniformly bounded by L^1 functions and by Lemma 7 they are converging pointwise almost everywhere. We apply Lebesgue's convergence theorem.

The proof of Lemma 3 is similar to the proof of the equality

$$\mathrm{vol}_n(K_1) = \frac{1}{n} \int_{\partial K_1} \langle x, N(x) \rangle \, d\mu(x)$$

PROOF OF LEMMA 4. We consider first the case of a polytope P. Let

$$P(t) = \bigcap_{x \in \partial P} \{ y \in \mathbb{R}^n \mid \langle N(x), y \rangle \leqq \langle N(x), x \rangle - t \}$$

where the intersection is taken only over those x where $N(x)$ exists. We observe that we have for t, $0 \leqq t \leqq 1$, that $(1-t)P \subset P(t)$. Therefore we have that

$$\mathrm{vol}_{n-1}(\partial P(t)) \geqq (1-t)^{n-1}\,\mathrm{vol}_{n-1}(\partial P)$$

Moreover, we have

$$\{x \in \partial P \mid r(x) \geqq t\} \supset \{y + tN(y) \mid y \in \partial P(t) \text{ and } y \text{ has a unique normal}\}$$

This implies that

$$\begin{aligned}&\mathrm{vol}_{n-1}\{x \in \partial P \mid r(x) \geqq t\}\\ &\geqq \mathrm{vol}_{n-1}\{y + tN(y) \mid y \in \partial P(t) \text{ and } y \text{ has a unique normal}\}\\ &= \mathrm{vol}_{n-1}(\partial P(t)) \geqq (1-t)^{n-1}\,\mathrm{vol}_{n-1}(\partial P)\end{aligned}$$

The general case follows by approximating K by polytopes. In the case of the n-dimensional cube B^n_∞ with side length 2 we have

$$\begin{aligned}&\mathrm{vol}_{n-1}\{x \in \partial B^n_\infty \mid r(x) \geqq t\} = 2n(2-2t)^{n-1} = n\,2^n(1-t)^{n-1}\\ &= (1-t)^{n-1}\,\mathrm{vol}_{n-1}(\partial B^n_\infty)\end{aligned}$$

Lemma 5 is an immediate consequence of Lemma 4. Indeed, Lemma 4 implies that

$$\mathrm{vol}_{n-1}\{x \mid r(x)^{-\alpha} \geqq t^{-\alpha}\} \leqq (n-1)t\,\mathrm{vol}_{n-1}(\partial K).$$

LEMMA 8 ([L1 p. 459]). *Let $C(\rho, \Delta)$ be a cap of a sphere with radius ρ and height Δ in $\mathbb{R}^n$. Then there is a continuous function g with $\lim_{t\to 0} g(t) = \sqrt{2}$ so that for $0 < \Delta < \rho$*

$$\mathrm{vol}_n(C(\rho, \Delta)) = g\left(\frac{\Delta}{\rho}\right)^{n+1} \frac{1}{n+1}\,\mathrm{vol}_{n-1}(B_2^{n-1}(0,1))\,\Delta^{\frac{n+1}{2}}\rho^{\frac{n-1}{2}}$$

PROOF OF LEMMA 6. There is $\alpha > 0$ so that

$$B_2^n\left(0, \frac{1}{\alpha}\right) \subset K \subset B_2^n(0, \alpha) \tag{2}$$

We choose δ_0 smaller than $\frac{1}{2}\mathrm{vol}_n\left(B_2^n\left(0, \frac{1}{4\alpha}\right)\right)$. We show first that

$$\begin{aligned}&\frac{1}{n}\delta^{-\frac{2}{n+1}}\langle x, N(x)\rangle\left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right)\\ &\leqq 2\alpha^2\left(\mathrm{vol}_{n-1}\left(B_2^{n-1}\left(0, \frac{1}{2\alpha}\right)\right)\right)^{-\frac{2}{n+1}} \|x - x_\delta\|^{-\frac{n-1}{n+1}}.\end{aligned} \tag{3}$$

Let H be a supporting hyperplane of ∂K_δ at x_δ that cuts off of K a set of volume δ. If H intersects $B_2^n\left(0,\frac{1}{2\alpha}\right)$ then $K\cap H^+$ and $K\cap H^-$ contain both a Euclidean ball of radius $\frac{1}{4\alpha}$. This cannot be since δ is strictly smaller than $\mathrm{vol}_n\left(B_2^n\left(0,\frac{1}{4\alpha}\right)\right)$. Therefore

$$H\cap B_2^n\left(0,\frac{1}{2\alpha}\right)=\emptyset \tag{4}$$

Let C be the cone that is the convex hull of x and $B\left(0,\frac{1}{2\alpha}\right)$ intersected by the hyperplane orthogonal to x and passing through 0. C is contained in K. Since H does not intersect $B_2^n\left(0,\frac{1}{2\alpha}\right)$ we have that $K\cap H^-$ contains $B_2^n\left(0,\frac{1}{2\alpha}\right)$ and $K\cap H^+$ contains the tip of the cone $C\cap H^+$ (or vice versa). Because of (4) the volume of $C\cap H^+$ would be minimal if H were orthogonal to x, i.e., if H were parallel to the base of the cone. Therefore we get

$$\delta=\mathrm{vol}_n(K\cap H^+)\geqq\frac{1}{n}\frac{\|x-x_\delta\|_2^n}{\|x\|_2^{n-1}}\mathrm{vol}_{n-1}\left(B_2^{n-1}\left(0,\frac{1}{2\alpha}\right)\right) \tag{5}$$

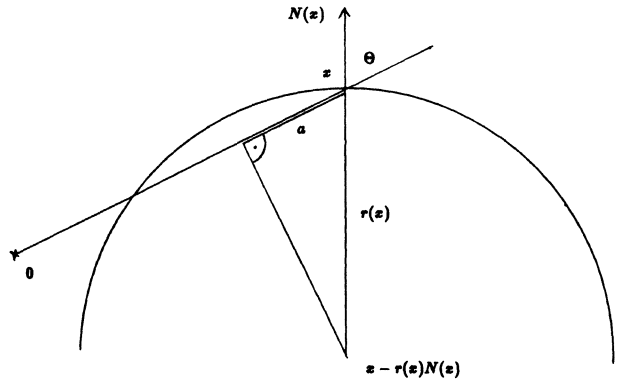
By (5), $\|x\|_2=\|x-x_\delta\|_2+\|x_\delta\|_2$ and $(1-s)^n\geqq 1-ns$ we get

$$\begin{aligned}
&\frac{1}{n}\delta^{-\frac{2}{n+1}}\langle x,N(x)\rangle\left(1-\left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right)\\
&\leqq\frac{1}{n}\delta^{-\frac{2}{n+1}}\|x\|_2\left(1-\left|1-\frac{\|x-x_\delta\|_2}{\|x\|_2}\right|^n\right)\\
&\leqq\delta^{-\frac{2}{n+1}}\|x-x_\delta\|_2\\
&\leqq n^{\frac{2}{n+1}}\left(\mathrm{vol}_{n-1}\left(B_2^{n-1}\left(0,\frac{1}{2\alpha}\right)\right)\right)^{-\frac{2}{n+1}}\|x\|_2^{\frac{2(n-1)}{n+1}}\|x-x_\delta\|_2^{-\frac{n-1}{n+1}}\\
&\leqq 2\alpha^2\left(\mathrm{vol}_{n-1}\left(B_2^{n-1}\left(0,\frac{1}{2\alpha}\right)\right)\right)^{-\frac{2}{n+1}}\|x-x_\delta\|_2^{-\frac{n-1}{n+1}}
\end{aligned}$$

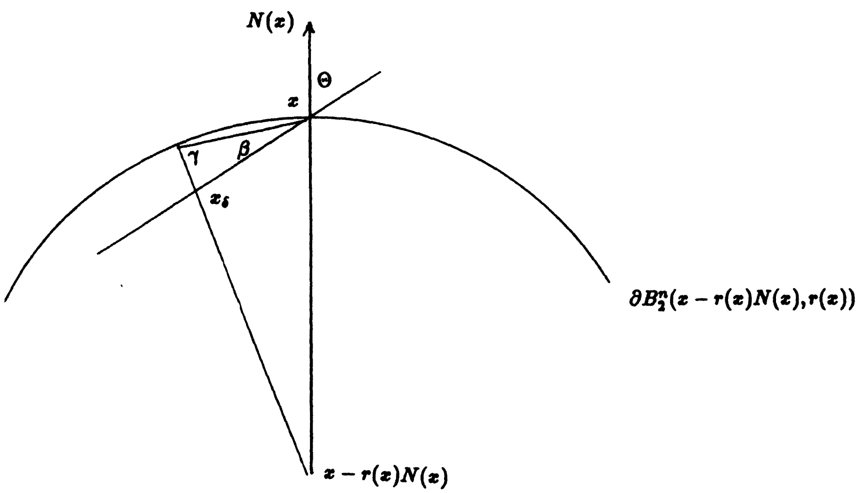
The inequality we just established certainly serves well if $\|x-x_\delta\|_2$ is not smaller than $\alpha^{-2}r(x)$. We consider now the case where $\|x-x_\delta\|_2\leqq\alpha^{-2}r(x)$. Now we observe that

$$\langle x,N(x)\rangle\geqq\frac{1}{\alpha} \tag{6}$$

because of (2) and the convexity of K. If we denote the angle between x and $N(x)$ by Θ we have $\cos\Theta \geqq \alpha^{-2}$. $B_2^n(x - r(x)N(x), r(x))$ is the maximal ball inside K that contains x.



According to the figure $a \geqq r(x)\alpha^{-2}$. Therefore we have for γ and β



that $\gamma + \beta \geqq \dfrac{\pi}{2}$.

According to the figure the distance $\tilde{\Delta}$ of x_δ to the boundary of $B_2^n(x - r(x)N(x), r(x))$ is

$$\tilde{\Delta} = \frac{\sin\beta}{\sin\gamma}\|x - x_\delta\|_2 \geqq \sin\beta\,\|x - x_\delta\|_2.$$

Because of $\gamma + \beta \geqq \frac{\pi}{2}$ and $\beta + \Theta = \gamma$ we have $2\beta \geqq \frac{\pi}{2} - \Theta$. We get

(7) $$\tilde{\Delta} \geqq \sin\left(\frac{\pi}{4} - \frac{\Theta}{2}\right) \|x - x_\delta\|_2$$

with $\alpha^{-2} \leqq \cos\Theta$. Therefore we get

(8) $$\delta = \mathrm{vol}_n(K \cap H^+) \geqq \mathrm{vol}_n(C(r(x), \tilde{\Delta}))$$

Now we get by (7), (8) and Lemma 8

$$\frac{1}{n}\delta^{-\frac{2}{n+1}} \langle x, N(x)\rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right)$$

$$\leqq \delta^{-\frac{2}{n+1}} \|x - x_\delta\|_2$$

$$\leqq 2g\left(\frac{\tilde{\Delta}}{r(x)}\right)^{-2} \mathrm{vol}_{n-1}(B_2^{n-1}(0,1))^{-\frac{2}{n+1}} \sin\left(\frac{\pi}{4} - \frac{\Theta}{2}\right)^{-1} r(x)^{-\frac{n-1}{n+1}}$$

This inequality together with (3) proves Lemma 6

A convex function ϕ on an open subset of R^n is said to be twice differentiable in a generalized sense at x_0 [Ba, L2, Sch] if there is a linear map $d^2\phi(x_0)$ from R^n into itself so that we have for all x in a neighborhood $U(x_0)$ and for all subdifferentials $d\phi(x)$

$$\|d\phi(x) - d\phi(x_0) - d^2\phi(x_0)(x - x_0)\|_2 \leqq c(\|x - x_0\|_2)\,\|x - x_0\|_2$$

where c is a properly chosen function from R^+ into itself with $\lim_{t\to 0} c(t) = 0$. Since a convex function is almost everywhere differentiable [R] we actually deal only with the case that ϕ is differentiable or that the subdifferentials are unique. Let K be a convex body in R^n with $0 \in \partial K$ and $N(0) = (0, \ldots, 0, -1)$ and for sufficiently small values x_n let ∂K be given by

$$x_n = \phi(x_1, \ldots, x_{n-1})$$

We say that the quadratic form

(9) $$(d^2\phi(0)(y_1, \ldots, y_{n-1}))(y_1, \ldots, y_{n-1}) = 1$$

is the indicatrix of Dupin [L2, Sch]. We put

(10) $$M_0 = \{y \in \mathrm{R}^{n-1} \mid (d^2\phi(0)(y))\, y \leqq 1\}$$

LEMMA 9. *Let K be a convex body in* R^n *with* $0 \in \partial K$ *and* $N(0) = (0, \ldots, 0, -1)$. *Let* $H_t = \{x \mid x_n = t\}$.

(i) *Suppose that the indicatrix of Dupin at* 0 *is a* $n - 1$ *dimensional sphere with* *radius* $\sqrt{\rho}$. *Then there is a monotone function* f *on* $\mathbb{R}^+$ *with* $\lim_{t\to 0} f(t) = 1$, *so that we have for sufficiently small* $t > 0$

$$\begin{aligned}&\{(f(t)^{-1}x_1,\ldots,f(t)^{-1}x_{n-1},t)\,|\,x\in B_2^n((0,\ldots,0,\rho),\rho) \text{ and } x_n = t\}\\ &\subset K\cap H_t\\ &\subset \{(f(t)x_1,\ldots,f(t)x_{n-1},t)\,|\,x\in B_2^n((0,\ldots,0,\rho),\rho) \text{ and } x_n = t\}\end{aligned}$$

(ii) *Suppose that the indicatrix of Dupin is a cylinder* $\mathbb{R}^k \times B_2^{n-k-1}(0,\sqrt{\rho})$ *and assume that* $\varepsilon > 0$. *Then we have for sufficiently small* $t > 0$

$$B_2^n((0\ldots,0,\rho-\varepsilon),\rho-\varepsilon)\cap H_t \subset K\cap H_t$$

Lemma 9 is a standard result that is e.g. implicity contained in [L2]. Let us comment briefly on it. As explained in [L2, p. 443 and 451] the surface ∂M_t of

$$M_t = \left\{\frac{1}{\sqrt{2t}}(y_1,\ldots,y_{n-1})\,|\,(y_1,\ldots,y_{n-1},t)\in K\right\} \tag{11}$$

converges in the case (i) uniformly towards the quadratic (9), i.e. there is a function f with $\lim_{t\to 0} f(t) = 1$ and

$$f(t)^{-1}M_t \subset M_0 \subset f(t)M_t$$

If (10) is actually a Eucidean sphere with radius $\sqrt{\rho}$ we get

$$\begin{aligned}K\cap H_t &= \{x\in\mathbb{R}^n\,|\,(x_1,\ldots,x_{n-1})\in\sqrt{2t}\,M_t,\ x_n = t\}\\ &\supset \{x\in\mathbb{R}^n\,|\,(x_1,\ldots,x_{n-1})\in f(t)^{-1}\sqrt{2t}\,M_0,\ x_n = t\}\\ &\supset \{x\in\mathbb{R}^n\,|\,(x_1,\ldots,x_n)\in f(t)^{-1}B_2^n(0,\ldots,0,\rho),\rho),\ x_n = t\}\end{aligned}$$

The other inclusion is shown in the same way.

LEMMA 10 ([L2, p. 450]). *Let* K *be a convex body in* $\mathbb{R}^n$.

(i) *If the indicatrix of Dupin at* x *is a* $n-1$ *dimensional sphere with radius* $\sqrt{\rho}$ *then*

$$\lim_{\delta\to 0} c_n \frac{\Delta(x,\delta)}{\delta^{\frac{2}{n+1}}} = \rho^{-\frac{n-1}{n+1}} \tag{12}$$

(ii) *If the indicatrix at* x *is an elliptic cylinder then*

$$\lim_{\delta\to 0} c_n \frac{\Delta(x,\delta)}{\delta^{\frac{2}{n+1}}} = 0 \tag{13}$$

LEMMA 11. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \partial K$ and $N(0) = (0, \dots, 0, -1)$. Suppose that the indicatrix of Dupin at 0 exists and is a $n-1$ dimensional sphere with radius $\sqrt{\rho}$. Let f be as in Lemma 9 and let ζ be an interior point of K.*

(i) *Let H be the hyperplane orthogonal to $N(0)$ and passing through $z \in [0, \zeta]$. Then we have for $0 \leqq z_n \leqq \rho$*

$$\mathrm{vol}_n(K \cap H^+) \leqq f(z_n)^{n-1} \mathrm{vol}_n(C(\rho, z_n))$$

(ii) *Let $d_0 = d(z, B_2^n((0, \dots, 0, \rho), \rho)^c)$. There is $\varepsilon > 0$ so that we have for all $z \in [0, \zeta]$ with $\|z\|_2 \leqq \varepsilon$*

$$d_0 \leqq z_n \leqq d_0 + \frac{2\, d_0^2}{\rho \left| \left\langle \frac{\zeta}{\|\zeta\|_2}, N(0) \right\rangle \right|^2}$$

(iii) *There is $\varepsilon > 0$ and $c > 0$ so that we have for all $z \in [0, \zeta]$ with $\|z\|_2 \leqq \varepsilon$ and all hyperplanes H passing through z*

$$\mathrm{vol}_n(K \cap H^+) \geqq f(\gamma)^{-n+1} \mathrm{vol}_n(C(\rho, d_0(1 - c(f(\gamma) - 1)))$$

where $\gamma = 4\sqrt{2\rho d_0}$.

PROOF. (i) Let $T_s(x) = (s x_1, \dots, s x_{n-1}, x_n)$. Because of Lemma 9 we have

$$\begin{aligned} \mathrm{vol}_n(K \cap H^+) &= \int_0^{z_n} \mathrm{vol}_{n-1}(K \cap H_t)\, dt \\ &\leqq \int_0^{z_n} \mathrm{vol}_{n-1}(T_{f(t)}(B_2^n((0, \dots, 0, \rho), \rho) \cap H_t)\, dt \\ &\leqq f(z_n)^{n-1} \mathrm{vol}_n(B_2^n((0, \dots, 0, \rho), \rho) \cap H^+) \end{aligned}$$

(ii) Since ζ is an interior point of K we have that $\beta = \left| \left\langle \frac{\zeta}{\|\zeta\|}, N(0) \right\rangle \right| > 0$. Therefore we can choose $\varepsilon > 0$ so small that $[0, z] \subset B_2^n((0, \dots, 0, \rho), \rho)$ and $2 d_0 \rho - d_0^2 < \rho^2 \beta^2$, where d_0 is the distance of z to the boundary of $B_2^n((0, \dots, 0, \rho)$. This means

$$d_0 = \rho - \|z - (0, \dots, 0, \rho)\|_2$$

or

$$|d_0 - \rho|^2 = \sum_{i=1}^{n-1} |z_i|^2 + |z_n - \rho|^2. \tag{14}$$

Also we have

$$z_n = \langle z, (0, \dots, 0, 1) \rangle = \|z\|_2 \left| \left\langle \frac{z}{\|z\|_2}, N(0) \right\rangle \right| = \|z\|_2 \left| \left\langle \frac{\zeta}{\|\zeta\|_2}, N(0) \right\rangle \right|.$$

That is

$$z_n^2(1/\beta^2 - 1) = \sum_{i=1}^{n-1} |z_i|^2.$$

This and (14) give

$$d_0^2 - 2\, d_0 \rho = \left| \frac{z_n}{\beta} \right|^2 - 2\, z_n \rho$$

$$z_n = \rho\beta^2 \left(1 - \left(1 - \frac{2 d_0 \rho - d_0^2}{\rho^2 \beta^2} \right)^{1/2} \right)$$

For $t < 1$ we have $(1 - t)^{1/2} \geqq 1 - \frac{1}{2}t - \frac{1}{2}t^2$. Therefore we get by the choice of ε

$$z_n \leqq \rho\beta^2 \left(\frac{1}{2} \frac{2 d_0 \rho - d_0^2}{\rho^2 \beta^2} + \frac{1}{2} \left(\frac{2 d_0 \rho - d_0^2}{\rho^2 \beta^2} \right)^2 \right)$$

$$\leqq d_0 + 2 \frac{d_0^2}{\rho \beta^2}$$

(iii) Let H be a hyperplane passing through z. Since ζ is an interior of K and $z \in [0, \zeta]$ we can choose ε so small that $K \cap H^+$, the part containing 0, has a smaller volume than $K \cap H^-$. Moreover, we assume that ε is so small that $8\, d_0 < \rho$. We show first that

$$B_2^n((0, \dots, 0, \rho)) \cap T_{1/f(\gamma)}^{-1}(H^+) \cap \{x \in \mathbb{R}^n \mid x_n \geqq \gamma\} = \emptyset \tag{15}$$

if $K \cap H^+$ has the smallest possible volume and if $\varepsilon > 0$ is so small that $f(\gamma) < \dfrac{11}{10}$. Assume that (15) is not true. The cap $B_2^n((0, \dots, 0, \rho), \rho) \cap T_{1/f(\gamma)}^{-1}(H^+)$ contains 0 because H^+ contains 0. Moreover, the radius of this cap is greater than $\gamma/2$ because we assume that (15) is not true. Therefore this cap contains a cap with radius equal to $\gamma/2$ and that also contains 0. This implies that this cap is contained in $\{x \in \mathbb{R}^n \mid x_n \leqq \gamma\}$. By this we get

$$\mathrm{vol}_n(K \cap H^+) \geqq \int_0^\gamma \mathrm{vol}_{n-1}(K \cap H^+ \cap H_t)\, dt$$

$$\geqq \int_0^\gamma \mathrm{vol}_{n-1}(T_{1/f(\gamma)}(B_2^n((0, \dots, 0, \rho)) \cap H^+ \cap H_t)\, dt$$

$$\geqq f(\gamma)^{-n+1} \int_0^\gamma \mathrm{vol}_{n-1}(B_2^n((0,\ldots,0,\rho),\rho) \cap T_{1/f(\gamma)}^{-1}(H^+) \cap H_t)\,dt$$

$$\geqq f(\gamma)^{-n+1}\,\mathrm{vol}_n(C(\rho, 4d_0)).$$

The last inequality follows because $\gamma = 4\sqrt{2\rho d_0}$ and therefore the height of a cap with radius $\gamma/2$ is greater than $4d_0$. Since we assumed that $\mathrm{vol}_n(K \cap H^+)$ is minimal we get a contradiction in view of (i) and (ii). Thus (15) must hold.

Assume now that H passes through z and $\mathrm{vol}_n(K \cap H^+)$ is minimal. Then we get

$$\mathrm{vol}_n(K \cap H^+) \geqq \int_0^\gamma \mathrm{vol}_{n-1}(K \cap H^+ \cap H_t)\,dt$$

$$\geqq \int_0^\gamma \mathrm{vol}_{n-1}(T_{1/f(\gamma)}(B_2^n((0,\ldots,0,\rho),\rho)) \cap H^+ \cap H_t)\,dt$$

$$\geqq f(\gamma)^{-n+1} \int_0^\gamma \mathrm{vol}_{n-1}(B_2^n((0,\ldots,0,\rho),\rho) \cap T_{1/f(\gamma)}^{-1}(H^+) \cap H_t)\,dt$$

By (15) we get

$$\mathrm{vol}_n(\mathrm{K} \cap \mathrm{H}^+) \geqq f(\gamma)^{-n+1}\,\mathrm{vol}_n(B_2^n((0,\ldots,\rho),\rho) \cap T_{1/f(\gamma)}^{-1}(H^+)$$

Since $z \in H$ we have $(f(\gamma)z_1,\ldots,f(\gamma)z_{n-1},z_n) \in T_{1/f(\gamma)}^{-1}(H)$. Therefore the height of the cap $B_2^n((0,\ldots,0,\rho),\rho) \cap T_{1/f(\gamma)}^{-1}(H^+)$ is at least

$$d_0 - (f(\gamma)-1)\left(\sum_{i=1}^{n-1} |z_i|^2\right)^{1/2}$$

As in the proof of part (ii) we see that this expressions equals

$$d_0 - (f(\gamma)-1)\,z_n\left(\left|\left\langle \frac{\zeta}{\|\zeta\|_2}, N(0)\right\rangle\right|^{-2} - 1\right)^{1/2}$$

Because of (ii) this is greater than or equal to

$$d_0\left(1 - (f(\gamma)-1)\left(\left|\left\langle \frac{\zeta}{\|\zeta\|_2}, N(0)\right\rangle\right|^{-2} - 1\right)^{1/2}\left(1 + \frac{2d_0}{\rho\left|\left\langle \frac{\zeta}{\|\zeta\|_2}, N(0)\right\rangle\right|^2}\right)\right)$$

Proof of Lemma 7. Let H be the hyperplane passing through x_δ and being orthogonal to $N(x)$. We shall show that there is a $c > 0$ so that

$$(16)\quad \left(1 - c\frac{\|x - x_\delta\|_2}{\|x\|_2}\right) n \frac{\Delta(x, H)}{\mathrm{vol}_n(K \cap H^+)^{\frac{2}{n+1}}} \leqq \delta^{-\frac{2}{n+1}} \langle x, N(x)\rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right)$$

Every hyperplane that cuts a set of volume δ off the set K has an empty intersection with the interior of K_δ. Therefore, a hyperplane passing through x_δ cuts off a set of volume equal to or greater than δ. Therefore $\mathrm{vol}_n(K \cap H^+)$ is equal to or greater than δ.

On the other hand, since x_δ is a multiple of x

$$\langle x, N(x)\rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right) = \langle x, N(x)\rangle \left(1 - \left|1 - \frac{\|x - x_\delta\|_2}{\|x\|_2}\right|^n\right)$$

$$= n \left\langle \frac{x}{\|x\|_2}, N(x)\right\rangle \|x - x_\delta\|_2 - \binom{n}{2} \left\langle \frac{x}{\|x\|_2}, N(x)\right\rangle \frac{\|x - x_\delta\|_2^2}{\|x\|_2} + \dots$$

$$\geqq n \left\langle \frac{x}{\|x\|_2}, N(x)\right\rangle \|x - x_\delta\|_2 \left(1 - c\frac{\|x - x_\delta\|_2}{\|x\|_2}\right)$$

$$= n\,\Delta(x, H)\left(1 - c\frac{\|x - x_\delta\|_2}{\|x\|_2}\right).$$

Now we consider the inverse inequality. We show that

$$(17)\quad \delta^{-\frac{2}{n+1}} \langle x, N(x)\rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right)$$

$$\leqq f^4(\gamma) \left|\frac{\mathrm{vol}_n\left(C\left(\rho, d_0 + \dfrac{2d_0^2}{\rho\left|\left\langle \frac{x}{\|x\|_2}, N(x)\right\rangle\right|^2}\right)\right)}{\mathrm{vol}_n(C(\rho, d_0(1 - c(f(\gamma) - 1))))}\right|^{\frac{2}{n+1}} n \frac{\Delta(x, H)}{\mathrm{vol}_n(K \cap H^+)^{\frac{2}{n+1}}}$$

provided that the indicatrix of Dupin is an Euclidean sphere with radius $\sqrt{\rho}$. In fact, if the indicatrix of Dupin is an ellipsoid we may assume that it is a Euclidean sphere with radius $\sqrt{\rho}$. To see this we apply first an operator T that is a composition of a translation and rotation with $T(x) = 0$ and $N(T(x)) = N(0) = (0, \dots, 0 - 1)$. Then we apply a linear transform S with $S(0, \dots, 0, 1) = (0, 0, \dots, 0, 1)$ and $\det S = 1$ so that $ST(K)$ has in $S(T(x)) = 0$ an indicatrix which is a Euclidean sphere. We have to show that

$$\langle x, N(x)\rangle \left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right)$$

$$= \langle ST(x) - ST(0), N(ST(x))\rangle \left(1 - \left|\frac{\|(ST(x))_\delta - ST(0)\|_2}{\|ST(x) - ST(0)\|_2}\right|^n\right)$$

where $(ST(x))_\delta \in \partial((ST(K))_\delta)$ lies on the line through $ST(x)$ and $ST(0)$. Since ST is affine and measure preserving we have $(ST(x))_\delta = ST(x_\delta)$. Moreover, since x_δ is a multiple of x the right factor of the left hand expression equals the right factor of the right hand expression. On the other hand,

$$\begin{aligned}
&\langle ST(x) - ST(0), N(ST(x))\rangle \\
&= \langle ST(x) - ST(0), (0, \ldots, 0, -1)\rangle \\
&= \langle T(x) - T(0), (0, \ldots, 0, -1)\rangle \\
&= \langle T(x) - T(0), N(T(x))\rangle \\
&= \langle x, N(x)\rangle
\end{aligned}$$

Therefore we may assume that the indicatrix of Dupin at x is a sphere with radius $\sqrt{\rho}$. Moreover, we may assume that 0 is an interior point. We have

$$\langle x, N(x)\rangle\left(1 - \left|\frac{\|x_\delta\|_2}{\|x\|_2}\right|^n\right) \leqq n\left\langle \frac{x}{\|x\|_2}, N(x)\right\rangle \|x - x_\delta\|_2 = n\,\Delta(x, H)$$

where H is the hyperplane orthogonal to $N(x)$ and passing through x_δ. Moreover, we have because of Lemma 11 (iii)

$$\delta \geqq f^{-n+1}(\gamma)\,\mathrm{vol}_n(C(\rho, d_0(1 - c(f(\gamma) - 1))))$$

From this and Lemma 11 (i) and (ii) the inequality (17) follows. It follows from (16), (17), and Lemma 10 that the left hand of (1) exists and is equal to the right hand if the indicatrix of Dupin exists and is an ellipsoid. Since the indicatrix of Dupin exists almost everywhere [L2] and $r(x) > 0$ (Lemma 4) the indicatrix of Dupin exists almost everywhere and is an ellipsoid or an elliptic cyclinder. It is left to consider the case of the elliptic cylinder.

By Lemma 10 the right hand limit of (1) is 0. We have to show that the left hand limit is also 0. As above we show that the elliptic cylinder can be assumed to be a spherical cylinder, i.e., the product of a k-dimensional plane and a $n - k - 1$ dimensional Euclidean sphere of radius ρ. In fact, by the same argument we can make ρ arbitrarily large. By Lemma 9 (ii) and similar considerations as used for proving (17) we can show that also the left hand limit in (1) equals 0.

We derive now Corollary 2 from Theorem 1. For this we need the following lemma.

LEMMA 12. *Let K be a convex body in $\mathbb{R}^n$ and suppose that $x \in \partial K$ is not an extreme point of K. Assume that $r(x) > 0$ (Lemma 4). Then*

$$\lim_{\delta \to 0} \frac{\Delta(x, \delta)}{\delta^{\frac{2}{n+1}}} = 0$$

PROOF. Since x is not an extreme point of K there are $y, z \in \partial K$ so that $x = \frac{1}{2}(y + z)$. We may assume that $x = 0$, $N(x) = (0, \ldots, 0, -1)$, $y = (\eta, 0, \ldots, 0)$, and $z = (-\eta, 0, \ldots, 0)$. By assumption $B_2^n((0, \ldots, 0, r), r) \subset K$ with $r = r(x)$.

Therefore the convex hull of y and $B_2^n((0, \ldots, 0, r), r)$ is contained in K.

$$[y, B_2^n((0, \ldots, 0, r))] \subset K$$

$$[z, B_2^n((0, \ldots, 0, r))] \subset K$$

It is easy to show that this implies

$$B_2^n\left(\left(\frac{\eta}{2}, 0, \ldots, 0, \frac{r}{2}\right), \frac{r}{2}\right) \subset K$$

$$B_2^n\left(\left(-\frac{\eta}{2}, 0, \ldots, 0, \frac{r}{2}\right), \frac{r}{2}\right) \subset K$$

Thus a hyperplane H orthogonal to $N(x)$ cuts off a set of volume δ from K that contains a cylinder whose base is a $n - 1$ dimensional cap. The rest follows from Lemma 8.

Let $1 \leqq p < \infty$ and $B_p^n = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^p \leqq 1\right\}$.

PROPOSITION 13. *Let $1 < p < \infty$. Then we have*

$$\lim_{\delta \to 0} c_n \frac{\operatorname{vol}_n(B_p^n) - \operatorname{vol}_n((B_p^n)_\delta)}{\delta^{\frac{2}{n+1}}}$$

$$= 2^n (p-1)^{\frac{n-1}{n+1}} p^{-n+1} \left(\Gamma\left(\frac{p+n-1}{p(n+1)}\right)\right)^n \left(\left(n \frac{p+n-1}{p(n+1)}\right)\right)^{-1}$$

PROOF. The Gauss-Kronecker curvature at $x \in \partial B_p^n$, $x > 0$, is

$$(p-1)^{n-1} (\Pi_{i=1}^n x_i^{p-2}) \left(\sum_{i=1}^n x_i^{2p-2}\right)^{-\frac{n+1}{2}}$$

Let $\partial B_p^{n^+} = \partial B_p^n \cap \{x \in \mathbb{R}^n \mid x > 0\}$. By Theorem 1 we get

$$\lim_{\delta \to 0} c_n \frac{\operatorname{vol}_n(B_p^n) - \operatorname{vol}_n((B_p^n)_\delta)}{\delta^{\frac{2}{n+1}}}$$

$$= \int_{\partial B_p^n} \lim_{\delta \to 0} c_n \frac{\Delta(x, \delta)}{\delta^{\frac{2}{n+1}}} d\mu(x)$$

$$= 2^n (p-1)^{\frac{n-1}{n+1}} \int_{\partial B_p^{n^+}} (\Pi_{i=1}^n x_i^{p-2})^{\frac{1}{n+1}} \left(\sum_{i=1}^n x_i^{2p-2}\right)^{-\frac{1}{2}} d\mu(x)$$

Putting $x_n = \left(1 - \sum_{i=1}^{n-1} x_i^p\right)^{1/p}$ the last expression equals

$$2^n(p-1)^{\frac{n-1}{n+1}} \int_{\partial B_p^{n-1+}} (\Pi_{i=1}^{n-1} x_i^{p-2})^{\frac{1}{n+1}} \left(1 - \sum_{i=1}^{n-1} x_i^p\right)^{\frac{n-np-1}{(n+1)p}} dx$$

By [GR, p. 621] we obtain

$$2^n(p-1)^{\frac{n-1}{n+1}} p^{-n+1} \left(\Gamma\left(\frac{\frac{p-2}{n+1}+1}{p}\right)\right)^{n-1} \Gamma\left(1+\frac{n-np-1}{(n+1)p}\right)\left(\Gamma\left(1+\frac{n-np-1}{(n+1)p}\right.\right.$$

$$\left.\left. + (n-1)\left(\frac{\frac{p-2}{n+1}+1}{p}\right)\right)\right)^{-1}$$

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