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THE INTEGRAL WEIGHT SYSTEM FOR TORUS ACTIONS ON SPHERES WITH NO FIXED POINT

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Introduction.

Let G be a compact connected Lie group and T a maximal torus. If φ is a representation of G on R^n, then it is well known that φ is determined by its restriction $\varphi|T$, the latter being completely described by its weight system, $\Omega(\varphi) = \Omega(\varphi|T)$. In the case of acyclic G-manifolds X, one obtains an analogous invariant, $\Omega(X)$, defined to be the weight system of the local representation of T at a fixed point. It is called the *geometric weight system* of (X, G), cf. Hsiang [4]. In a similar manner, if X is a (cohomology) sphere and T has fixed point, then one obtains such an invariant by local linearity, and it is a global invariant (i.e., independent of $x \in F(T)$; a result due to Atiyah and Bott when $F(T) = S^0$).

Now, assume $H^*(X; \mathsf{Z}) = H^*(S^n; \mathsf{Z})$ and $F(T) = \emptyset$. From Borel's formula, [1, p. 182], there is still a "connected" version ($p = 0$) as well as a p-version (p prime), $\Omega_p(X)$, of the geometric weight system. An alternative definition follows from the "splitting principle" in equivariant cohomology, see Hsiang [3]. In $\Omega_0(X)$ the weights are just directions in the weight lattice of T. In the past, efforts have been made to associate length to the weights in $\Omega_0(X)$, in a way which is consistent with local representations of subtori, see e.g. Sullivan [9]. Truly, the existence of such an integral weight system would be very pleasant, as it would simplify a lot of calculations of the orbit structural data, cf. Hsiang [3].

The cohomological method actually gives $\Omega_0(X)$ together with an "integral content" $C \in \mathsf{Z}$. When $F(T) \neq \emptyset$, C is the product of the coefficients of all nonzero weights, when they are expressed as multiples of primitive weights. In the opposite direction, however, one cannot always find a "good" integral weight system, as a refinement of $\Omega_0(X)$, by factorizing C and distributing the factors as coefficients of weights. The reason is that the above mentioned consistency condition may fail. This is actually an interesting phenomenon which explains some of the exotic character of the action in question, see Sec. 4.

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The purpose of this paper is to understand what is the "correct" notion of an integral weight system, $\Omega(X)$, and to establish some basic properties and its limitations. In Sec. 1 we review the definitions of $\Omega_0(X)$ and $\Omega_p(X)$, in Sec. 2–3 we discuss $\Omega(X)$. Applications of geometric weight systems, such as $\Omega_0(X)$, $\Omega_p(X)$ (and to some extent $\Omega(X)$), can be found in the literature, we refer to [4], [5], [7], [8]. In a forthcoming paper, Hsiang, Straume [6], $\Omega(X)$ is needed to its full extent, and is playing the key rôle.

As an illustrative application of $\Omega(X)$ we shall classify all actions of $S^1 \times SO(n)$, $n = 2k + 1$, on spaces $X \sim_Z S^{2n-1}$ with $F(S^1) = \varnothing$. This will also cover a case left out by Wang [10] in his classification of actions of compact connected Lie groups on spheres with one-dimensional orbit space. The calculations are presented in Sec. 4.

1. The rational weight system and the p-weight system.

Let $X \sim_0 S^n$ be a compact cohomology sphere and T a torus acting on X. (We assume differentiability if necessary). The well known Borel formula (cf. [1]) gives

$$\dim X - \dim X^T = \sum_{i=1}^{k} (\dim X^{H_i} - \dim X^T) = \sum m(H_i), \tag{1}$$

where $H_i \subset T$ runs over all corank 1 subtori, and $m(H_i)$ may be called the multiplicity of H_i. The multiset of those H_i contributing to the sum,

$$\Omega'_0(X) = \{H_i; \text{ mult. of } H_i = m(H_i) > 0\},$$

is called the (reduced) *rational weight system* of the T-space X. The "zero weight" (0) is given multiplicity $m_0 = \dim X^T + 1 \geqq 0$, and we write

$$\Omega_0(X) = \Omega'_0(X) + m_0\{0\} \quad \text{(non-reduced)} \tag{2}$$

Similarly, let $T_p \simeq (\mathbf{Z}_p)^r$ be a p-torus, p prime, acting on a space $X \sim_p S^n$. Then there is a Borel formula analogous to (1)

$$\dim X - \dim X^{T_p} = \sum_i (\dim X^{K_i} - \dim X^{T_p}), \tag{3}$$

where $K_i \subset T_p$ runs over subgroups $\simeq (\mathbf{Z}_p)^{r-1}$. Those K_i with multiplicity $m(K_i) = (\dim X^{K_i} - \dim X^{T_p}) > 0$ are called the nonzero p-weights, and the total multiset of p-weights, $\Omega_p(X)$, is the *p-weight system.*

Let T be a torus of rank r and let $T_p \subset T$ denote its p-torus of rank r. Now, if $H \subset T$ is a subtorus of rank $r - 1$, then $H \cap T_p = K$ is a p-torus of rank $r - 1$. Hence there is a well defined restriction map

$$\Omega'_0(X) \to \Omega'_0(X) \mid T_p, \quad H \to H \cap T_p = K \tag{4}$$

between multisets. Here the multiplicity of K is $m(K) = \sum m(H)$, sum over all H such that $H \cap T_p = K$.

LEMMA 1.1. $\Omega'_p(X) \subset \Omega'_0(X) \mid T_p$ *(inclusion of multisets), if* $X \sim_p S^n$.

Proof. (i) Suppose $K \subset T_p$ is a corank 1 p-torus which is *not* of type $H_i \cap T_p$, $H_i \in \Omega'_0(X)$. We shall show $K \notin \Omega'_p(X)$.

By assumption, $KH_i = T_pH_i$, $\forall\ H_i \in \Omega'_0(X)$, and Borel's formula for the action of T on X^K gives

$$\dim X^K - \dim X^T = \sum [\dim (X^K)^{H_i} - \dim X^T],$$

where we can assume $H_i \in \Omega'_0(X)$, since otherwise $X^{H_i} = X^T$ and hence $(X^K)^{H_i} = (X^{H_i})^K = X^T$. But $(X^K)^{H_i} = X^{KH_i} = (X^{T_p})^{H_i}$ and so

$$\begin{aligned}\dim X^K - \dim X^T &= \sum [\dim (X^{T_p})^{H_i} - \dim (X^{T_p})^T] \\ &\leqq [\dim X^{T_p} - \dim X^T]\end{aligned}$$

This implies $\dim X^K = \dim X^{T_p}$, so $X^K = X^{T_p}$ and K is *not* a p-weight.

(ii) Let $K \in \Omega'_p(X)$, and let H_i, $i = 1, 2, \dots, q$, be those $H \in \Omega'_0(X)$ satisfying $H_i \cap T_p = K$. We show $m(K) \leqq \sum_{i=1}^{q} m(H_i)$:

Apply Borel's formula as follows. First, the T-action on X^K gives

$$\begin{aligned}\dim X^K - \dim X^T &= \sum_{i \leqq q} [\dim X^{H_i} - \dim X^T] \\ &\quad + \sum_{i > q} [\dim (X^K)^{H_i} - \dim X^T] \\ &= \sum_{i \leqq q} [\dim X^{H_i} - \dim X^T] + \sum_{i > q} [\dim (X^{T_p})^{H_i} - \dim X^T]\end{aligned}$$

Next, the T-action on X^{T_p} gives

$$\dim X^{T_p} - \dim X^T = \sum_j [\dim (X^{T_p})^{H_j} - \dim X^T].$$

Combining the two formulas we find

$$\begin{aligned}m(K) = (\dim X^K - \dim X^{T_p}) &= \sum_{i \leqq q} (\dim X^{H_i} - \dim X^{H_i T_p}) \\ &\leqq \sum_{i \leqq q} [\dim X^{H_i} - \dim X^T] = \sum_{i \leqq q} m(H_i).\end{aligned} \tag{5}$$

REMARK 1.2. If we had $X^{T_p} = \varnothing$ then $\Omega_p(X) = \Omega_0(X) \mid T_p$.

Let T_p be a torus ($p = 0$) or p-torus (p prime). If $T_p' \subset T_p$ is a p-subtorus (or torus, $p = 0$), then we can calculate from $\Omega_p(X)$ its restriction to T_p', $\Omega_p(X) \mid T_p'$ similar to (4). Another application of the Borel formula gives the following identity

(6) $$\Omega_p(X \mid T_p') = \Omega_p(X) \mid T_p',$$

where $(X \mid T_p')$ means X regarded as T_p'-space.

2. On the definition of the integral weight system $\Omega'(X)$.

The weight lattice of the torus T is

$$\Gamma(T) \simeq H^1(T; \mathbf{Z}) \simeq H^2(B_T; \mathbf{Z}) = \mathbf{Z}[t_1, t_2, \ldots, t_r].$$

Let Ω be a multiset of weight pairs $(\pm\omega_i)$, each of multiplicity m_i, and possibly the zero weight of multiplicity m_0, say,

$$\Omega = m_1\{\pm\omega_1\} + m_2\{\pm\omega_2\} + \ldots + m_k\{\pm\omega_k\} + m_0\{0\}.$$

The restriction of Ω, $\Omega \mid T'$, where $T' \subset T$, is the "projection" of Ω, as multiset, into the weight lattice $\Gamma(T')$. For example, the multiplicity of $\{0\}$ in $\Omega \mid T'$ is $m_0 + 2\sum m_j$, where j runs over indices such that $\omega_j \mid T' = 0$.

We shall also regard the rational weight system $\Omega_0(X)$ of a T-space $X \sim S^n$ as an integral weight collection

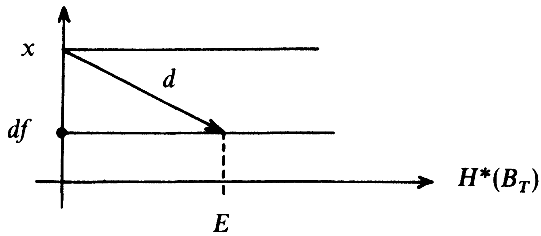
$$\Omega_0 = m_1\{\pm\bar{\omega}_1\} + m_2\{\pm\bar{\omega}_2\} + \ldots + m_k\{\pm\bar{\omega}_k\} + m_0\{0\},$$

where $(\pm\bar{\omega}_i)$ is the unique pair of *primitive weights* such that $\bar{\omega}_i^{\perp} = H_i$, and $m_i = \frac{1}{2}(\dim X^{H_i} - \dim X^T)$ is its multiplicity.

We shall arrive at an appropriate definition of integral weight system by following the cohomological description of $\Omega_0'(X)$ (or $\Omega_p'(X)$). Recall the Borel construction of a fibration $X \to X_T \to B_T$, $X_T = E_T \times_T X$, and consider the Leray-Serre spectral sequence of

$$(X, X^T) \to (X_T, B_T \times X^T) \to B_T,$$

where $X \sim_0 S^n$, $X^T \sim_0 S^r (r \geqq -1)$. Let $x \in H^n(X) = H^n(X, X^T)$, $df \in H^{r+1}(X, X^T)$ be generators (rational cohomology). The E_2-term $H^*(B_T) \otimes H^*(X, X^T)$ has two lines:



, the transgression of x is

$$dx = E_T(X) \otimes df \in H^{n-r}(B_T) \otimes H^{r+1}(X, X^T).$$

By the well known splitting principle (cf. [3]) the so-called Euler class splits as follows

$$E_T(X) = \prod_{i=1}^{k} (\bar{\omega}_i)^{m_i} \text{ (mod constant factor)},$$

and so we recover $\Omega'_0(X) = \sum m_i\{\pm\bar{\omega}_i\}$.

Now, assume $X \sim_{\mathbf{Z}} S^n$ and take cohomology with $\mathbf{Z}$ as coefficients in the above spectral sequence. Then we obtain the (equivariant) Euler class

$$E_T(X) = \pm C_X \cdot \prod_{i=1}^{k} (\bar{\omega}_i)^{m_i}, \quad C_X \in \mathbf{Z}^+,$$

and we shall call C_X the *integral content*. It contains information beyond $\Omega'_0(X)$. In fact, we can do better, since C_X has a canonical splitting into k factors, as will be explained below.

Regard the spaces X^{H_i}, $H_i = \bar{\omega}_i^{\perp}$, as T-spaces, and observe that $\Omega'_0(X^{H_i}) = m_i\{\pm\bar{\omega}_i\}$, using Borel's formula. Therefore $E_T(X^{H_i}) = \pm C_i(\bar{\omega}_i)^{m_i}$ for some $C_i \in \mathbf{Z}^+$, and in Section 3 it will be seen that $C_X = \prod C_i$.

DEFINITION 2.1. Assume $X \sim_{\mathbf{Z}} S^n$ is a T-space. The (reduced) *integral weight system*, denoted $\Omega'(X)$, consists of its components, each component is an Euler class, namely

$$\Omega'(X) = \{E_T(X^{H_i})\}_{i=1}^{k} = \{\pm C_i(\bar{\omega}_i)^{m_i}\}.$$

In particular, X^{H_i} regarded as T-space, has only one (nonzero) component

$$\Omega'(X^{H_i}) = E_T(X^{H_i}) = \pm C_i(\bar{\omega}_i)^{m_i}.$$

$C_i \in \mathbf{Z}^+$ is the *integral content* of the i-th component $\Omega'(X^{H_i})$ of $\Omega'(X)$.

(In general, let $m_0 = \dim X^T + 1$. Then $\Omega(X)$ is $\Omega'(X)$ plus a "zero-component" $m_0\{0\}$. This is the full (or unreduced) weight system. In particular, $\Omega'(X) = \Omega(X)$ iff $X^T = \varnothing$.)

REMARK 2.2. If T acts linearly on $X = S^n \subset \mathbf{R}^{n+1}$ via the representation ϕ or if T acts smoothly on $X \sim S^n$ with $X^T \neq \emptyset$ and $\phi = \phi_x$ is the local representation at $x \in X^T$, then

$$E_T(X) = \pm \prod \omega_i,$$

where $\Omega'(\phi) = \{\pm \omega_i\}$ is the (reduced) weight system of ϕ.

REMARK 2.3. Let $X \sim_Z S^n$ be a G-space, and G is a compact Lie group with maximal torus T and Weyl group W. Then W acts on $H^*(B_T)$ and it is easily seen that W permutes the classes $\pm E_T(X^{H_i})$, that is, the integral weight system is W-invariant.

We now turn to the special case where T has fixed point. In view of Remark 2.2 the following definition is consistent with Definition 2.1, and is clearly a "refinement" of the latter.

DEFINITION 2.4. Assume the action of T on $X \sim S^n$ is differentiable, and $X^T \sim S^{d-1} \neq \emptyset$. Let ϕ_x be the local representation of T at a fixed point x, and denote its weight system by $\Omega(\phi_x)$. Then the collection

$$\Omega(X) = \Omega(\phi_x) + \{0\} = \Omega'(\phi_x) + d\{0\}$$

is defined to be the *integral weight system.*

Let X be a G-space as above. Then $\Omega_0'(X) = \sum_j m_j\{\pm\bar{\omega}_j)$ splits into W-orbits

$$\Omega_0'(X) = \mu_1 \Lambda_1 + \mu_2 \Lambda_2 + \ldots + \mu_q \Lambda_q, \tag{7}$$

where the multiplicity μ_l is the common multiplicity m_j of all $(\pm\bar{\omega}_j) \in \Lambda_l$. Suppose now $X^T \neq \emptyset$, and the action of G is differentiable. Then the integral weight system $\Omega'(X) = \Omega'(\phi_x)$, where ϕ_x is the local representation of T at some $x \in X^T$, is a "refinement" of $\Omega_0'(X)$ which is clearly W-invariant (as multiset in the weight lattice $\Gamma(T)$), namely

$$\Omega'(X) = \delta_1 \Sigma_1 + \delta_2 \Sigma_2 + \ldots + \delta_s \Sigma_s \quad (s \geqq q), \tag{8}$$

where $\Sigma_i \subset \Gamma(T)$ is a W-orbit of pairs $(\pm\omega)$.

If $T \subset G_x$, G_x is an isotropy group, then the following identity holds

$$\Omega'(X \mid T) = \Omega'(i_x \mid T) + \Omega'(\mathscr{S}_x \mid T) = \Delta'(G \mid T) - \Delta'(G_x \mid T) + \Omega'(\mathscr{S}_x) \tag{9}$$

where i_x is the isotropy representation of G/G_x, $\Delta'(G \mid T) = \Omega'(\mathrm{Ad}_G \mid T)$ is the nonzero root system of G (with respect to T) and $\mathscr{S}_x$ is the slice representation of G_x. There is an analogous formula for p-weights and p-roots, $p = 0$ or prime. Application of formula (9) is demonstrated in and Sec. 4.

Clearly, one obtains $\Omega'_0(X)$ from $\Omega'(X)$ by identifying weights which are colinear, and $\Omega'(X)$ is a "refinement" of $\Omega'_0(X)$, both in Definition 2.1 and 2.4. In the case of 2.1, where $F(T)$ is empty, one may wonder if it is possible to do something better, say, by somehow associate appropriate length to m_i weight pairs parallel to $(\pm \bar{\omega}_i)$. Then the resulting multiset $\bar{\Omega}$ in the weight lattice could be regarded as the integral weight system $\Omega(X)$. Problems related to choosing such a collection, such as the validity of (8), (9), are postponed until § 5.

3. Further analysis of $\Omega(X)$ and equivariant Euler classes.

Consider the family of p-groups $T_{p,\alpha} \subset T$, $\alpha \geqq 1$, where $T_{p,\alpha} = (\mathbf{Z}_{p^\alpha})^r$, $T = (S^1)^r$. The following result will be useful.

THEOREM 3.1. (Golber [2]). *Let $X \sim_{\mathbf{Z}} S^n$ be a T-space (assume differentiability to avoid technical conditions). Let $C_X \in \mathbf{Z}^+$ be the integral content of the Euler class $E_T(X)$, and for a fixed prime p write $C_X = p^{e_p} \cdot C'$, where $p \nmid C'$. Then*

$$\sum_{\alpha \geqq 1} [\dim X^{T_{p,\alpha}} - \dim X^T] = 2e_p. \tag{10}$$

THEOREM 3.2. *In the above situation, let*

$$\Omega'(X) = \{E_T(X^{H_i})\}_{i=1}^k$$

be the integral weight system (cf. Def. 2.1). *Then*

$$\pm C_X \prod (\bar{\omega}_i)^{m_i} = E_T(X) = \prod_{i=1}^k E_T(X^{H_i}),$$

namely, $C_X = \prod_{i=1}^k C_i$, where C_X, C_i is the integral content of $E_T(X)$, $E_T(X^{H_i})$, respectively.

PROOF. Write $C_i = p^{e_{p,i}} \cdot C'_i$, where $p \nmid C'_i$ is a fixed prime. Borel's formula for the T-action on $X^{T_{p,\alpha}}$ gives

$$\begin{aligned} \dim X^{T_{p,\alpha}} - \dim X^T &= \sum_i [\dim (X^{T_{p,\alpha}})^{H_i} - \dim X^T] \\ &= \sum_i [\dim (X^{H_i})^{T_{p,\alpha}} - \dim X^T] \end{aligned}$$

Golber's formula for the T-action on X^{H_i} gives

$$\sum_\alpha [\dim (X^{H_i})^{T_{p,\alpha}} - \dim X^T] = 2e_{p,i},$$

hence

$$\sum_i \sum_\alpha [\dim (X^{H_i})^{T_{p,\alpha}} - \dim X^T] = 2\sum e_{p,i} = \sum_\alpha [\dim X^{T_{p,\alpha}} - \dim X^T]$$

Now, comparing with Golber's formula for the T-action on X, cf. (10), we get $e_p = \sum e_{p,\alpha}$. Consequently $C_X = \prod C_i$.

In order to study the behavior of the integral weight system with respect to restriction to subtori $T' \subset T$, we shall first look more closely at (equivariant) Euler classes. Suppose $X^T \subset Y \subsetneqq X$, where $Y \sim_{\mathbf{Z}} S^q$ is T-invariant. In the spectral sequence of

$$(X, Y) \to (X_T, Y_T) \to B_T,$$

the transgression of a generator $x \in H^n(X, Y; \mathbf{Z})$ is $dx = E_T(X, Y) \otimes dy \in H^{n-q}(B_T) \otimes H^{q+1}(X, Y)$. This gives the Euler class $E_T(X, Y) \in H^{n-q}(B_T; \mathbf{Z})$; it is uniquely determined up to sign. By definition, $E_T(X, X^T) = E_T(X)$.

Proposition 3.3. *Let $X^T \subset Z \subset Y \subset X$ be T-invariant $\mathbf{Z}$-cohomology spheres. Then*

$$E_T(X, Z) = E_T(X, Y) \cdot E_T(Y, Z).$$

Proof. Consider the three spectral sequences of the pairs

$$(X_T, Y_T) \supset (X_T, Z_T) \supset (Y_T, Z_T)$$

With the previous notation we may write

$$\begin{aligned} dx' &= E_T(X, Y) \otimes dy' \\ dx &= E_T(X, Z) \otimes dz \\ dy &= E_T(Y, Z) \otimes dz' \end{aligned} \tag{11}$$

By the naturality of the above construction, induced mappings and (geometric) interpretation of transgression, we may identify x and x', y and y', z and z', and then

$$\begin{aligned} dx' = E_T(X, Y) \otimes dy' &= E_T(X, Y) \cdot [E_T(Y, Z) \otimes dz'] \\ &= E_T(X, Y) \cdot E_T(Y, Z) \otimes dz'. \end{aligned}$$

Comparison with the second equation in (11) gives the product formula (essentially!).

Corollary 3.4. *If $X^T \subset X^{T'} \subset X$, $T' \subset T$, then*

$$E_T(X) = E_T(X^{T'}) \cdot E_T(X, X^{T'}).$$

Suppose $T' \subset H_j$ for $j \leqq l$, and $T' \not\subset H_j$ for $j > l$. By Theorem 3.2

$$E_T(X^{T'}) = \prod_{j \leqq l} E_T(X^{H_j}),$$

so by Corollary 3.4

$$E_T(X, X^{T'}) = \prod_{j > l} E_T(X^{H_j}).$$

Let $i^*: H^*(B_T) \to H^*(B_{T'})$ be the map induced by the inclusion $T' \xrightarrow{i} T$. It follows that

$$E_{T'}(X) = E_{T'}(X, X^{T'}) = i^*(E_T(X, X^{T'})) = \prod_{j > l} \pm C_j(i^* \bar{\omega}_j)^{m_j}$$

Let H'_j, $1 \leqq j \leqq q$, be the different rational weights of the T'-action on X; these are the corank 1 subtori of T' of type $(H_t \cap T')^0$, $t > l$. We also know $E_{T'}(X)$ splits into the product of all $E_{T'}(X^{H'_j})$, by Theorem 3.2.

COROLLARY 3.5. *Let $T' \xrightarrow{i} T$, $X \sim_Z S^n$ a T-space. Then the integral weight system of the T'-space X*

$$\Omega'(X \mid T') = \{E_{T'}(X^{H'_j})\}_{j=1}^{q}$$

can be calculated from $\Omega'(X) = \{E_T(X^{H_i})\}_{i=1}^{k}$ by

$$E_{T'}(X^{H'_j}) = \prod_t i^*(E_T(X^{H_t}) = \pm \prod_t C_t(i^* \bar{\omega}_t)^{m_t},$$

where t runs over indices such that $(H_t \cap T')^0 = H'_j$, for each fixed j.

COROLLARY 3.6. *Suppose all $m_i = 1$, and hence we may write*

$$\Omega(X) = \{\pm\omega_1, \pm\omega_2, \ldots, \pm\omega_k\} + m_0\{0\}, m_0 \geqq 0.$$

If $\omega_i \mid T'$ and $\omega_j \mid T'$ are colinear when $i \neq j$, then by Corollary 2

$$\Omega(X \mid T') = \Omega(X) \mid T'.$$

4. Exotic actions on spheres of cohomogeneity one.

In this section we shall illustrate the usefulness of the integral weight system by studying some actions on spheres with 1-dimensional orbit space. The actions considered in Theorem 4.1 will also lead to more insight with regard to the integral weight system. We also refer to § 5.

Consider the following orthogonal representation of $G = S^1 \times SO(n)$, $n = 2k + 1 \geqq 3$,

$$\phi = [u_1^2]_{\mathbf{R}} \oplus [\mu_1^l \otimes_{\mathbf{C}} \rho_n^{\mathbf{C}}]_{\mathbf{R}}: \mathbf{C} \oplus \mathbf{C}^n = \mathbf{R}^{2n+2} \supset S^{2n+1}.$$

Let $(\pm\theta)$ be the unit weights for S^1, and $\Omega(\rho_n) = \{\pm\tau_i; i \leqq k\} + \{0\}$, so

$$\Omega(S^{2n+1}) = \Omega(\phi) = \{\pm 2\theta\} + \{\pm l\theta\} + \{\pm(l\theta \pm \tau_i)\}.$$

Consider the Brieskorn variety $\Sigma^{2n-1} \subset S^{2n+1}$ defined by

$$\Sigma^{2n-1}\colon \begin{array}{l} z_0^l + z_1^2 + \ldots + z_n^2 = 0 \\ |z_0|^2 + |z_1|^2 + \ldots + |z_n|^2 = 1. \end{array} \tag{12}$$

It is clearly G-invariant, and as is well known, $\Sigma^{2n-1} \sim_Z S^{2n-1}$ when l is odd. (This does not hold if l is even, see proof of 4.1.) It is easily seen that $\Omega_0(\Sigma^{2n-1}) = \{\pm(l\theta \pm \tau_i)\} + \{\pm\theta\}$, and consequently

$$\Omega(\Sigma^{2n-1}) = \{\pm q(l\theta \pm \tau_i)\} + (\pm d\theta\}, \text{ cf. (8).} \tag{13}$$

By restriction to $SO(n)$ we first find $q = 1$. Let $\Sigma^1 = (\Sigma^{2n-1})^H \approx S^1$, $H = \theta^\perp \subset SO(n)$. Then $\Omega(\Sigma^1) = \{\pm d\theta\}$ and

$$\Sigma^1\colon \begin{array}{l} z_1 = z_2 = \ldots = z_{n-1} = 0 \\ z_0^l + z_n^2 = 0, \quad \Sigma|z_i|^2 = 1. \end{array}$$

Let $Z_p \subset S^1 = T/H$, and observe that $(\Sigma^1)^{Z_p} = \emptyset$, so $p \nmid d$ and hence $d = 1$,

$$\Omega(\Sigma^{2n-1}) = \{\pm\theta\} + \{\pm(l\theta \pm \tau_i)\}. \tag{14}$$

Further analysis of this action will appear in Example 5.12.

In the remainder of this section we shall give a detailed proof of the following theorem.

THEOREM 4.1. *Let $G = S^1 \times SO(n)$, $n = 2k + 1 \geqq 3$, and let $X \sim_Z S^{2n-1}$ be a faithful smooth G-manifold, and assume $X^{S^1} = \varnothing$. Then*

(i) *The integral weight system is*

$$\Omega(X) = \Omega(\Sigma^{2n-1}), \text{ as in (14),} \tag{15}$$

l is odd, and X is homeomorphic to S^{2n-1}.

(ii) *X can be imbedded G-equivariantly*

$$X \xrightarrow{\approx} \Sigma^{2n-1} \subset S^{2n+1} \subset R^{2n+2},$$

where (G, R^{2n+2}) is the linear group described above, and Σ^{2n-1} is the Brieskorn variety, (12). [*In particular, X is the Kervaire sphere if $l \equiv \pm 3 \pmod 8$ and $bP_{2n} \simeq Z_2$* (i.e., *the Kervaire invariant is trivial*), *and X is the standard sphere otherwise.*]

REMARK 4.2. (i) The theorem also gives that the Brieskorn variety Σ^{2n-1} is not an integral cohomology sphere when l is even (well known, of course!).

(ii) The major part of the proof consists of establishing (15). Knowing $\Omega(X)$ we can determine precisely the isotropy types. Moreover, $\dim X/G = 1$ (X/G is an interval), and the associated triple of isotropy groups, $(K_1 \supset H \subset K_2)$, is uniquely determined modulo simultaneous conjugation in G. It follows that X and Σ^{2n-1} are isomorphic as smooth G-manifolds.

First of all, since $F(S^1) = \varnothing$, every rational weight is of type $\pm(s\theta + \omega)$, where ω is a SO(n)-weight, possibly $\omega = 0$. By Weyl group invariance, and since $\sharp\,\Omega(X) = 2n$, it follows easily that

$$(16) \qquad \Omega(X) = \{\pm q(l\theta \pm m\tau_i)\} + \{\pm d\theta\}, \text{ cf. (8)}$$

for suitable positive integers, q, l, m, d.

$T = S^1 \times T^k \subset S^1 \times \mathrm{SO}(2k) \subset G$ is the maximal torus in question, and

$$\Omega(X)\,|\,\mathrm{SO}(n) = \Omega(X)\,|\,T^k = \{\pm(q^2m^2)(\tau_i)^2\}_{i=1}^k + 2\{0\},$$

i.e., $(q^2m^2) = ab$ is the integral content of $\{\pm a\tau_i, \pm b\tau_i\}$ in the local representation of T^k ($F(T^k) \simeq S^1$). If $n \geqq 5$ then it is not too difficult to see, e.g., by (9), that $\Omega_0(X\,|\,\mathrm{SO}(n)) = \Omega_0(2\rho_n)$ implies $\Omega(X\,|\,\mathrm{SO}(n)) = \Omega(2\rho_n)$. If $n = 3$ then the SO(3)-action has weight system $\{\pm a\tau, \pm b\tau\} + 2\{0\}$, and we can show $a = b = 1$ or $a = 1, b = 2$ (cf. [6]). In any case, $q = m = 1$ in (16), so

$$(17) \qquad \Omega(X) = \{\pm(l\theta \pm \tau_i)\} + \{\pm d\theta\}, \ (d, l) = 1.$$

To see why $(d, l) = 1$, observe that $\Omega_p(X)$ is calculable from $\Omega(X)$ (Lemma 5.4, Prop. 5.5), showing that $\mathbf{Z}_p \subset S^1$ would act trivially if $p\,|\,(d, l)$, cf. also Prop. 5.8.

Let $T_1 = (l\theta - \tau_1)^\perp \subset G_{x_1} = K_1$. K_1^0 is calculated precisely as in Example 5.11, using the corresponding formula (35). Now, the only multiple of $(\pm\theta)$ in $\Delta(G)\,|\,T_1$ is $(\pm\tau_1) \equiv (\pm l\theta)$, and $\Delta(K_1)$ cannot contain $\{\pm l\theta)$ since $\mathrm{SO}(n) \not\subset K_1$. Consequently, $\{(\pm l\theta), (\pm 2d\theta)\} \subset \Omega(X\,|\,T_1)$ and hence also by (34) (with X instead of Σ^{2n-1})

$$(18) \qquad \Omega(\mathscr{S}_1) = \{\pm 2d\theta\}, \ (\mathscr{S}_1 = \text{slice repr. at } x_1).$$

LEMMA A. *K_1 is connected.*

Proof.

$$\begin{array}{l} 1 \to K_1^0 \to K_1 \to E_1 \to 0 \\ \quad\ \ \ \cap\!| \\ \quad\ NK_1^0 = S^1 \times \mathrm{SO}(2) \times \mathrm{SO}(n-2). \end{array}$$

Suppose $\mathbf{Z}_p \subset E_1$ with p prime. Then $K_1 \supset T_p = \mathbf{Z}_p \times T_p^k$, where $(Z_p)^k \simeq T_p^k \subset T^k$. Therefore $X^{T_p} \simeq S^1$ and $p\,|\,d$ by (17). On the other hand, by (9), (17)

$$(19) \qquad X^{T_p} \subset X^{T_p^k} = X^{T^k} = X^{\mathrm{SO}(2k)}.$$

Consequently, $X^{K_1} \subset X^{\mathrm{SO}(2k)}$ and $K_1 \supset \mathrm{SO}\,(2k)$, and this is a contradiction.

The slice representation of K_1 is $[\mu_1^{2d}]_{\mathbb{R}}$, cf. (18), and therefore the principal isotropy group is

$$(20)\qquad H = \left\{((\xi), \begin{pmatrix} \xi^l & 0 \\ 0 & \mathrm{SO}(n-2) \end{pmatrix}); \xi^{2d} = 1\right\} \simeq \mathbf{Z}_{2d} \times \mathrm{SO}(n-2).$$

We note that $\mathbf{Z}_{2d}$ sits "diagonally", namely, $\xi^l \neq 1$ if $\xi = \exp(2\pi i/2d)$, except in the case $d = 1$ and l even, cf. (17).

Next, let $K_2 \supset H$ be the other isotropy group of singular type; we can assume $\theta^\perp = T^k \subset K_2$, by the "Torus Algorithm" (cf. [5]), and (19) implies $K_2^0 = \mathrm{SO}(2k)$,

$$(21)\qquad \begin{array}{c} 1 \to \mathrm{SO}(2k) \to K_2 \to E_2 \to 1 \\ \cap\vert \\ \mathrm{NSO}(2k) = S^1 \times S[0(2k) \times 0(1)]. \end{array}$$

Now,

$$H \subset K_2 \Rightarrow [\exists h \in \mathbf{Z}^+\colon\ 2d \mid h, K_2 \supset \mathbf{Z}_h \times \mathrm{SO}(2k) \subset S^1 \times \mathrm{SO}(2k)].$$

The slice representation $\mathscr{S}_2$ of K_2 has restriction $\mathscr{S}_2 \mid \mathrm{SO}(2k) = \rho_{2k}$, and $\mathscr{S}_2$, having cohomogeneity one, is also irreducible

$$(22)\qquad \begin{cases} \mathscr{S}_2 \mid \mathbf{Z}_h \times \mathrm{SO}(2k) = \varphi_1 \otimes_{\mathbf{R}} \rho_{2k}, \dim \varphi_1 = 1 \\ 1 \to \mathbf{Z}_{h/2} \to \mathbf{Z}_h \xrightarrow{\varphi_1} \mathbf{Z}_2 \to 1 \end{cases}$$

Suppose $p \mid d$ with p an odd prime. We see from $\mathscr{S}_2$, (22), that this does not give the "diagonal" group $\mathbf{Z}_{2d} \subset H$. Hence, the only possibility is $d = 2^s$ ($s \geqq 0$).

LEMMA B. $d = 1$.

PROOF. Consider the 2-weight system $\Omega_2(X)$, with respect to the maximal 2-torus of G

$$(23)\qquad S = Z_2 \times S' = \mathbf{Z}_2 \times \mathbf{Z}_2^{n-1} = \left\{(\varepsilon, \begin{pmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ & & \ddots \\ 0 & & \varepsilon_n \end{pmatrix}); \varepsilon_1 \varepsilon_i = \pm 1\right\}.$$

Since $\Omega(X \mid \mathrm{SO}(n)) = \Omega(2\rho_n)$, we can show (see e.g. [7])

$$\Omega_2(X \mid \mathrm{SO}(n)) = \Omega_2(2\rho_n) = 2\{\varepsilon_i; 1 \leqq i \leqq n\}.$$

If we write

$$\Omega_2(X) = a\{\varepsilon\} + b\{\varepsilon\varepsilon_i\} + c\{\varepsilon_i\} + \delta(0)$$

then restriction to $\mathrm{SO}(n)$ implies $b + c = 2$, $a + \delta = 0$, so

$$(24)\qquad \Omega_2(X) = b\{\varepsilon\varepsilon_i\} + c\{\varepsilon_i\},\ b + c = 2.$$

Moreover, we can show $d > 1 \Rightarrow b \geqq 1$ (by considering the "nonplitting" isotropy groups H and K_1). From (24)

$$X^S = X^{S'} = \varnothing.$$

Consider the 2-torus $S'' = \mathbf{Z}_2 \times T_2^k \subset S^1 \times T^k \subset S^1 \times \mathrm{SO}(2k)$. It is given by $\varepsilon_1 = \varepsilon_2, \varepsilon_3 = \varepsilon_4, \ldots, \varepsilon_{2k-1} = \varepsilon_{2k}, \varepsilon_n = 1$. We can calculate $\dim X^{S''}$ in two days:

$$\text{From } \Omega(X)\text{: } \dim X^{S''} = \begin{cases} 1, & d > 1 \\ -1, & d = 1 \end{cases}$$

$$\text{From } \Omega_2(X)\text{: } \dim X^{S''} = \begin{cases} -1, & b = 2 \\ 0, & b = 1 \\ 1, & b = 0 \end{cases}$$

As observed above, $b = 0$ is impossible if $d > 1$, so the only possibility is $d = 1$, $b = 2$.

Suppose l were even. Then $H \cap S' = \mathbf{Z}_2$ acts trivially. This is impossible, so l must be odd.

Finally, it is easily seen that E_2 in (21) is a 2-group, and in fact $E_2 \simeq \mathbf{Z}_2$. Assuming $H \subset K_2$ we have more precisely

$$K_2 = \left\{\left(\varepsilon, \begin{pmatrix} \varepsilon & 0 \\ 0 & O(n-1) \end{pmatrix}\right)\right\} \simeq S[O(1) \times O(n-1)] \simeq O(n-1).$$

This completes the proof of Theorem 4.1, cf. Remark 4.2.

REMARKS 4.3. (a) In the case of $n = 7$, Theorem 4.1 also holds when $SO(7)$ is replaced by the subgroup G_2.

(b) The above actions are exotic (i.e., nonlinear) actions on homotopy spheres of cohomogeneity one. A complete discussion will appear in [6]. Wang [10] claims that such actions on spheres are of linear type, assuming $\dim X = N > 31$ if N is odd, or $N > 4$ and is even. However, this is not true, and the above cases with $G = S^1 \times SO(n)$ or $S^1 \times G_2$ are missing in [10].

5. Integral liftings and the consistency of $\Omega(X)$.

Let Γ be a multiset consisting of k different pairs of primitive weights $\pm \bar{\omega}_i$ of multiplicity m_i, and suppose each $\pm \bar{\omega}_i$ is associated with some integer $C_i > 0$. Then we write

$$\Gamma = \{\pm C_1(\bar{\omega}_1)^{m_1} + \ldots \pm C_k(\bar{\omega}_k)^{m_k}\}. \tag{25}$$

C_i is called the *integral content* along $\pm \bar{\omega}_i$, and $C = \Pi C_i$ is the *total* integral content. If we choose m_i integers c_{ij} with product C_i, and replace $\pm C_i(\bar{\omega}_i)^{m_i}$ in (25) by $\{\pm c_{i1}\bar{\omega}_i, \pm c_{i2}\bar{\omega}_i, \ldots\}$, then we obtain a multiset $\bar{\Gamma}$ in the weight lattice, and $\bar{\Gamma}$ is called an *integral lifting* of Γ.

Suppose T acts on X with $X^T = \emptyset$, with $\Omega'(X) = \Omega(X)$ defined as in Definition 2.1. Then $\Omega(X)$ is an object of type (25), with total integral content C_X. By definition, we also request that an integral lifting $\bar{\Omega}$ of $\Omega(X)$ must be W-invariant; in particular $\bar{\Omega}$ may be presented as in (8). In the special case where each $m_i = 1$, $\bar{\Omega}$ is clearly unique and we may write $\bar{\Omega} = \Omega(X)$ without ambiguity. (In fact, we did so in §4.)

Now we turn to the following central problem.

PROBLEM. Suppose $X \sim_Z S^n$ is a T-space with $X^T = \emptyset$. Is it possible to obtain a suitable integral lifting $\bar{\Omega}$ of $\Omega(X)$ containing additional information about the orbit structure, similar to those of $\Omega'(X)$ in the case $X^T \neq \emptyset$?

REMARK 5.1. It turns out that in general there is no unique "best possible" candidate $\bar{\Omega}(X)$ of integral lifting of $\Omega(X)$. In fact, even when it is unique it may still fail to fulfill certain consistency conditions which are known to hold if $X^T \neq \emptyset$, see below.

Let $\Omega(X) = \{\pm\omega_1, \pm\omega_2, \ldots\} + m_0\{0\}$ be the weight system of an orthogonal action on $\mathsf{R}^{n+1} \supset S^n = X$. Then all representations $\phi \mid K$, $K \subset T$ can be calculated. In particular, $\forall K \subset T$

$$\dim X^K = \#\{(\pm\omega_j);\ \omega_j \mid K = 0\} + (m_0 - 1). \tag{26}$$

Clearly, if $X \sim S^n$ is a differentiable T-space and $X^T \neq \emptyset$, then the same formula (26) also holds (at least locally at $x \in X^T$).

Henceforth, assume $X \sim_Z S^n$, $X^T = \emptyset$ (i.e., $m_0 = 0$). Among the various integral liftings $\bar{\Omega}$ of $\Omega(X)$ we hope to find a "good" candidate from which invariants like $\dim X^K$ can be calculated. So, we say $\dim X^K$ is *calculable* from $\bar{\Omega}$ if formula (26) holds. The question is whether such a $\bar{\Omega}$ actually exists.

DEFINITION 5.2. Let $\bar{\Omega}$ be an integral lifting of $\Omega(X)$.

(i) $\bar{\Omega}$ is *0-consistent* if for each subtorus $T' \subset T$ with $X^{T'} \neq \emptyset$, $\Omega(X \mid T') = \bar{\Omega} \mid T'$.

(ii) $\bar{\Omega}$ is *p-consistent* if all numbers $\dim (X^{H_i})^{T_{p,\alpha}}$, $\alpha \geqq 1$, $H_i = \bar{\omega}_i^{\perp}$ a rational weight, are calculable from $\bar{\Omega}$.

(iii) $\bar{\Omega}$ is *consistent* if all numbers $\dim X^K$ are calculable from $\bar{\Omega}$ whenever K/K^0 has prime power order, $K \subset T$ a closed subgroup.

Write $\bar{\Omega} = \bar{\Omega}(X)$, and for $T' \subset T$ a subtorus, define

$$\bar{\Omega}(X^{T'}) = \{(\pm\omega_i) \in \bar{\Omega}(X);\ \omega_i \mid T' = 0\}. \tag{27}$$

Clearly, $\bar{\Omega}(X^{T'})$ is an integral lifting of $\Omega(X^{T'})$.

LEMMA 5.3. (i) If $\bar{\Omega} = \bar{\Omega}(X)$ is 0-consistent, then $\bar{\Omega}(X^{T'})$ is also 0-consistent.

(ii) $\bar{\Omega}(X)$ is 0-consistent if and only if $\Omega(X \mid H_i) = \bar{\Omega}(X) \mid H_i$ for each rational weight H_i.

PROOF. (i) Let $T'' \subset T$ be another torus and assume $(X^{T'})^{T''} \neq \emptyset$. Then $S = T' \cdot T'' \subset T$ is a torus and $X^S \neq \emptyset$. Hence, if $\bar{\Omega}$ is 0-consistent, then $\Omega(X \mid S) = \bar{\Omega} \mid S$, and from the local representation of S

$$\begin{aligned}\Omega(X^{T'} \mid T'') &= \Omega(X^{T'} \mid S) \mid T'' = \bar{\Omega}(X^{T'}) \mid S \mid T'' \\ &= \bar{\Omega}(X^{T'}) \mid T''.\end{aligned}$$

(ii) This is clear since $X^{T'} \neq \emptyset$ implies $T' \subseteq H_i$ for some i.

LEMMA 5.4. *If* $\bar{\Omega} = \bar{\Omega}(X)$ *is* 0-*consistent, then* $\Omega_p(X)$ *is calculable from* $\bar{\Omega}$, *namely*

$$\Omega_p(X) = \bar{\Omega}(X) \mid T_p, \ \forall p (T_p \subset T).$$

In particular, $\dim X^{T_p}$ *is calculable from* $\bar{\Omega}$.

PROOF. By Lemma 1.1 each p-weight $K_i \subset T_p$ is a group of type $H_j \cap T_p$ for some rational weight H_j. Therefore $\dim X^{K_i}$ is calculable from $\bar{\Omega} \mid H_j = \Omega(X \mid H_j)$. Borel's formula gives

$$\dim X - \dim X^{T_p} = \sum_j (\dim X^{H_j \cap T_p} - \dim X^{T_p}),$$

and this gives $\dim X^{T_p}$ as well.

Now we turn to p-consistency (p fixed prime).

LEMMA 5.5. *Suppose* $\bar{\Omega} = \bar{\Omega}(X)$ *is p-consistent. Then* $\Omega_p(X)$ *is calculable from* $\bar{\Omega}$, *namely*

$$\Omega_p(X) = \bar{\Omega}(X) \mid T_p.$$

PROOF. There is a decomposition

$$\bar{\Omega}(X) = \sum_{i=1}^{k} \bar{\Omega}(X^{H_i}) = \sum_{i=1}^{k} \sum_{\alpha \geqq 0} P_{i,\alpha},$$

where $P_{i,\alpha} = \{(\pm\omega) \in \bar{\Omega}(X^{H_i}); \ \omega \text{ is divisible by } p^\alpha, \text{ but not by } p^{\alpha+1}\}$, and we put $p_{i,\alpha} = (\frac{1}{2}) \cdot \# P_{i,\alpha}$ (number of pairs).

(i) We show $\dim X^{T_p}$ is calculable from $\bar{\Omega}$: We claim

$$\dim X^{T_p} + 1 = \sum_{i=1}^{k} [\dim X^{H_i \cdot T_p} + 1] = 2 \sum_{i=1}^{k} \sum_{\alpha > 0} p_{i,\alpha}. \tag{28}$$

The first equality follows from Borel's formula, the second from the fact that $\dim X^{H_i \cdot T_p}$ is calculable. The "outer" identity just means that $\dim X^{T_p}$ is calculable.

NOTE. In the same manner, each $\dim X^{T_{p,\alpha}}$ is calculable cf. section 3.

(ii) Let $K \subset T_p$ be a p-weight. By Lemma 1.1, $K = H_i \cap T_p$ for some i, and K is a nonzero p-weight if and only if $(\dim X^K - \dim X^{T_p}) = m(K) > 0$. We need to show that the numbers $m(K)$ are calculable.

Fix K, and we may re-index $\{H_i\}$ so that

$$i \leqq q\colon\ K = H_i \cap T_p, \quad (X^K)^{H_i} = X^{H_i}$$

$$i > q\colon\ K \neq H_i \cap T_p, \quad (X^K)^{H_i} = X^{H_i \cdot T_p}.$$

Using Borel's formula for the T-action on X^K:

$$\dim X^K + 1 = \sum_{i=1}^{q} (\dim X^{H_i} + 1) + \sum_{i=q+1}^{k} (\dim X^{H_i \cdot T_p} + 1)$$
$$= 2\sum_{i=1}^{q} m_i + 2\sum_{i=q+1}^{k} \sum_{\alpha>0} p_{i,\alpha}.$$

Now, $K \subset H_i$ for $i \leqq q$, and the double sum gives the number of pairs $(\pm\omega)$ such that $\bar{\omega}^{\perp} \not\supset H_i$ and $\omega \mid K = 0$. Consequently, $\dim X^K$ is calculable.

PROPOSITION 5.6. *For each fixed prime p, $\Omega'(X)$ has a p-consistent integral lifting $\bar{\Omega}$. Moreover, $\bar{\Omega}$ is unique modulo divisors of weights which are relatively prime to p.*

PROOF. Let $C_i = p^{e_{p,i}} \cdot C_i'$ and $p \mid C_i'$, where C_i is the integral content of $\Omega'(X^{H_i})$. By Golber's formula (10)

$$e_{p,i} = \sum_{\alpha \geqq 1} \sum_{\beta \geqq \alpha} p_{i,\beta} = \sum_{\alpha \geqq 1} \alpha \cdot p_{i,\alpha} \tag{29}$$

Define $\bar{\Omega}(X^{H_i})$ by taking m_i copies of $(\pm\bar{\omega}_i)$ and let $p_{i,\beta}$ of them have coefficient p^{β}. This is possible since $m_i \geqq \sum_{\beta \geqq 1} p_{i,\beta}$. In order to have the integral content of $\bar{\Omega}(X^{H_i})$ equal to C_i, we need only choose any factorization of C_i' and distribute the factors as additional factors of coefficients of any chosen weights.
Finally, put

$$\bar{\Omega} = \sum_i \bar{\Omega}(X^{H_i}).$$

It is easily checked that $\dim (X^{H_i})^{T_{p,\alpha}}$ is calculable from $\bar{\Omega}$. The uniqueness of $\bar{\Omega}$ (modulo non-p-primary factors) is also obvious.

EXAMPLE 5.7. Let $X = \mathrm{SO}(3)/I \sim_Z S^3$ be the Poincaré sphere with the transitive $\mathrm{SO}(3)$ action. The icosahedral group I has order $60 = 4 \cdot 3 \cdot 5$. $T = \mathrm{SO}(2)$ has no fixed point, whereas $Z^{Z_2} \simeq S^1, X^{Z_3} \simeq S^1, X^{Z_5} \simeq S^1, X^{Z_{p^2}} = \varnothing$ for p prime. We infer from (10),

$$\Omega(X) = \pm 30 \cdot (\gamma^2), \quad \gamma = \text{unit weight of } T.$$

REMARK 5.8. In general, one cannot tell from $\Omega'(X)$ whether $T_{p,\alpha} \subset T$ acts trivially, or what is $\dim X^{T_{p,\alpha}}$. However, if each integral content C_i of the components of $\Omega'(X)$ satisfies $p^2 \nmid C_i$, then $\dim X^{T_{p,\alpha}}$ is clearly calculable. So, in the above example, knowledge of the content $C = 30$ gives all $\dim X^{Z_{p^\alpha}}$. It is also not generally possible to decide which of the groups $(T_p \cap H_i)$ are actually p-weights $\not\supset (0)$. The following proposition gives some partial results.

PROPOSITION 5.9. *Let $\Omega'(X^{H_i}) = \pm C_i(\bar{\omega}_i)^{m_i}$ be the components of $\Omega'(X)$, $i = 1, 2, \ldots, k$. (We assume $X^T = \varnothing$, so $\Omega'(X) = \Omega(X)$.*

(i) *Suppose each $m_i = 1$. If $p^\alpha \mid C_i$, $\forall\, i$, then $T_{p,\alpha}$ acts trivially on X.*

(ii) *If for some i, the highest power of p dividing C_i is p^q and $q < m_i$, then T_p does not act trivially on X. Furthermore, $K = T_p \cap H_i$ is a p-weight $\neq (0)$.*

PROOF. (i) Clearly $X^{H_i} \simeq S^1$, and from formula (10) we deduce $X^{H_i \cdot T_{p,\alpha}} = X^{H_i}$, that is, $X^{H_i} \subset X^{T_{p,\alpha}}$. Using Borel's formula for the T-action on X and $X^{T_{p,\alpha}}$,

$$\dim X + 1 = \sum (\dim X^{H_i} + 1) = \dim X^{T_{p,\alpha}} + 1,$$

consequently $X^{T_{p,\alpha}} = X$.

(ii) If $T_p = T_{p,1}$ acts trivially on X^{H_i}, then

$$\begin{aligned} 2m_i = (\dim X^{H_i} + 1) &\leqq \sum_{\alpha \geqq 1} (\dim X^{H_i \cdot T_{p,\alpha}} + 1) \\ &= 2e_{p,i} = 2q < 2m_i, \end{aligned} \tag{30}$$

a contradiction. Hence, T_p acts nontrivially on X^{H_i}.

It remains to show $m(K) = (\dim X^K - \dim X^{T_p}) > 0$. But this number satisfies the relations in (5), and since there exists $i \leqq q$ such that T_p acts nontrivially on X^{H_i}, we infer $m(K) > 0$.

Let $G = G_1 \times G_2 \times \ldots \times G_k$, $k > 1$, be a compact connected Lie group acting almost effectively on X, $T = T_1 \times T_2 \times \ldots \times T_k \subset G$ a maximal torus. A subgroup $K \subset G$ is *splitting* if $K = K_1 \times K_2 \times \ldots K_k$, $K_i \subset G_i$. In general, a closed subgroup K is contained in a unique smallest splitting subgroup $\bar{K}$, called the *splitting closure* of K.

Observe that each T_i-weight may be regarded as a T-weight, in view of the projection $T \to T_i$ given by the above decomposition of T. We say the weight system $\Omega_0(X)$ is *splitting* if

$$\Omega_0'(X) = \sum_{i=1}^{k} \Omega_0'(X_i) = \sum_i \Omega_0'(X \mid T_i)), \tag{31}$$

where $X_i = X^{\hat{T}_i}$, $\hat{T}_i = T_1 \times \ldots \hat{T}_i \times \ldots \times T_k$. X_i can be regarded as a T_i-space. Hence each weight has "support" in one of the T_i. By Corollary 3.5 we also have

$$\Omega'(X) = \sum_i \Omega'(X_i) \quad \text{(cf. Definition 2.1).} \tag{32}$$

On the other hand, $X^{T_i} \neq \varnothing$ since $k > 1$, and therefore $\Omega'(X_i)$ has a well defined integral lifting given by the nonzero weights of the local representation; this lifting is also denoted $\Omega'(X_i)$. Hence $\bar{\Omega} = \Omega'(X)$ in (32) may be regarded as a "naturally" defined integral lifting of $\Omega'(X)$, via representation theory, although T itself has no fixed point.

THEOREM 5.10. *Assume $G = \prod G_i$ acts on $X \sim_Z S^n$ with splitting weight system $\Omega(X)$, cf.* (32), *and assume $\Omega(X)$ is consistent* (*cf. Definition* 5.2, (iii)). *Then all isotropy groups are splitting subgroups.*

REMARK 5.11. The consistency assumption on $\Omega(X)$ in the above theorem is not really necessary; the complete proof is based upon Lemma 1.1, but we omit more details. Note that $\Omega(X)$ is consistent if $X^T \neq \emptyset$. The corresponding theorem for acyclic G-spaces X is proved in Hsiang [4], and the proof below is similar to this since, at the technical level, "consistency" plays the role of having $X^T \neq \emptyset$.

PROOF. Suppose there is some nonsplitting isotropy group G_x. Then $\exists k \in G_x$ with $\pi_j(k) \notin G_x$, where $\pi_j \colon G \to G_j$. Let $K = \langle k \rangle$ be the closed subgroup generated by k, hence

$$K = K^0 \times \mathbf{Z}_m, \quad \pi_j(K) \not\subset G_x.$$

Let $K^p \subset \mathbf{Z}_m$ be the p-primary subgroup, and note that for some p (p prime or zero) $K^p \not\subset G_x$, and denote this group K^p by Q. We may assume $k \in T$, so $Q \subset \bar{Q} \subset T$, where $\bar{Q}$ is the splitting closure of Q.

Now, observe that $X^{\bar{Q}} \subsetneqq X^Q$ since $x \notin X^{\bar{Q}}$. On the other hand, Q and $\bar{Q}$ are either tori or p-groups, so $\dim X^{\bar{Q}} < \dim X^Q$ by Smith theory. However, both numbers are calculable from $\Omega(X)$, and since the latter is splitting it is easy to see that the subgroup Q and its splitting closure $\bar{Q}$ have $\dim X^Q = \dim X^{\bar{Q}}$. This is a contradiction.

We conclude this section with a closer look at the concept of 0-consistency.

QUESTION. Suppose all multiplicity coefficients $m_i = 1$, so in particular $\Omega(X)$ has a unique integral lifting $\bar{\Omega} = \Omega(X)$. Is $\bar{\Omega}$ 0-consistent?

Perhaps, the intuitive answer to the above question is "yes". However, as will be demonstrated below, the answer is generally "no".

To explain the situation, let H_i be a (rational) weight in $\Omega_0(X)$, and let $\bar{\Omega}_i$ be the multiset of all nonzero weights in $\bar{\Omega} \mid H_i$. For a fixed weight direction (i.e. rational weight) $\pm\rho$ in $\bar{\Omega}_i$, let m_ρ be the number of pairs in $\bar{\Omega}$ whose image in $\bar{\Omega}_i$ has the direction of $\pm\rho$, and let C_ρ be the integral content of these image vectors. As a consequence of Corollary 3.5, $\bar{\Omega}$ will be 0-consistent if for each H_i and choice of $\pm\rho$ one of the two conditions holds:

(i) $m_\rho = 1$ (cf. Coroll. 3.6)
(ii) $C_\rho = 1$ or a prime.

In contrast to the above, the following example has some $m_\rho = 2$ and the corresponding $C_\rho = 2l$, l odd. Explicit calculations show $\Omega(X)$ is not 0-consistent.

EXAMPLE 5.12. Consider again the linear group $(S^1 \times SO(n), \phi)$ described in the beginning of §4, and its restriction to the Brieskorn variety Σ^{2n-1}, see (12) and (14).

Let $T_1 = \omega_1^\perp \subset T$, $\omega_1 = (l\theta - \tau_1)$. T_1 is a maximal torus of some isotropy group G_x, since $\omega_1 \notin \Delta'(G)$ (= root system), according to the "Torus Algorithm", cf. [5]. From (14) we calculate weights $\neq 0 \bmod T_1$,

$$\text{(33)} \qquad \Omega(\Sigma^{2n-1}) \mid T_1 : \ \pm\theta, \ \pm(l\theta + \tau_1) \equiv \pm 2l\theta$$
$$\pm(l\theta \pm \tau_i) \equiv \pm(\tau_1 \pm \tau_i), \ i > 1$$

Hence, by Corollary 3.5, $\Omega'(\Sigma^{2n-1} \mid T_1)$ coincides with the collection in (33), except that the subset $\{\pm\theta, \pm 2l\theta\}$ may be replaced by some $\{\pm a\theta, \pm b\theta\}$, $ab = 2l$. However, by formula (9)

$$\text{(34)} \qquad \Omega'(\Sigma^{2n-1} \mid T_1) = \Delta'(G \mid T_1) - \Delta'(G_x) + \Omega'(\mathscr{S}_x),$$

with $\Delta'(G) = \{\pm(\tau_i \pm \tau_j)\} + \{\pm\tau_i\}$, we find $\Delta'(G_x) \supset \{\pm(\tau_i \pm \tau_j);\ 1 < i < j\} + \{\pm\tau_i, i > 1\}$, and from this

$$\text{(35)} \qquad G_x^0 \simeq \{(e^{i\theta}, \begin{pmatrix} e^{il\theta} & 0 \\ 0 & \begin{smallmatrix} ** \\ ** \end{smallmatrix} \end{pmatrix})\} \simeq \mathrm{SO}(2) \times \mathrm{SO}(n-2)$$

Now, $\Delta'(G \mid T_1) - \Delta'(G_x)$ contains $\pm\tau_1 \equiv \pm l\theta$, so $a = 2$, $b = l$, consequently

$$\Omega(\Sigma^{2n-1} \mid T_1) \neq \Omega(\Sigma^{2n-1}) \mid T_1.$$

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