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HYPERGROUPS AND DISTANCE DISTRIBUTIONS OF RANDOM WALKS ON GRAPHS

KENTA ENDO, IPPEI MIMURA and YUSUKE SAWADA

Abstract

Wildberger's construction enables us to obtain a hypergroup from a random walk on a special graph. We will give a probability theoretic interpretation to products on the hypergroup. The hypergroup can be identified with a commutative algebra whose basis is transition matrices. We will estimate the operator norm of such a transition matrix and clarify a relationship between their matrix products and random walks.

1. Introduction

The concept of hypergroup is a probability theoretic extension of the one of locally compact group and established by Dunkl [6], Jewett [10] and Spector [15]. We refer the reader to Bloom and Heyer's monograph [1] for details of the general theory of (locally compact) hypergroups. A discrete hypergroup is a generalizations of a discrete group. As was the case with groups, we can completely determine structures of hypergroups of low order, see e.g. [18] and [14]. In this paper, we shall only treat hermitian discrete hypergroups which are generalizations of $\mathbb{Z}/2\mathbb{Z}$.

In [16] and [17], Wildberger introduced a method to construct a hermitian finite hypergroup from a random walk on a special graph. This construction produces a $*$-algebra with structure constants determined by pairs of successive jumps in the graph. To obtain a hypergroup from such a random walk requires the underlying graph to satisfy certain conditions. A hermitian hypergroup automatically becomes a commutative algebra, however the well-definedness of products, the associativity or the commutativity may fail, depending on the choice of graph. A non-associative hypergroup is called a pre-hypergroup. Recently, Ikkai and Sawada [9] have studied what kind of graph induces a hermitian discrete hypergroup via Wildberger's construction. Especially, they showed that all self-centered graphs and all distance regular (not necessarily finite) graphs induce pre-hypergroups and hermitian discrete hypergroups, respectively. In this paper, we treat only graphs inducing pre-hypergroups.

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We shall give an outline of this paper. In §2, we will prepare the basic concept of graphs and recall Wildberger's construction.

We will explicitly define a probability space and random variables which describe a time evolution of distances given by random walks on a Cayley graph in §3. We will also discuss the Markov property of the time evolution in the finite graph case.

In §4, we will find a probability theoretic interpretation of m-th products on the pre-hypergroup derived from a random walk on a graph with some symmetry conditions. For a graph Γ, which is not necessarily distance regular, let $H(\Gamma)$ denote the pre-hypergroup with a basis $x_0, x_1, x_2, \ldots$. For the case $m = 2$, by Wildberger's method, the probability theoretic meaning of $x_i \circ x_j$ is clear. However, for $m > 2$, the meaning of a product formed as $((\cdots((x_{i_1} \circ x_{i_2}) \circ x_{i_3}) \circ \cdots) \circ x_{i_{m-1}}) \circ x_{i_m}$ had not been clarified yet.

We shall explain §5. Any discrete hermitian hypergroup is isomorphic to a commutative matrix algebra whose basis consists of transition matrices. The matrix algebra associated with the hypergroup derived from a distance regular graph Γ is close to the Bose-Mesner algebra of Γ in [2]. We can apply the construction of transition matrices to a pre-hypergroup. We will estimate the operator norms of these transition matrices.

In this paper, let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the set of all real numbers and $\mathbb{C}$ the set of all complex numbers. For a set S, let $\mathbb{C}S$ denote the free vector space of S over the field $\mathbb{C}$.

2. Preliminaries

In this section, we prepare definitions, notations and facts related to graph theory and Wildberger's construction of hermitian (discrete) hypergroups from some graphs.

2.1. Graphs

We refer the reader to [7] for general graph theory. Let Γ be a graph with a vertex set V. When a base point $v_0 \in V$ is fixed, the pair (Γ, v_0) is called a *pointed* graph. For $v, w \in V$, let $d(v, w)$ denote the *distance* between v and w, that is, the length of the shortest path from v to w. In particular, we denote by $|v| = d(v_0, v)$ the distance between the base point v_0 and a vertex $v \in V$. We also define

$$I(\Gamma) = I(\Gamma, v_0) = \{n \in \mathbb{N}_0 \mid |v| = n \text{ for some } v \in V\},$$

$$M(\Gamma) = M(\Gamma, v_0) = \sup I(\Gamma, v_0),$$

$$S_n(w) = \{w' \in V \mid d(w, w') = n\}$$

for each $n \in \mathbb{N}_0$ and $w \in V$. In particular, the set $S_n(v_0)$ is sometimes denoted by S_n simply. Note that $\sup_{n\in I(\Gamma)} |S_n| < \infty$ if and only if $\sup_{n\in I(\Gamma)} |S_n(v)| < \infty$ for all $v \in V$. In this paper, assume that any graph Γ is

(i) simple, connected and locally finite,

(ii) has at most countable vertices, and

(iii) satisfies $S_{M(\Gamma)}(v) \neq \varnothing$ for any vertex v if $M(\Gamma) < +\infty$.

Note that the third condition in our assumption is weaker than self-centered.

Now, we shall introduce some symmetric conditions for graphs as follows:

Definition 2.1. Let Γ be a graph with a base point v_0. We define two conditions:

(S1) the function $|S_i(\cdot)|$ is constant on the vertex set of Γ for each $i \in I(\Gamma, v_0)$;

(S2) the function $|S_i(\cdot) \cap S_j(v_0)|$ is constant on $S_k(v_0)$ for each $i, j, k \in I(\Gamma)$.

Definition 2.2. A graph Γ with a vertex set V is said to be *distance regular* if for every $i, j, k \in \tilde{I}(\Gamma) := \{n \in \mathbb{N}_0 \mid d(v, w) = n$ for some $v, w \in V\}$, the cardinality $|\{x \in V \mid d(v, x) = i, d(x, w) = j\}|$ is independent of the choice $v, w \in V$ with $d(v, w) = k$. The above cardinality is denoted by $Q(\Gamma)^k_{i,j}$.

Note that the above definition of distance regular graph is not the original definition but the definition via association schemes. It is known that they are equivalent (for example, see [9, Proposition 2.4]). We refer the reader to [4] and [5] for the general theory of distance regular graphs. It is clear that any distance regular graph satisfies the conditions (S1) and (S2). Of course, any connected graph Γ equipped with condition (S1) automatically satisfies the assumption (iii) $S_{M(\Gamma)}(v) \neq \varnothing$ for any vertex v if $M(\Gamma) < +\infty$.

We shall recall the definition of Cayley graph and define some notations related to graphs. For the general theory of Cayley graphs, see [11]. For a discrete group G and a symmetric finite subset $S \subset G$ not containing the unit element e in G and generating G, the *Cayley graph* $\mathrm{Cay}(G, S)$ of the pair (G, S) is a graph whose vertex set is G and edges are defined by the follows: a vertex $v \in G$ is adjacent to a vertex $w \in G$ if $v^{-1}w \in S$. The Cayley graph $\mathrm{Cay}(G, S)$ is sometimes denoted by G simply when we need not specify the subset S. In this paper, we always assume any base point in a Cayley graph G is the unit element e in G. For a Cayley graph $\mathrm{Cay}(G, S)$, we define a condition (S3) by

(S3) $w \in S_i(v_0)$ if and only if $vw \in S_i(v)$ for all $v \in G$ and $i \in I(G)$.

Lemma 2.3. *Any Cayley graph* $\mathrm{Cay}(G, S)$*, with base point* $v_0 = e$*, satisfies the conditions* (S1) *and* (S3).

PROOF. For $\ell \geq 2$, the vertices v and $w \in G$ are said to have a path of length ℓ if there exists a tuple $\mathbf{w}_\ell = (w_1, w_2, \ldots, w_{\ell-1}) \in G^{\ell-1}$ satisfying the condition $P_\ell(w_0, w)$:

$$P_\ell(w_0, w) \quad \begin{cases} 0 \leq j \leq \ell - 1 \implies w_j^{-1} w_{j+1} \in S, \\ 0 \leq j \leq k \leq \ell,\ k - j \geq 2 \implies w_j^{-1} w_k \notin S, \\ w_j \in G \setminus \{w_0, w_1, \ldots, w_{j-1}, w_\ell\} \quad \text{for any } j = 1, 2, \ldots, \ell - 1, \end{cases}$$

where $w_0 = v$ and $w_\ell = w$.

Fix $i \in I(G)$ and $v \in G$. It suffices to show that the map $F_v \colon S_i(v_0) \ni w \mapsto vw \in S_i(v)$ is bijective. The case $i = 1$ is trivial. First we check the well-definedness of F_v. Take $w \in S_i(v_0)$. Then there exists $\mathbf{w}_i = (w_1, w_2, \ldots, w_{i-1}) \in G^{i-1}$ such that $\mathbf{w}_i$ satisfies $P(v_0, w)$, and that, for any $j = 1, 2, \ldots, i-2$ and $\mathbf{w}_j \in G^{j-1}$, $\mathbf{w}_j$ does not satisfy $P_j(v_0, w)$. Note that these two conditions are equivalent to $w \in S_i(v_0)$. Now we find that $v\mathbf{w}_j = (vw_1, vw_2, \ldots, vw_{i-1})$ satisfies $P_i(v, vw)$ and that, for any $j = 1, 2, \ldots, i-2$ and $\mathbf{w}_j \in G^{j-1}$, $\mathbf{w}_j$ does not satisfy $P_j(v, vw)$, which give the well-definedness of F_v. For the bijectivity, we can easily check that the map $F_{v^{-1}} \colon S_i(v) \ni u \mapsto v^{-1}u \in S_i(v_0)$ is also well-defined and that this map is the inverse mapping of F_v.

In general, the condition (S2) is weaker than distance regularity, however these symmetry conditions are equivalent for a Cayley graph as follows:

LEMMA 2.4. *Let* $\mathrm{Cay}(G, S)$ *be a Cayley graph with the base point* $e \in G$. *If* $\mathrm{Cay}(G, S)$ *satisfies the condition* (S2) *then it is distance regular.*

PROOF. Fix $i, j, k \in \tilde{I}(G)$ and $g, g', h, h' \in G$ with $d(g, h) = k = d(g', h')$. If we define the action $\varphi_x \colon G \ni y \mapsto xy \in G$ for each $x \in G$, by Lemma 2.3, then we have $\varphi_{h^{-1}}(S_i(g) \cap S_j(h)) = S_i(h^{-1}g) \cap S_j(e)$. By Lemma 2.3 again, we have $d(h^{-1}g, e) = d(g, h) = k = d(g', h') = d(h'^{-1}g', e)$, and hence the condition (S2) implies that

$$\begin{aligned} |S_i(g) \cap S_j(h)| &= |\varphi_{h^{-1}}(S_i(g) \cap S_j(h))| = |S_i(h^{-1}g) \cap S_j(e)| \\ &= |S_i(h'^{-1}g') \cap S_j(e)| = |\varphi_{h'^{-1}}(S_i(g') \cap S_j(h'))| \\ &= |S_i(g') \cap S_j(h')|. \end{aligned}$$

This proves that G is distance regular.

EXAMPLE 2.5. The 1-dimensional lattice $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$ is distance regular, and the 2-dimensional lattice $\mathrm{Cay}(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\})$ satisfies condition (S1) and does not satisfy condition (S2).

EXAMPLE 2.6. For each natural number $n > 1$, let S denote a set with cardinality $2n - 1$. The odd graph $\mathcal{O}_n$ with degree n is a distance regular graph whose vertices are the subsets of S and an edge between two vertexes v and w is drawn if and only if $v \cap w = \varnothing$. It is known that $\mathcal{O}_n$ is not a Cayley graph if $n > 2$ by [8].

EXAMPLE 2.7. The Cayley graph $\mathcal{P}_n = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \{(\pm 1, 0), (0, 1)\})$, called the n-gonal prism graph, is distance regular if and only if $n = 4$. For $n \neq 4$, the graph $\mathcal{P}_n$ with an arbitrary base point does not satisfy the condition (S2).

EXAMPLE 2.8. The pair $(\mathcal{B}, v_0)$ consisting of the binary tree $\mathcal{B}$ and a base point v_0 in $\mathcal{B}$ defined as Figure 1, satisfies condition (S2) and does not satisfy condition (S1).

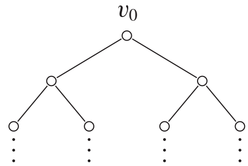

FIGURE 1. Binary tree

2.2. Hypergroups derived from graphs

We shall recall Wildberger's construction of hypergroups from some graphs. We refer the reader to [17] and [9] for details of the construction.

Let (Γ, v_0) be a pointed graph and put $H(\Gamma, v_0) = \{x_i\}_{i \in I(\Gamma, v_0)}$ with dummy symbols x_i for $i \in I(\Gamma, v_0)$. If Γ is an infinite graph, then $I(\Gamma, v_0) = \mathbb{N}_0$. For $i, j, k \in I(\Gamma, v_0)$, we define

$$p_{i,j}^k = \frac{1}{|S_i(v_0)|} \sum_{v \in S_i(v_0)} \frac{|S_j(v) \cap S_k(v_0)|}{|S_j(v)|}. \tag{2.1}$$

The set $\{p_{i,j}^k\}_{k \in I(\Gamma, v_0)}$ is the distribution of distances between the base point v_0 and a random vertex $w \in S_j(v)$ after a jump to a random vertex $v \in S_i(v_0)$. By our assumptions on the graphs, the probability $p_{i,j}^k$ is well-defined for any $i, j, k \in I(\Gamma, v_0)$ (see also [9, Proposition 3.1]). Also, we define a product on the free vector space $\mathbb{C}H(\Gamma, v_0)$ by

$$x_i \circ x_j = \sum_{k \in I(\Gamma, v_0)} p_{i,j}^k x_k$$

for each $x_i, x_j \in H(\Gamma, v_0)$. Note that $|\{k \in I(\Gamma, v_0) \mid p_{i,j}^k \neq 0\}| < \infty$ for all $i, j \in I(\Gamma, v_0)$ even if Γ is infinite.

When the graph Γ has a "good symmetry", defining $x_i^* = x_i$ for each $i \in I(\Gamma)$, the triple $(H(\Gamma, v_0), \circ, *)$ forms a discrete hermitian hypergroup in the following sense.

DEFINITION 2.9. Let $H = \{x_i\}_{i \in I(H)}$ be a countable set whose elements are parametrized by a index set $I(H) = \{0, \dots, N\}$ for some $N \in \mathbb{N}_0$ or $\mathbb{N}_0$. Suppose $\circ$ and $*$ are a binary operation and an involution, respectively, on the free vector space $\mathbb{C}H$. We give the following three definitions:

The triple $(H, \circ, *)$ is called a *discrete hypergroup* if the following conditions are satisfied.

(1) The triple $(\mathbb{C}H, \circ, *)$ is a $*$-algebra with the unit $x_0 \in H$.
(2) For $i, j \in I(H)$, if $x_i \circ x_j = \sum_{k=0}^m p_{i,j}^k x_k$ and $p_{i,j}^k \in \mathbb{C}$ $(k = 0, \dots, m)$, we have $p_{i,j}^k \geq 0$ for all $k = 0, \dots, m$ and $\sum_{k=0}^m p_{i,j}^k = 1$.
(3) For all $i, j \in I(H)$, one has $p_{i,j}^0 \neq 0$ if and only if $x_i = x_j^*$.
(4) The restriction $*|_H$ maps H onto H.

We call the above numbers $\{p_{i,j}^k\}_{i,j,k \in I(H)}$ the structure constants of H.

The hypergroup $(H, \circ, *)$ is said to be

(a) *finite* if $I(H)$ is finite,
(b) *commutative* if $(\mathbb{C}H, \circ)$ is a commutative algebra,
(c) *hermitian* if the restriction $*|_H$ is the identity map.

In this paper, the pair $(H, \circ)$ is called a *pre-hypergroup* if $(\mathbb{C}H, \circ)$ is an algebra with the unit $x_0 \in H$, which may fail the associativity, satisfing (2) and

(3′) for all $i, j \in I(H)$, one has $p_{i,j}^0 \neq 0$ if and only if $x_i = x_j$.

We will denote a hypergroup $(H, \circ, *)$, or a pre-hypergroup $(H, \circ)$, simply by H.

We refer the reader to [13] for details of the general theory of discrete commutative hypergroups. Note that a discrete hermitian hypergroup is automatically commutative. In this paper, we only treat discrete hermitian hypergroups and pre-hypergroups. Thus, if we simply say hypergroup, then it means discrete hermitian hypergroup. Now, we note that, for any graph Γ with a base point, one can get a pre-hypergroup $H(\Gamma)$, and it forms a hypergroup if and only if the conditions of the associativity and the commutativity hold:

$$\sum_{\ell \in I(\Gamma)} p_{h,i}^\ell p_{\ell,j}^k = \sum_{\ell \in I(\Gamma)} p_{i,j}^\ell p_{h,\ell}^k, \quad p_{i,j}^k = p_{j,i}^k$$

for all $h, i, j, k \in I(\Gamma)$. Then we say that the graph Γ produces a hypergroup $H(\Gamma)$. Also, the hypergroup derived from an infinite graph is a polynomial hypergroup in the sense of [12].

REMARK 2.10. Our construction of hypergroups is a slight extension of the one referred to in [9] in which we treat only self-centered graphs. Thanks to our construction, we can sometimes get a hypergroup from a graph equipped with few symmetries as in Example 2.16.

Not all graphs produce hypergroups, however we have a sufficient condition on graphs for producing hypergroups as follows:

THEOREM 2.11 ([9, Theorem 3.3]). *If* (Γ, v_0) *is a pointed distance regular graph, then* $H(\Gamma, v_0)$ *is a hermitian discrete hypergroup with the structure constants* $\{p^k_{i,j}\}_{i,j,k\in I(\Gamma,v_0)}$. *Moreover, the structure is independent of the choice of* v_0.

EXAMPLE 2.12 ([9, Corollary 3.8]). A typical example of graphs producing a hypergroup is the 1-dimensional lattice $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$, in which case the hypergroup $H(\mathbb{Z})$ has the structure given by $x_i \circ x_j = \frac{1}{2}x_{|i-j|} + \frac{1}{2}x_{i+j}$ for each $i, j \in \mathbb{N}_0$. The hypergroup $H(\mathbb{Z})$ is the polynomial hypergroup with respect to the Chebyshev polynomials.

In general, let F_n be the n-free group with the generator $A = \{a_1, a_2, \ldots, a_n\}$. The Cayley graph $\mathrm{Cay}(F_n, A \cup A^{-1})$ is distance regular, and hence it produces a hypergroup

EXAMPLE 2.13. The 2-dimensional lattice $\mathbb{Z}^2$ does not produce a hypergroup. Indeed, we can check that $(x_1 \circ x_2) \circ x_3 \neq x_1 \circ (x_2 \circ x_3)$.

EXAMPLE 2.14. The binary tree $\mathscr{B}$ with the base point v_0 defined in Example 2.8 does not produce a hypergroup. Indeed, the commutativity fails.

Note that distance regularity and conditions (S1) and (S2) are not necessary conditions for producing hypergroups, and we give such examples below.

EXAMPLE 2.15 ([9, §4.2]).

(1) For $n \neq 4$, the n-gonal prism graph $\mathscr{P}_n$ in Example 2.7 is not distance regular, however it produces a hypergroup.

(2) The complete bipartite graph $\mathscr{K}_{n,m}$ with partitions (n, m) produces a hypergroup for each $n, m \in \mathbb{N}$. For example, $\mathscr{K}_{2,3}$ is drawn in Figure 2 (a). The graph $\mathscr{K}_{n,m}$ is not regular if and only if $n \neq m$ and then a pointed graph $(\mathscr{K}_{n,m}, v_0)$ with an arbitrary base point v_0 satisfies condition (S2).

(3) An important example of a graph producing a hypergroup is one drawn in Figure 2 (b) which is not distance regular and satisfies condition (S1). This graph Γ produces different hypergroups depends on base points w_0, w'_0. Note that (Γ, w_0) satisfies condition (S2) and (Γ, w'_0) does not satisfy it.

EXAMPLE 2.16. The pointed graph (Γ, u_0) drawn in Figure 2 (c) is not a Cayley graph, does not satisfy (S1) or (S2), but produces a hypergroup.

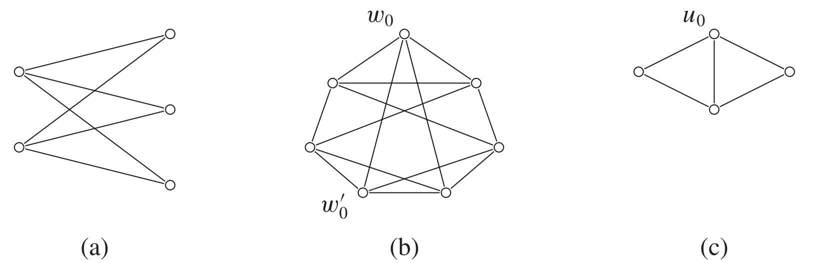

FIGURE 2. Three example graphs

REMARK 2.17. Let Γ and Γ' be two distance regular graphs. If the two hypergroups $H(\Gamma)$ and $H(\Gamma')$ are isomorphic, that is, there is a bijective $*$-homomorphism $\Phi\colon \mathbb{C}H(\Gamma) \to \mathbb{C}H(\Gamma')$, then the constants $Q(\Gamma)_{i,j}^k$ associated with Γ in Definition 2.2, coincide with $Q(\Gamma')_{i,j}^k$. Indeed, the hypergroup structures are represented as $p_{i,j}^k = Q(\Gamma)_{j,k}^i / Q(\Gamma)_{j,j}^0$ and $Q(\Gamma)_{j,0}^j = 1$.

3. Distances distribution obtained from random walks on Cayley graphs

As we mentioned in the previous section, in Wildberger's construction a random walker leaves a fixed base point and moves about a distance greater than or equal to 1 in each observing time. We define a probability measure and random variables related with such a random walk on a Cayley graph.

DEFINITION 3.1. Let $\mathrm{Cay}(G, S)$ be a Cayley graph with the base point $v_0 = e$ and let $\{\alpha_i\}_{i \in I(G,v_0)}$ a sequence of non-negative numbers with

$$\sum_{i \in I(G,v_0)} \alpha_i |S_i(v_0)| = 1.$$

We define a probability measure $\mathbb{P}_0$ on G by $\mathbb{P}_0(\{v\}) = \alpha_{|v|}$ for each $v \in G$ and denote by $\mathbb{P}$ the probability measure on $\Omega = G^{\mathbb{N}}$ obtained by Kolmogorov's extension theorem. For each $m \in \mathbb{N}$, we define a G-valued random variable X_m on Ω, which describes the distance of the n-th jump, by $X_m((\omega_n)_{n=1}^{\infty}) = \omega_m$ for $(\omega_n)_{n=1}^{\infty} \in \Omega$.

We call $\{\alpha_i\}_{i \in I(G,v_0)}$ a *distribution* of G. Also, if G is finite and $\alpha_i = \alpha_j$ for all $i, j \in I(G, v_0)$, we say that G has the *uniform distribution*.

For each $n \in \mathbb{N}$, we also define a $\mathbb{N}_0$-valued random variables Z_n on Ω, which describes the distance between the unit element and a random vertex at the time n, by $Z_n = |X_1 X_2 \cdots X_n|$. Since we assume that a random walker leaves the unit element, we suppose $Z_0 = 0$.

For $n \in \mathbb{N}$, we have $\mathbb{P}(X_n = v) = \alpha_{|v|}$ and $\mathbb{P}(|X_n| = i) = \alpha_i |S_i(v_0)|$ for any $v \in G$ and $i \in I(G)$.

We shall discuss the Markov property of the process $\{Z_n\}_{n=0}^{\infty}$ defined in Definition 3.1 and its stationary distribution. We refer the reader to [3] for the general theory of Markov chains.

Let $\mathrm{Cay}(G, S)$ be a not necessarily finite Cayley graph with a distribution $\{\alpha_n\}_{n \in I(G)}$ satisfying $\alpha_n > 0$ for all $n \in I(G)$. Suppose $\{X_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ are the processes defined in Definition 3.1. By condition (S3), we can calculate

$$\begin{aligned}\mathbb{P}(Z_0 = i_0, Z_1 = i_1, \ldots, Z_k = i_k) \\ = \sum_{w_1 \in S_{i_1}(v_0)} \sum_{w_2 \in S_{i_2}(w_1^{-1})} \sum_{w_3 \in S_{i_3}(w_2^{-1} w_1^{-1})} \cdots \sum_{w_k \in S_{i_k}(w_{k-1}^{-1} \cdots w_1^{-1})} \alpha_{|w_1|} \cdots \alpha_{|w_k|}\end{aligned}$$

for each $i_1, \ldots, i_k$, where we suppose $i_0 = 0$. Hence, in general, the conditional probability

$$\mathbb{P}\big(Z_{n+1} = i_{n+1} \;\big|\; Z_0 = i_0, Z_1 = i_1, \ldots, Z_{n-1} = i_{n-1}, Z_n = i_n\big)$$

depends on $i_1, \ldots, i_{n-1} \in I(G)$, that is, the process $\{Z_n\}_{n=0}^{\infty}$ is not always a Markov chain. If G is finite and has the uniform distribution, that is, $\alpha_n = 1/|G|$ for all $n \in I(G)$, then we can show that $\{Z_n\}_{n=0}^{\infty}$ is a Markov chain and, moreover, is independently identically distributed with

$$\mathbb{P}(Z_0 = i_0, Z_1 = i_1, \ldots, Z_k = i_k) = \frac{1}{|G|^k} \prod_{\ell=1}^{k} |S_{i_\ell}| \tag{3.1}$$

by condition (S1). Note that $\mathbb{P}(Z_0 = i_0, \ldots, Z_n = i_n) \neq 0$ because $S_\ell(v_0) \neq \varnothing$ for all $\ell \in I(G)$. We have $\mathbb{P}\big(Z_{n+1} = i_{n+1} \;\big|\; Z_0 = i_0, \ldots, Z_n = i_n\big) = |S_{i_{n+1}}|/|G|$ which is independent of the choice of $i_1, \ldots, i_{n-1}, i_n$ and coincides with $\mathbb{P}(Z_n = i_{n+1})$.

In such a situation, we denote by $P = (p_{ij})_{i,j \in I(G)}$ the transition probability matrix associated with the ergodic Markov chain $\{Z_n\}_{n=0}^{\infty}$. Then $p_{ij} = |S_j|/|G|$ depends only j, and hence we have

$$P = \frac{1}{|G|} \begin{pmatrix} 1 & |S_1| & |S_2| & \cdots & |S_{M(G)}| \\ 1 & |S_1| & |S_2| & \cdots & |S_{M(G)}| \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & |S_1| & |S_2| & \cdots & |S_{M(G)}| \end{pmatrix},$$

which is idempotent, with a stationary distribution

$$\pi_G = \frac{1}{|G|}(1, |S_1|, |S_2|, \ldots, |S_{M(G)}|). \tag{3.2}$$

4. Hypergroup products

We consider a random walk, used in Wildberger's construction, on a pointed graph (Γ, v_0) equipped with conditions (S1) and (S2). Suppose a random walker leaves v_0 and jumps m-times as $v_0 \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdots \xrightarrow{i_m} \cdot$, where the superscript i_j means the jumping distance. Let $\tilde{p}^k_{i_1,\ldots,i_m}$ denote the conditional probability that the distance between v_0 and a vertex which the random walker reaches is k, under the m-times jumps. (We give the precise definition of $\tilde{p}^k_{i_1,\ldots,i_m}$ later.) In this section, we show that the coefficient of x_k in $((\cdots((x_{i_1} \circ x_{i_2}) \circ x_{i_3}) \circ \cdots) \circ x_{i_{m-1}}) \circ x_{i_m}$ on the pre-hypergroup $H(\Gamma, v_0)$ coincides with $\tilde{p}^k_{i_1,\ldots,i_m}$. When $m = 2$, it is clear by the construction.

First, the familiy $\{p^k_{i,j}\}$, given by the hypergroup product (2.1) with respect to a Cayley graph G, can be represented by conditional probabilities as follows:

Proposition 4.1. *Let* $\mathrm{Cay}(G, S)$ *be a Cayley graph with a distribution* $\{\alpha_i\}$. *For every* $i, j, k \in I(G)$ *with* $\alpha_i \neq 0$ *and* $\alpha_j \neq 0$, *we have* $p^k_{i,j} = \mathbb{P}\big(Z_2 = k \;\big|\; |X_1| = i, |X_2| = j\big)$.

Proof. Recall that G satisfies the conditions (S1) and (S3) by Lemma 2.3. Condition (S3) implies that there is a bijection between $\{w \in S_j(v_0) \mid |vw| = k\}$ and $S_j(v) \cap S_k(v_0)$ for all $v \in S_i(v_0)$. Thus, we have

$$\begin{aligned}
&\mathbb{P}(|X_1X_2| = k, |X_1| = i, |X_2| = j) \\
&\quad = \sum\{\mathbb{P}(X_1 = v,\ X_2 = w) \mid (v, w) \in S_i(v_0) \times S_j(v_0),\ vw \in S_k(v_0)\} \\
&\quad = \sum\{\alpha_i\alpha_j \mid (v, w) \in S_i(v_0) \times S_j(v_0),\ vw \in S_k(v_0)\} \\
&\quad = \alpha_i\alpha_j \sum_{v \in S_i(v_0)} \sum\{1 \mid w \in S_j(v_0),\ vw \in S_k(v_0)\} \\
&\quad = \alpha_i\alpha_j \sum_{v \in S_i(v_0)} \sum\{1 \mid w \in S_j(v) \cap S_k(v_0)\} \\
&\quad = \alpha_i\alpha_j \sum_{v \in S_i(v_0)} |S_j(v) \cap S_k(v_0)|.
\end{aligned}$$

Since $\mathbb{P}(|X_1| = i,\ |X_2| = j) = \alpha_i\alpha_j|S_i(v_0)||S_j(v_0)|$, condition (S1) implies that

$$\mathbb{P}\big(|X_1X_2| = k \;\big|\; |X_1| = i, |X_2| = j\big)$$
$$= \frac{1}{|S_i(v_0)|} \sum_{v \in S_i(v_0)} \frac{|S_j(v) \cap S_k(v_0)|}{|S_j(v_0)|}$$
$$= \frac{1}{|S_i(v_0)|} \sum_{v \in S_i(v_0)} \frac{|S_j(v) \cap S_k(v_0)|}{|S_j(v)|} = p_{i,j}^k.$$

Note that the distribution $\{\alpha_n\}$ does not appear in a conditional probability of the form in the theorem. Now, we give the following definitions:

DEFINITION 4.2. Let (Γ, v_0) be a pointed graph with a vertex set V and $\mathrm{Cay}(G, S)$ a (pointed) Cayley graph.

(1) For $i_1, i_2, \ldots, i_m \in I(\Gamma)$, we define

$$PL(i_1, i_2, \ldots, i_m) = ((\cdots((x_{i_1} \circ x_{i_2}) \circ x_{i_3}) \circ \cdots) \circ x_{i_{m-1}}) \circ x_{i_m}, \quad (4.1)$$

$$J(i_1, i_2, \ldots, i_m) = \sum_{v_1 \in S_{i_1}(v_0)} \sum_{v_2 \in S_{i_2}(v_1)} \cdots \sum_{v_m \in S_{i_m}(v_{m-1})} \frac{1}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_{|v_m|}. \quad (4.2)$$

Let $\tilde{p}_{i_1,\ldots,i_m}^k$ denote the coefficient of x_k in $J(i_1, i_2, \ldots, i_m)$.

(2) Let $\mathbb{P}$ be the probability measure given in Definition 3.1 with respect to a sequence $\{\alpha_i\}_{i \in I(G, v_0)}$ of positive numbers. For $i_1, i_2, \ldots, i_m \in I(G)$ and $k \in I(G)$, we define

$$p_{i_1,i_2,\ldots,i_m}^k = \mathbb{P}\big(Z_m = k \;\big|\; |X_1| = i_1, |X_2| = i_2, \ldots, |X_m| = i_m\big). \quad (4.3)$$

REMARK 4.3. (1) The right hand side of (4.1) means the $m-1$ times product from the left, step by step.

(2) The coefficient $\tilde{p}_{i_1,\ldots,i_m}^k$ is the conditional probability that the distance between v_0 and a vertex which the random walker reaches is k, under the m-times jumps $v_0 \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdots \xrightarrow{i_m} \cdot$.

(3) By Proposition 4.1, the probability $p_{i,j}^k$ in (4.3) is well-defined for each $i, j \in I(G)$.

Definition (4.2) will play a role in giving a probability theoretic interpretation to m-th products on $H(\Gamma)$ for a graph which is not necessary a Cayley graph. For a Cayley graph, it is shown that (4.3) coincides with the coefficient of x_k in (4.2) as in the following theorem.

THEOREM 4.4. *Let* Cay(G, S) *be a Cayley graph. For all* $m \geq 2$ *and* $i_1, \ldots, i_m, k \in I(G)$*, we have*

$$\sum_{v_1 \in S_{i_1}(v_0)} \sum_{v_2 \in S_{i_2}(v_1)} \cdots \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} |S_{i_m}(v_{m-1}) \cap S_k(v_0)|$$
$$= \left| \{(v_1, \ldots, v_m) \in \prod_{k=1}^{m} S_{i_k}(v_0) \mid |v_1 \cdots v_m| = k\} \right|, \quad (4.4)$$

and $\tilde{p}^k_{i_1,\ldots,i_m} = p^k_{i_1,\ldots,i_m}$ *holds.*

PROOF. We shall prove (4.4). By the proof of Proposition 4.1, there exists a bijection between $S_{i_m}(v_{m-1}) \cap S_k(v_0)$ and $\{v_m \in S_{i_m}(v_0) \mid |v_{m-1}v_m| = k\}$, and hence we have

$$\sum_{v_1 \in S_{i_1}(v_0)} \cdots \sum_{v_{m-2} \in S_{i_{m-2}}(v_{m-3})} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} |S_{i_m}(v_{m-1}) \cap S_k(v_0)|$$
$$= \sum_{v_1 \in S_{i_1}(v_0)} \cdots \sum_{v_{m-2} \in S_{i_{m-2}}(v_{m-3})} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} |\{v_m \in S_{i_m}(v_0) \mid |v_{m-1}v_m| = k\}|$$
$$= \sum_{v_1 \in S_{i_1}(v_0)} \cdots \sum_{v_{m-2} \in S_{i_{m-2}}(v_{m-3})} |\{(v_{m-1}, v_m) \in S_{i_{m-1}}(v_{m-2}) \times S_{i_m}(v_0) \mid |v_{m-1}v_m| = k\}|$$
$$= \sum_{v_1 \in S_{i_1}(v_0)} \cdots \sum_{v_{m-2} \in S_{i_{m-2}}(v_{m-3})} |\{(v_{m-1}, v_m) \in S_{i_{m-1}}(v_0) \times S_{i_m}(v_0) \mid |v_{m-2}v_{m-1}v_m| = k\}|. \quad (4.5)$$

Repeating this argument, (4.5) equals to $|\{(v_1, \ldots, v_m) \in \prod_{k=1}^m S_{i_k}(v_0) \mid |v_1 \cdots v_m| = k\}|$.

By (4.4) and (S1), we have

$$\tilde{p}^k_{i_1,\ldots,i_m} = \frac{\left|\{(v_1, \ldots, v_m) \in \prod_{k=1}^m S_{i_k}(v_0) \mid |v_1 \cdots v_m| = k\}\right|}{\prod_{j=1}^m |S_{i_j}(v_0)|} = p^k_{i_1,\ldots,i_m}.$$

This completes the proof.

Next, we shall consider the definitions (4.1) and (4.2). In the following theorem, it will be shown that $PL(i_1, \ldots, i_m) = J(i_1, \ldots, i_m)$ for a pointed graph (Γ, v_0) with conditions (S1) and (S2). However, the case for two jumps ($m = 2$) can be shown without both of the conditions (S1) and (S2) as follows:

for a pointed graph (Γ, v_0), and $i, j \in I(\Gamma, v_0)$, by the definitions, we have

$$J(i, j) = \sum_{k\in I(\Gamma)} \sum_{v_1\in S_i(v_0)} \sum_{v_2\in S_j(v_1)\cap S_k(v_0)} \frac{1}{|S_i(v_0)||S_j(v_1)|} x_k$$

$$= \sum_{k\in I(\Gamma)} \frac{1}{|S_i(v_0)|} \sum_{v_1\in S_i(v_0)} \frac{|S_j(v_1)\cap S_k(v_0)|}{|S_j(v_1)|} x_k \tag{4.6}$$

$$= \sum_{k\in I(\Gamma)} p^k_{i,j} x_k = PL(i, j).$$

THEOREM 4.5. *Suppose that* (Γ, v_0) *is a pointed graph equipped with conditions* (S1) *and* (S2). *Then we have* $PL(i_1, i_2, \ldots, i_m) = J(i_1, i_2, \ldots, i_m)$ *for* $i_1, i_2, \ldots, i_m \in I(\Gamma)$.

PROOF. We shall show the theorem by induction. Assume that $PL(i_1, i_2, \ldots, i_{m-1}) = J(i_1, i_2, \ldots, i_{m-1})$. Then it holds that

$$PL(i_1, \ldots, i_m) = (x_{i_1} \circ \ldots \circ x_{i_{m-1}}) \circ x_{i_m} = J(i_1, \ldots, i_{m-1}) \circ x_{i_m}$$

$$= \left(\sum_{v_1\in S_{i_1}(v_0)} \cdots \sum_{v_{m-1}\in S_{i_{m-1}}(v_{m-2})} \frac{1}{\prod_{j=1}^{m-1} |S_{i_j}(v_{j-1})|} x_{|v_{m-1}|} \right) \circ x_{i_m}$$

$$= \sum_{v_1\in S_{i_1}(v_0)} \cdots \sum_{v_{m-2}\in S_{i_{m-2}}(v_{m-3})} \sum_{k\in I(\Gamma)} \sum_{v_{m-1}\in S_{i_{m-1}}(v_{m-2})\cap S_k(v_0)} \frac{1}{\prod_{j=1}^{m-1} |S_{i_j}(v_{j-1})|} x_k \circ x_{i_m}$$

$$= \sum_{v_1\in S_{i_1}(v_0)} \cdots \sum_{v_{m-2}\in S_{i_{m-2}}(v_{m-3})} \sum_{k\in I(\Gamma)} \frac{|S_{i_{m-1}}(v_{m-2})\cap S_k(v_0)|}{\prod_{j=1}^{m-1} |S_{i_j}(v_{j-1})|} x_k \circ x_{i_m}.$$

Now, by (S1), (S2) and equation (4.6), we have

$$\sum_{k\in I(\Gamma)} \frac{|S_{i_{m-1}}(v_{m-2})\cap S_k(v_0)|}{\prod_{j=1}^{m-1} |S_{i_j}(v_{j-1})|} x_k \circ x_{i_m}$$

$$= \sum_{k\in I(\Gamma)} \frac{|S_{i_{m-1}}(v_{m-2})\cap S_k(v_0)|}{\prod_{j=1}^{m-1} |S_{i_j}(v_{j-1})|} \sum_{\ell\in I(\Gamma)} \frac{1}{|S_k(v_0)|} \sum_{v\in S_k(v_0)} \frac{|S_{i_m}(v)\cap S_\ell(v_0)|}{|S_{i_m}(v)|} x_\ell$$

$$= \sum_{\ell\in I(\Gamma)} \sum_{k\in I(\Gamma)} \frac{|S_{i_{m-1}}(v_{m-2})\cap S_k(v_0)|}{\prod_{j=1}^{m} |S_{i_j}(v_{j-1})|} \frac{1}{|S_k(v_0)|} \sum_{v\in S_k(v_0)} |S_{i_m}(v)\cap S_\ell(v_0)| x_\ell$$

$$= \sum_{\ell\in I(\Gamma)} \sum_{k\in I(\Gamma)} \sum_{v_{m-1}\in S_{i_{m-1}}(v_{m-2})\cap S_k(v_0)} \sum_{v\in S_k(v_0)} \frac{1}{|S_k(v_0)|} \frac{|S_{i_m}(v)\cap S_\ell(v_0)|}{\prod_{j=1}^{m} |S_{i_j}(v_{j-1})|} x_\ell$$

$$= \sum_{\ell \in I(\Gamma)} \sum_{k \in I(\Gamma)} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2}) \cap S_k(v_0)} \sum_{v \in S_k(v_0)} \frac{1}{|S_k(v_0)|} \frac{|S_{i_m}(v_{m-1}) \cap S_\ell(v_0)|}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_\ell$$

$$= \sum_{\ell \in I(\Gamma)} \sum_{k \in I(\Gamma)} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2}) \cap S_k(v_0)} \frac{|S_{i_m}(v_{m-1}) \cap S_\ell(v_0)|}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_\ell$$

$$= \sum_{\ell \in I(\Gamma)} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} \frac{|S_{i_m}(v_{m-1}) \cap S_\ell(v_0)|}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_\ell$$

$$= \sum_{\ell \in I(\Gamma)} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} \sum_{v_m \in S_{i_m}(v_{m-1}) \cap S_\ell(v_0)} \frac{1}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_\ell$$

$$= \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} \sum_{\ell \in I(\Gamma)} \sum_{v_m \in S_{i_m}(v_{m-1}) \cap S_\ell(v_0)} \frac{1}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_{|v_m|}$$

$$= \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} \sum_{v_m \in S_{i_m}(v_{m-1})} \frac{1}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_{|v_m|},$$

and hence

$$\sum_{v_1 \in S_{i_1}(v_0)} \sum_{v_2 \in S_{i_2}(v_1)} \cdots \sum_{v_{m-2} \in S_{i_{m-2}}(v_{m-3})} \sum_{k \in I(\Gamma)} \frac{|S_{i_{m-1}}(v_{m-2}) \cap S_k(v_0)|}{\prod_{j=1}^{m-1} |S_{i_j}(v_{j-1})|} x_k \circ x_{i_m}$$
$$= \sum_{v_1 \in S_{i_1}(v_0)} \sum_{v_2 \in S_{i_2}(v_1)} \cdots \sum_{v_{m-2} \in S_{i_{m-2}}(v_{m-3})} \sum_{v_{m-1} \in S_{i_{m-1}}(v_{m-2})} \sum_{v_m \in S_{i_m}(v_{m-1})} \frac{1}{\prod_{j=1}^m |S_{i_j}(v_{j-1})|} x_{|v_m|}.$$

This completes the proof.

EXAMPLE 4.6. All distance regular graphs satisfy the assumption, conditions (S1) and (S2), in the previous theorem. We already know that the pointed graph (Γ, w_0) in Example 2.15 (3) is not distance regular and satisfies the conditions (S1) and (S2). We present other examples of such pointed graphs drawn in Figure 3 (a) and (b). They all produce hypergroups.

On the other hand, the authors could not find any example of pointed graphs not producing hypergroups and satisfying conditions (S1) and (S2). We propose the following conjecture:

CONJECTURE 4.7. *The conditions* (S1) *and* (S2) *imply that hypergroups are produced.*

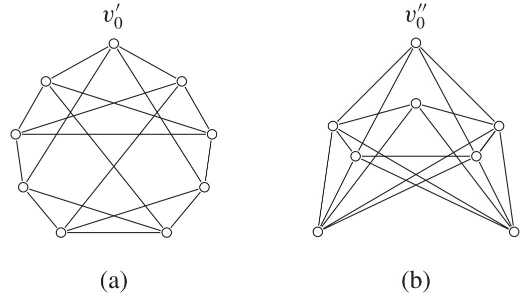

FIGURE 3. Two graphs satisfying (S1) and (S2) which are not distance regular.

EXAMPLE 4.8. The 3-gonal prism graph $\mathscr{P}_3$ in Example 2.7 satisfies

$$PL(1,2,1) = \frac{6}{27}x_0 + \frac{10}{27}x_1 + \frac{11}{27}x_2 \neq \frac{2}{9}x_0 + \frac{1}{3}x_1 + \frac{4}{9}x_2 = J(1,2,1).$$

The binary tree $\mathscr{B}$ with the base point v_0 of $\mathscr{B}$ defined in Example 2.8, satisfies

$$PL(1,1,2) = \frac{1}{9}x_0 + \frac{4}{9}x_2 + \frac{4}{9}x_4 \neq \frac{1}{6}x_0 + \frac{1}{6}x_2 + \frac{2}{3}x_4 = J(1,1,2).$$

COROLLARY 4.9. *If a Cayley graph* $\mathrm{Cay}(G,S)$ *satisfies condition* (S2) *then we have* $PL(i_1,\dots,i_m) = J(i_1,\dots,i_m) = \sum_{k\in I(G)} p^k_{i_1,\dots,i_m} x_k$ *for all* $k, i_1, \dots, i_m \in I(G)$. *We have also* $p^k_{i_{\sigma(1)},\dots,i_{\sigma(m)}} = p^k_{i_1,\dots,i_m}$ *for every permutation* $\sigma \in \mathfrak{S}_m$ *and all* $k, i_1, \dots, i_m \in I(G)$.

PROOF. By Lemma 2.4, G is distance regular and produces a hypergroup $H(G)$. Thus, Theorem 4.4 and Theorem 4.5 imply the first assertion, and the commutativity and the associativity of $H(G)$ imply the second one.

EXAMPLE 4.10. For the Cayley graph $\mathbb{Z}^2$, we have $PL(1,2,3) \neq PL(2,3,1)$.

To end this section, we present a formula with respect to the transition probabilities giving the Markov chain discussed in §3, and constants of the pre-hypergroup structure derived from G as follows:

PROPOSITION 4.11. *Let* $\mathrm{Cay}(G,S)$ *be a Cayley graph which is not necessarily finite and* $\{p^k_{i,j}\}_{i,j,k\in I(G)}$ *the constants giving the structure of the pre-hypergroup* $H(G)$ *derived from* G. *For* $i, j \in I(G)$ *and* $n \in \mathbb{N}$, *we have* $\mathbb{P}(Z_2 = j \mid Z_1 = i) = \sum_{k\in I(G)} p^j_{i,k}\alpha_k |S_k|$. *When* G *is finite and has the uniform distribution, we have* $|S_j| = \sum_{k=0}^{M(G)} p^j_{i,k}|S_k|$.

PROOF. We have

$$\frac{\mathbb{P}(Z_2 = j,\ Z_1 = i)}{\mathbb{P}(Z_1 = i)} = \sum_{k\in I(G)} \frac{\mathbb{P}(Z_2 = j,\ |X_1| = i,\ |X_2| = k)}{\mathbb{P}(|X_1| = i)}$$

$$= \sum_{k\in I(G)} \frac{\mathbb{P}(Z_2 = j,\ |X_1| = i,\ |X_2| = k)}{\mathbb{P}(|X_1| = i,\ |X_2| = k)} \mathbb{P}(|X_2| = k) = \sum_{k\in I(G)} p_{i,k}^{j} \alpha_k |S_k|.$$

The second assertion is implied from (3.1).

5. Transition matrices associated with hypergroups derived from graphs

Any (hermitian) hypergroup is identified with a commutative matrix algebra whose basis consists of transition matrices. We can also apply the construction of transition matrices for a pre-hypergroup. In this section, we will estimate the operator norms of the transition matrices.

A finite hypergroup can be identified with a commutative matrix algebra. This fact is true for the discrete infinite case. In other words, for a not necessarily finite discrete hypergroup $H = \{x_i\}_{i\in I(H)}$ with structure constants $\{p_{i,j}^k\}_{i,j,k\in I(H)}$, a family $\mathscr{P}_H = \{P_k\}_{k\in I(H)}$ of transition matrices $P_k = (p_{k,i}^j)_{i,j\in I(H)}$ satisfies that $P_i P_j = \sum_{k\in I(H)} p_{i,j}^k P_k$ for all $i, j \in I(H)$.

REMARK 5.1. The associativity of the hypergroup $H(\Gamma, v_0)$ derived from a pointed graph (Γ, v_0) is characterized by the commutativity of the transition matrices $\mathscr{P}_{H(\Gamma,v_0)}$. That is, the commutative pre-hypergroup $H(\Gamma, v_0)$ forms a hypergroup if and only if all of the transition probability matrices in $\mathscr{P}_{H(\Gamma,v_0)} = \{P_k\}_{k\in I(\Gamma,v_0)}$ mutually commute.

We can also define the transition matrices $\mathscr{P}_H$ from a pre-hypergroup H in the same way. Then, matrices in $\mathscr{P}_H$ can be regarded as linear operators on the Hilbert space $\ell^2(H) := \{(\xi_n)_{n\in I(H)} \mid \xi_n \in \mathbb{C},\ \sum_{n\in I(H)} |\xi_n|^2 < \infty\}$ as follows: for $k \in I(H)$, we define an operator, which is denote by P_k too, on $\ell^2(H)$ by $P_k(\xi)_n = \sum_{\ell\in I(H)} p_{k,\ell}^n \xi_\ell$ for $\xi = (\xi_n) \in \ell^2(H)$ with $\sum_{n\in I(H)} |\sum_{\ell\in I(\Gamma)} p_{k,\ell}^n \xi_\ell|^2 < \infty$. The actions can be regarded as the matrix products of row vectors in $\ell^2(H)$ and matrix P_k's.

If $\Gamma = \mathrm{Cay}(G, S)$ is a finite Cayley graph with the uniform distribution, then Proposition 4.11 implies that the distribution π_G defined as (3.2) is a stationary distribution of all transition matrices in $\mathscr{P}_{H(G)}$. However, in the infinite case, P_k does not always have a stationary distribution. Indeed, the transition matrix P_1 associated with the hypergroup $H(\mathbb{Z})$ in Example 2.12 has no stationary distribution. For the irreducibility, there are the case in which a transition matrix P_k associated with a hypergroup derived from a graph is irreducible and the case in which it is reducible as follows.

EXAMPLE 5.2. Let P_1, P_2 be the transition matrices associated with the hypergroup $H(\mathscr{C}_4)$ derived from the 4-cycle graph $\mathscr{C}_4 = \mathrm{Cay}(\mathbb{Z}/4\mathbb{Z}, \{\pm 1\})$. Then P_1 is irreducible and P_2 is reducible.

Now, we shall estimate the operator norm of P_k associated with the pre-hypergroup derived from a pointed graph. We define sets

$$\begin{aligned} \mathrm{Supp}(k) &= \{(i,j) \in I(\Gamma)^2 \mid p_{k,i}^j \neq 0\} \\ \mathrm{Supp}_i(k) &= \{j \in I(\Gamma) \mid p_{k,i}^j \neq 0\} \subset \{|i-k|, |i-k|+1, \ldots, i+k\}, \\ \mathrm{Supp}^j(k) &= \{i \in I(\Gamma) \mid p_{k,i}^j \neq 0\} \subset \{|j-k|, |j-k|+1, \ldots, j+k\}. \end{aligned}$$

Then, we have the following theorem.

THEOREM 5.3. *Let Γ be a pointed graph. For all $k \in I(\Gamma)$, the operator P_k is a bounded operator on $\ell^2(H(\Gamma))$. Moreover, if we define constants $c_k = \sup_{j \in I(\Gamma)} \sum_{i \in \mathrm{Supp}^j(k)} (p_{k,i}^j)^2$ and $d_k = \sup_{i \in I(\Gamma)} |\mathrm{Supp}_i(k)|$ the operator norm of P_k is estimated as $1 \le \|P_k\| \le \sqrt{c_k d_k}$.*

PROOF. For every $\xi = (\xi_n) \in \ell^2(H(\Gamma))$, we have

$$\begin{aligned} \|P_k(\xi)\|^2 &= \sum_{j \in I(\Gamma)} \Big| \sum_{i \in I(\Gamma)} p_{k,i}^j \xi_i \Big|^2 = \sum_{j \in I(\Gamma)} \Big| \sum_{i \in \mathrm{Supp}^j(k)} p_{k,i}^j \xi_i \Big|^2 \qquad (5.1) \\ &\le \sum_{j \in I(\Gamma)} \Big(\sum_{i \in \mathrm{Supp}^j(k)} (p_{k,i}^j)^2 \Big) \Big(\sum_{i \in \mathrm{Supp}^j(k)} |\xi_i|^2 \Big) \le c_k \sum_{j \in I(\Gamma)} \sum_{i \in \mathrm{Supp}^j(k)} |\xi_i|^2. \end{aligned}$$

Interchanging of the order of the above sums, we have

$$\sum_{j \in I(\Gamma)} \sum_{i \in \mathrm{Supp}^j(k)} |\xi_i|^2 = \sum_{(i,j) \in \mathrm{Supp}(k)} |\xi_i|^2 = \sum_{i \in I(\Gamma)} \sum_{j \in \mathrm{Supp}_i(k)} |\xi_i|^2 \le d_k \sum_{i \in I(\Gamma)} |\xi_i|^2.$$

By (5.1), the inequality $\|P_k\| \le \sqrt{c_k d_k}$ has been proved. Also, if $\xi = (1, 0, \ldots) \in \ell^2(H(\Gamma))$, we have $\|P_k(\xi)\| = 1$, and hence $\|P_k\| \ge 1$ for all $k \in I(H(\Gamma))$.

EXAMPLE 5.4. Let $\mathscr{P}_{H(\mathbb{Z})} = \{P_k\}_{k \in I(H(\mathbb{Z}))}$ be the family of transition matrices associated with the hypergroup $H(\mathbb{Z})$. The matrix P_1 has the form

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

For all vectors $\xi = (\xi_n) \in \ell^2(H(\mathbb{Z}))$, we have

$$\begin{aligned}\|P_1(\xi)\|^2 &= \left|\frac{1}{2}\xi_1\right|^2 + \left|\xi_0 + \frac{1}{2}\xi_2\right|^2 + \left|\frac{1}{2}\xi_1 + \frac{1}{2}\xi_3\right|^2 + \left|\frac{1}{2}\xi_2 + \frac{1}{2}\xi_4\right|^2 + \cdots \\ &\leq \left|\frac{1}{2}\xi_1\right|^2 + 2|\xi_0|^2 + 2\left|\frac{1}{2}\xi_2\right|^2 + \cdots \leq 2\|\xi\|^2,\end{aligned}$$

and hence $\|P_1\| \leq \sqrt{2} < \sqrt{10}/2 = \sqrt{c_1 d_1}$. By the similar argument with the above, we can check that $\|P_k\| \leq \sqrt{2} < \sqrt{10}/2 = \sqrt{c_k d_k}$.

Remark that the norm of a transition matrix in Theorem 5.3 may be strictly greater than 1. Indeed, taking $\xi_n = 1/2^n$ for each $n \in \mathbb{N}_0$, the vector $\xi = (\xi_n) \in \ell^2(H(\mathbb{Z}))$ has the norm $2/\sqrt{3}$ and $\|P_1(\xi)\| = \sqrt{2}$, and hence we have $\|P_1\| \geq \sqrt{3/2} > 1$.

Example 5.5. Let $\mathscr{P}_{H(F_2)} = \{P_k\}_{k \in I(H(F_2))}$ be the family of transition matrices associated with the hypergroup $H(F_2)$. It turns out that $d_k = k+1$ and $c_k > 1$. The matrix P_1 has the form

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

By a similar argument to the Example 5.4, we can check that $\|P_k\| \leq \sqrt{k} < \sqrt{k+1} < \sqrt{c_k d_k}$.

We shall discuss uniform boundedness for the operators P_k $(k \in I(\Gamma))$ for a pointed infinite graph Γ in the following corollary. (Obviously, when Γ is a finite graph the set $\{\|P_k\|\}_{k \in I(\Gamma)}$ is bounded.)

Corollary 5.6. *Let Γ be an infinite graph with a vertex set V and $v_0 \in V$ a base point. If $S(\Gamma) := \sup_{v \in V} \sup_{k \in I(\Gamma)} |S_k(v)| < \infty$ then we have $\|P_k\| \leq S(\Gamma)^2$.*

Proof. It is enough to show that $c_k, d_k \leq S(\Gamma)^2$ by Theorem 5.3. For $i, j, k \in I(\Gamma)$, we have

$$\begin{aligned}\mathrm{Supp}^j(k) &= \bigcup_{v \in S_k(v_0)} \{\ell \in \mathbb{N}_0 \mid S_\ell(v) \cup S_j(v_0) \neq \varnothing\} \\ &= \bigcup_{v \in S_k(v_0)} \bigcup_{w \in S_j(v_0)} \{\ell \in \mathbb{N}_0 \mid w \in S_\ell(v)\},\end{aligned}$$

$$\begin{aligned}\mathrm{Supp}_i(k) &= \bigcup_{v\in S_k(v_0)} \{m \in \mathbb{N}_0 \mid S_i(v) \cup S_m(v_0) \neq \varnothing\} \\ &= \bigcup_{v\in S_k(v_0)} \bigcup_{w\in S_i(v)} \{m \in \mathbb{N}_0 \mid w \in S_m(v_0)\}\end{aligned}$$

by the definition of $p^j_{k,i}$. These imply that

$$|\,\mathrm{Supp}^j(k)| \leq \sum_{v\in S_k(v_0)} \sum_{w\in S_j(v_0)} 1 = S(\Gamma)^2$$

and

$$|\,\mathrm{Supp}_i(k)| \leq \sum_{v\in S_k(v_0)} |S_i(v)| \leq S(\Gamma)^2,$$

and hence we have $c_k, d_k \leq S(\Gamma)^2$.

EXAMPLE 5.7. It is easy to check that the 1-dimensional lattice $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$ and the infinite ladder graph $\mathscr{L} = \mathrm{Cay}(\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), \{(\pm 1, 0), (0, 1)\})$ satisfy the assumption of Corollary 5.6 and $S(\mathbb{Z}) = 2,\ S(\mathscr{L}) = 4$.

COROLLARY 5.8. *Let* (Γ, v_0) *be a pointed graph producing a hypergroup* $H(\Gamma)$ *and* $\mathscr{P}_{H(\Gamma)}$ *the transition matrices associated with the hypergroup* $H(\Gamma)$. *If* Γ *satisfies conditions* (S1) *and* (S2) *then we have* $(P_{i_1} P_{i_2} \cdots P_{i_m})_{i,j} = \sum_{k\in I(G)} \tilde{p}^k_{i_1,i_2,\ldots,i_m} p^j_{k,i}$ *for all* $m \in \mathbb{N}$ *and* $i, j, i_1, \ldots, i_m \in I(\Gamma)$.

Under the assumption of the previous corollary, the k-th coefficient in $P_{i_m} \cdots P_{i_1} \xi^0$ coincides with $\tilde{p}^k_{i_1,\ldots,i_m}$, where $\xi^0 = (1, 0, 0, \ldots)$. In other words, a matrix product of P_k's describes a distribution of distances between and the base point and a vertex to which a random walker reaches from the base point by some steps (see Remark 4.3 (2)).

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GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
CHIKUSAKU
NAGOYA, 464-8602
JAPAN
E-mail: m16010c@math.nagoya-u.ac.jp
m16052v@math.nagoya-u.ac.jp
m14017c@math.nagoya-u.ac.jp