INFINITE TENSOR PRODUCTS OF UPPER TRIANGULAR MATRIX ALGEBRAS

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Let \( n \geq 2 \) be an integer and let \( T(n) \) be the algebra of \( n \times n \) complex matrices which have zero entries below the main diagonal. Under the operator norm, \( T(n) \) is a Banach algebra, and for a sequence \( (n_k) \) of such integers there is a natural way to associate a unital Banach algebra

\[ T((n_k)) = T(n_1) \otimes T(n_2) \otimes \ldots \]

which is an infinite tensor product in the sense of inductive limits.

In what follows we determine the group \( \text{Aut} T((n_k)) \) of Banach algebra automorphisms of \( T((n_k)) \). The quotient group \( \text{Out} T((n_k)) \), obtained from the normal subgroup of pointwise inner automorphisms, turns out to be the discrete group of permutations \( \pi \) such that \( n_k = n_{\pi(k)} \), \( k = 1, 2, \ldots \). Thus, up to composition by pointwise inner automorphisms the set of outer automorphisms may be uncountable, finite, or even trivial. In fact we describe all isomorphisms and epimorphisms between these Banach algebras.

We also determine the structure of the complete lattice \( \text{Id} T((n_k)) \) of all closed two-sided ideals of \( T((n_k)) \), with the natural lattice operations. The abstract framework needed concerns primary approximately finite lattices, and we develop a little general theory in this direction, inspired by Arveson's unique factorization theory for primary completely distributive metric lattices. It turns out that the unordered set \( \{n_1, n_2, \ldots\} \) is a complete lattice isomorphism invariant for the AF lattice \( \text{Id} T((n_k)) \) and hence a complete Banach algebra isomorphism invariant for the algebras.

The algebras \( T((n_k)) \) can be regarded as the approximately finite versions of reflexive operator algebras associated with certain commutative subspace lattices defined on an infinite tensor product Hilbert space. Such algebras were introduced and studied by Arveson [1, Chapter 3]. He obtained complete

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similarity invariants for these algebras as a consequence of a unique factorization theory mentioned above. We obtain a similar result in the class of approximately finite lattices and our proof derives directly from Arveson’s arguments. However the arguments simplify considerably in our setting.

We can define $T((n_k))$ as a subalgebra of the well known Glimm algebra, or UHF C*-algebra,

$$M((n_k)) = M(n_1) \otimes M(n_2) \otimes \ldots .$$

Here $M(n)$ indicates the full $n \times n$ complex matrix algebra and the infinite tensor product is the C*-algebra direct limit of the direct injective unital system $M(n_1) \to M(n_1 n_2) \to \ldots$, under natural embeddings. The isomorphism theory and automorphism groups of these algebras are well understood (see [4], [5], [7], [9], for example) and, being approximately finite C*-algebras, $K_0$ theory is also available as a complete invariant. Thus $M((n_k))$ and $M((m_k))$ are isomorphic if and only if the sequences of partial products $n_1 n_2 \ldots n_k$ and $m_1 m_2 \ldots m_k$, satisfy the Glimm divisibility criterion: each term from one sequence must divide some term of the other. In other language, $(n_k)$ and $(m_k)$ must determine the same supernatural number. It follows then that $T((n_k))$ and $T((m_k))$ may fail to be isomorphic even though their associated UHF algebras are isomorphic, just as with finite tensor products. We note that the $K_0$ group of $T((n_k))$ coincides with the $K_0$ group of the diagonal subalgebra, from which it follows that $K$-theory provides poor invariants for the algebras $T((n_k))$. However unlike the UHF algebras, which are simple, there is a rich ideal structure, and this structure can serve to study morphisms and the automorphism group. For example the automorphisms that fix the ideal lattice are precisely the pointwise inner automorphisms.

The results above and related matters are organized in the following way. In section one we define approximately finite lattices and note relevant examples and key properties such as zero-one laws for factorizations. In section two we determine the ideal lattice of $T((n_k))$ as an AF lattice. Here we use standard approximation techniques associated with natural expectation mappings on the containing UHF algebra. We have used similar methods in [8] to study ideals in another class of non-self-adjoint subalgebras of AF algebras, namely in nest subalgebras associated with a maximal projection nest in the diagonal. Sections three and four use ideas of Arveson and develop the structure of prime elements in finite and approximately finite primary lattices, respectively. In section five we determine the nature of isomorphisms, epimorphisms and the automorphism group. In the final section we compute $K_0$.

For general lattice theory the reader may consult the standard reference Birkhoff [3], where ideal completions of lattices are discussed a little. Arveson’s results are also described in his lecture notes [2].

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1. Approximately finite lattices.

Let $L_0$ be a lattice with respect to meet and join operations $\wedge$ and $\vee$ respectively. An ideal of $L_0$ is a subset $J$ which is closed under joins and is such that if $a \leq b$, with $a \in L_0$ and $b \in J$, then $a \in J$.

The collection of all ideals, including the empty set, forms a complete lattice $\hat{L}_0$ known as the ideal completion of $L_0$. The lattice $L_0$ is injectively embedded in $\hat{L}_0$ as the sublattice of principal ideals of $L_0$.

We say that a complete lattice $L$ is approximately finite if there is a countable sublattice $L_0 \subset L$ such that $L$ is isomorphic to $\hat{L}_0$ as a lattice. More precisely we require that the natural injection $L_0 \to \hat{L}_0$ extends to an isomorphism $L \to \hat{L}_0$.

Let $L_1 \subset L_2 \subset \ldots$ be a chain of finite sublattices of $L_0$ with union equal to $L_0$. Then there is a one-to-one correspondence between elements $J$ of $\hat{L}_0$ and certain chains of ideals $J_1 \subset J_2 \subset \ldots$, where each $J_k$ is an ideal in $L_k$. The correspondence is given by

$$J \to J \cap L_1, J \cap L_2, \ldots,$$

and so we require that the chain have the fullness property,

$$J_k = (\cup J_m) \cap L_k, \text{ for all } k.$$  

Approximately finite lattices often arise naturally as the direct limit of a direct system of finite lattices. In fact the class of such limit lattices, which we shall define in terms of an ideal completion, coincides with the class of AF lattices, as we now indicate.

An injective direct system of finite lattices is a sequence of finite lattices $M_1, M_2, \ldots$ together with injective embeddings

$$M_1 \to M_2 \to \ldots.$$  

The collection $M_{00}$ of increasing sequences $(m_k)$, with $m_k \in M_k$, and which are eventually constant, forms a lattice in a natural way. Identifying eventually equal sequences we obtain a countable lattice $M_0$ in which each lattice $M_i$ is naturally and injectively embedded, say $M_i \to \alpha(M_i)$. Moreover $M_0$ is the union of the chain $\alpha(M_1) \subset \alpha(M_2) \subset \ldots$. We define the direct limit $L$ of the original system to be the ideal completion of $M_0$, and we write $L = \lim_k M_k$.

We usually consider lattices which possess both a first and last element, denoted by 0 and 1 respectively, and refer to such as unital lattices. A morphism between unital lattices is said to be unital if it maps 0 to 0 and 1 to 1.
An element $c$ of a lattice is **join-irreducible**, or **prime**, if $c = a \lor b$ implies that $a = c$ or $b = c$, and a unital lattice is primary if the unit 1 is prime. An element $c$ is **meet-irreducible** if $c = a \land b$ implies $a = c$ or $b = c$. If the first element 0 of a unital lattice is meet-irreducible then we say that the lattice itself is meet-irreducible. There is an elementary duality between the theory of primary lattices and meet-irreducible lattices that arises through the converse lattice, $(L, <)$ say, of the lattice $(L, \leq)$; $a < b$ in $(L, <)$ if and only if $b \leq a$ in $(L, \leq)$, $a \land b$ in $(L, \leq)$ is the supremum $a \lor b$ in $(L, \leq)$, and $a \lor b$ in $(L, <)$ is the infimum $a \land b$ in $(L, \leq)$. It is easy to check that $(L, <)$ is primary if and only if $(L, \leq)$ is meet-irreducible.

A finite lattice is primary if the supremum of all elements strictly less than 1 is also strictly less than 1, and is meet-irreducible if the infimum of all elements strictly greater than zero is also strictly greater than zero.

We now give some examples to illustrate the concepts above.

**Examples 1.** For $n = 2, 3, \ldots$ write $L(n)$ for the totally ordered unital lattice $\{0, 1, \ldots, n - 1\}$. In particular $L(2)$ is the trivial unital lattice. These lattices are primary and meet-irreducible.

2. For $n, m = 2, 3, \ldots$ let $L(n) \times L(m)$ be the product lattice of $L(n)$ and $L(m)$ with the product partial ordering. For $n, m > 2$ these lattices are neither primary nor meet-irreducible.

3. A subset $A$ of the product set $\{1, \ldots, n - 1\} \times \{1, \ldots, m - 1\}$ for $n, m \geq 2$, is said to be increasing if $(j_1, j_2)$ belongs to $A$ whenever $j_1 \leq k_1$ and $j_2 \leq k_2$ for some element $(k_1, k_2)$ in $A$. The totality of increasing sets, together with the empty set (which is also regarded as an increasing set), forms a lattice of sets (under the set operations) which we denote by $\text{Inc}(n, m)$. Thus $\text{Inc}(n, 2)$ and $\text{Inc}(2, n)$ are just copies of $L(n)$. Similarly we can define $\text{Inc}(n_1, \ldots, n_r)$ for integers $n_1, \ldots, n_r$ that are greater than unity, and there are natural unital injections

$$\text{Inc}(n_1, \ldots, n_r) \rightarrow \text{Inc}(n_1, \ldots, n_s)$$

for $r < s$. Here the increasing set $A$ gets mapped to the increasing set $A \times N_{r+1} \times \ldots \times N_s$, where $N_j = \{1, \ldots, n_j - 1\}$. Note that the lattice $\text{Inc}(n_1, \ldots, n_r)$ is generated by $r$ sublattices $L_1, \ldots, L_r$ where $L_k$ is a copy of the nest lattice $L(n_k)$. These lattices are primary and meet-irreducible.

4. For a sequence $(n_k)$, of integers $n_k \geq 2$, we can define the direct limit AF lattice associated with the system

$$\text{Inc}(n_1, n_2) \rightarrow \text{Inc}(n_1, n_2, n_3) \rightarrow \ldots$$

We see later that such lattices are primary and meet-irreducible. The lattice can be thought of as the infinite tensor product of the nest lattices $L(n_1), L(n_2), \ldots$.

5. Let $A$ be a partially ordered set with a last element $a$, and let $L$ be a unital lattice. Then the collection, $\text{Inc}(A, L)$ say, of increasing functions from $A$ to $L$,
forms a unital lattice. Thus \( f \) belongs to \( \text{Inc}(A, L) \) if \( f: A \rightarrow L \) and \( f(b) \leq f(c) \) if \( b \leq c \). If \( L \) is a finite meet-irreducible lattice then \( \text{Inc}(A, L) \) is also meet-irreducible. For if \( 0^+ \) is the unique successor of 0 in \( L \) then the function \( f \), such that \( f(1) = 0^+ \) and \( f(b) = 0 \) for all \( b \neq 1 \), is the unique successor of the zero function.

For example, if \( L \) is a lattice then \( \text{Inc}(L, L(2)) \) is the lattice of increasing subsets of \( L \).

The lattice structure that we will be concerned with in later sections is the lattice \( \text{Id} A \) of closed ideals of a unital Banach algebra \( A \). Here the join operation is closed linear span and meet is intersection. Clearly \( \text{Id} A \) is a complete unital lattice. We shall look at a class of inductive limit Banach algebras where the ideal lattice \( \text{Id} A \) can be identified as a direct limit of explicit finite lattices. This identification is fairly standard analysis, but the analysis of the structure of \( \text{Id} A \) requires quite a bit of lattice theory. The payoff is that the structure of meet-irreducible elements can be made quite explicit (see Theorem 4.2) and this has considerable implications for the nature of isomorphisms and automorphisms of the algebra \( A \).

We complete the present section by considering complete distributivity and factorizations in the context of AF-lattices.

This information will be needed for the lattice theory in section 4.

**Proposition 1.1.** Let \( L \) be an AF lattice and let \( c_1, c_2, \ldots \) and \( b \) be elements of \( L \).

Then \( \bigvee_{j=1}^{\infty} (b \wedge c_j) = b \wedge \left( \bigvee_{j=1}^{\infty} c_j \right) \).

**Proof.** This is immediate because \( L \) is a complete lattice.

**Definition 1.2.** Sublattices \( L_1 \) and \( L_2 \) of a lattice are said to be independent if the following property holds: if \( a \wedge b \leq a' \vee b' \), with \( a, a' \) in \( L_1 \) and \( b, b' \) in \( L_2 \), then \( a \leq a' \) or \( b \leq b' \).

**Definition 1.3** (Arveson [1]). Let \( L \) be a complete unital lattice. A factorization of \( L \) is a sequence of sublattices \( L_1, L_2, \ldots \) such that

(i) \( L = L_1 \vee L_2 \vee \ldots \)

(ii) For every \( j \) the lattices \( L_j \) and \( \bigvee_{k+j} L_k \) are independent

(iii) \( \bigcap_{n=1}^{\infty} (L_n \vee L_{n+1} \vee \ldots) = \{0, 1\} \).

Similarly we shall say that \( L_1, \ldots, L_n \) is a factorization of \( L \) if (i) and (ii) hold. Property (iii) is called the zero-one law for the sequence \( L_1, L_2, \ldots \). The next proposition shows how zero-one laws arise naturally in certain direct limit AF lattices.
Proposition 1.4. Let $L_1, L_2, \ldots$ be unital sublattices of a lattice $L$ such that for each $n$ the lattices $L_1, \ldots, L_n$ form a factorization of the lattice that they generate. If $L = \lim_n (L_1 \vee \cdots \vee L_n)$ then $L_1, L_2, \ldots$ is a factorization of $L$.

Proof. Let $M_k = L_1 \vee \cdots \vee L_k$ so that $L$ is (isomorphic to) the AF lattice $\lim_k M_k$. This means that $L$ is identified with the lattice of ideals of the countable sublattice $L_0 = \bigcup_{k=1}^{\infty} M_k$. Moreover each such ideal $\beta$ of $L_0$ is associated uniquely with the increasing sequence $\beta \cap M_1, \beta \cap M_2, \ldots$. In view of this correspondence we can establish properties of elements $\beta$ in $L$ by arguing locally with the finite lattice of ideals in $M_k$.

First we obtain property (ii) of Definition 1.3. Let $N_r = \vee_{j+r} L_j$, and note that $N_r$ is simply the sublattice of ideals $\beta$ of $L_0$ such that $\beta \cap M_n = \beta \cap N_r^n$, where $N_r^n = L_1 \vee \cdots \vee L_{r-1} \vee L_{r+1} \vee \cdots \vee L_n$, for $n \geq r$. Let $\beta, \beta'$ be elements of $N_r$, regarded as ideals of $L_0$, and let $\alpha, \alpha'$ be principal ideals of $L_0$ determined by $p, p'$ in $L_r$ respectively. Furthermore assume that $\alpha \wedge \beta \leq \alpha' \vee \beta'$. Let $q \in \beta \cap M_n$. Since $(\alpha \wedge \beta) \cap M_n \subseteq (\alpha' \vee \beta') \cap M_n$ there exists $q' \in \beta' \cap M_n$ and $p''$ in $\alpha' \cap M_n$ such that $p \wedge q = p'' \vee q' \leq p' \vee q'$. By the independence of $L_r$ and $N_r^n$, if $p \not\leq p'$ then $q \leq q'$. Thus if $p \not\leq p'$, or equivalently, $\alpha \not\leq \alpha'$, then $\beta' \cap M_n \subset \beta' \cap M_n$. Since $n$ is arbitrary property (ii) now follows.

Similarly it can be shown that if $\beta \in L_n \vee L_{n+1} \vee \ldots$, then, for $n \leq m$, $\beta \cap M_m$ is an ideal in $L_n \vee \cdots \vee L_m$ and for $n > m$, $\beta \cap M_m = \{0\}$ or $M_m$. Hence for $\gamma \in \bigcap_{n=1}^{\infty} (L_n \vee L_{n+1} \vee \ldots)$ we have $\gamma \cap M_m$ for all $m$, and so property (iii) holds.

Definition 1.5. We say that the factorization $L_1, L_2, \ldots$ of the AF lattice $L$ is a coherent factorization if $L$ is isomorphic to the approximately finite lattice $\lim_n (L_1 \vee \cdots \vee L_n)$, as in the statement of Proposition 1.4.

Proposition 1.6. Let $L_1, L_2, \ldots$ be a coherent factorization of the unital AF lattice $L$, and let $p_k \in L_k$ for $k = 1, 2, \ldots$. Then either $\wedge_k p_k$ is the zero element or $p_k = 1$ for all but a finite number of $k$.

Proof. Let $\beta = \wedge_k p_k$ which is identified with the ideal $\{x \in L_0 : x \leq p_k$ for all $k\}$, where $L_0$ is as in the proof of Proposition 1.4. Let $x \in \beta \cap M_r$, where $M_r = L_1 \vee \cdots \vee L_r$, as before. Then $x \wedge 1 \leq 0 \vee p_k$ for all $k$, and so, by the independence of the lattices $M_r$ and $L_k$ for $k > r$, it follows that $x \leq 0$ or $1 \leq p_k$. Thus if $p_k \not= 1$ for an infinity of $k$, then $x = 0$. Hence $\beta = 0$.

Our last proposition in this section is also an elementary consequence of local arguments. A similar assertion holds with primary replaced by meet-irreducible.

Proposition 1.7. Let $L = \lim_k L_k$ be the AF lattice determined by finite primary unital lattices $L_k$. Then $L$ is primary.
2. \textit{Id} \( T((n_k)) \) as an AF lattice.

The following notation will be useful. Let \((n_k)\) be a sequence of integers, with \(n_k \geq 2\) for all \(k\), to avoid trivialities. Let \( A = T((n_k)) = \bigotimes_{k=1}^{\infty} T(n_k) \), \( B = M((n_k)) = \bigotimes_{k=1}^{\infty} M(n_k) \), \( C = C((n_k)) = \bigotimes_{k=1}^{\infty} C(n_k) \), where \(C(n_k)\) is the diagonal algebra \( T(n_k) \cap T(n_k)^* \). Also, for \( r = 1, 2, \ldots\), let us write \( A_r, B_r \) and \( C_r \) for the finite tensor product algebras associated with the \( r \)-tuple \( n_1, \ldots, n_r \), regarded as the canonical subalgebras of \( A, B \) and \( C \) respectively.

We now define some important expectation maps on the algebra \( B \). For \( r < s \) let \( U_{r,s} \) be the unitary group of the diagonal algebra \( C(n_{r+1}) \otimes \ldots \otimes C(n_s) \subseteq C \), and let \( du \) denote Haar measure on \( U_{r,s} \). The linear contractive map \( \Phi_{r,s} \) defined on \( B_s \) by

\[
\Phi_{r,s}(x) = \int_{U_{r,s}} u^*xu \, du, \quad x \text{ in } B_s,
\]

is a projection and has range equal to the subalgebra \( M(n_1) \otimes \ldots \otimes M(n_r) \otimes C(n_{r+1}) \otimes \ldots \otimes C(n_s) \). Since \( \Phi_{r,t} \) extends \( \Phi_{r,s} \) when \( s < t \), we can define \( \Phi_r \) on \( B \) as the pointwise limit

\[
\Phi_r(x) = \lim_{n \to \infty} \Phi_{r,r+n}(x).
\]

The map \( \Phi_r \) is a contractive projection onto the subalgebra \( B_r \otimes C(n_{r+1}) \otimes \ldots \). In particular \( \Phi_r(x) \to x \) as \( r \to \infty \) for every \( x \) in \( B \).

**Proposition 2.1.** Let \( J \) be a closed subspace of \( B \) that is a \( C \)-module. Then \( J \) is the closed union of the subspaces \( J \cap B_n, n = 1, 2, \ldots \). In particular this holds true for ideals \( J \) of the subalgebra \( A \).

**Proof.** Note that if \( x \) belongs to \( J \) then so does \( \Phi_r(x) \) for every \( r \). On the other hand the bimodule \( \Phi_r(J) \) is finitely generated over \( C \), and the proposition is easily obtained in this case.

The synthesis property expressed in the last proposition is required to identify the ideal lattice of \( A \). In fact the same feature holds for appropriate modules in general approximately finite \( C^* \)-algebras (see [8]).

Let us introduce a twisted partial ordering on the set of pairs

\[
\delta(n) = \{(i,j) : 1 \leq i \leq j \leq n\},
\]

which reflects the ideal structure of \( T(n) \). We write \((i,j) \leq (k,l)\) when \( i \geq k \) and \( j \leq l \). If \( S \) is an increasing subset of \( \delta(n) \) with respect to this ordering then the set \( J \) of matrices in \( M(n) \) supported by \( S \) is an ideal. Conversely every ideal arises in this way. More generally we have the following elementary proposition.
We write 2 for the trivial unital lattice \( L(2) \), and we use the notation of example 5 in section 1.

**Proposition 2.2.** (i) The ideal lattice \( \text{Id} T(n) \) is isomorphic to \( \text{Inc}(\delta(n), 2) \).

(ii) If \( A \) is any complex algebra then the ideal lattice \( \text{Id} (T(n) \otimes A) \) is isomorphic to the lattice \( \text{Inc}(\delta(n), \text{Id} A) \).

In particular \( T(n_1) \otimes T(n_2) \) has an ideal lattice which is isomorphic to \( \text{Inc}(\delta(n_1), \text{Inc}(\delta(n_2), 2)) \), and we write this more simply as \( \text{Inc}(\delta(n_1), \delta(n_2), 2) \). Similarly the \( r \)-fold tensor product \( T(n_1) \otimes \ldots \otimes T(n_r) \) has an ideal lattice denoted by \( \text{Inc}(\delta(n_1), \ldots, \delta(n_r), 2) \).

There are natural embeddings

\[
\text{Inc}(\delta(n_1), \ldots, \delta(n_r), 2) \to \text{Inc}(\delta(n_1), \ldots, \delta(n_s), 2),
\]

when \( r \leq s \), which are most easily identified by checking first that \( \text{Inc}(\delta(n_1), \ldots, \delta(n_r), 2) \) is isomorphic to \( \text{Inc}(\delta(n_1) \times \ldots \times \delta(n_r), 2) \), the lattice of increasing subsets of the partially ordered product space \( \delta(n_1) \times \ldots \times \delta(n_r) \). The embeddings above correspond precisely to the embedding \( \text{Id} A_r \to \text{Id} A_{r+1} \) of the ideal lattice of \( \text{Id} A_r \). (Here an ideal \( J \) in \( \text{Id} A_r \) is identified with the ideal \( \bar{J} \) in \( \text{Id} A_{r+1} \) that it generates).

**Theorem 2.3.** The ideal lattice of \( T((n_k)) \) is isomorphic to the approximately finite lattice \( \lim_k \text{Inc}(\delta(n_1) \times \ldots \times \delta(n_k), 2) \).

**Proof.** We have observed that the limit lattice in the statement of the theorem is isomorphic to \( \lim_k \text{Id} A_k \).

By Proposition 2.1 we can identify \( \text{Id} A \) with the set of sequences \( J \cap A_1, J \cap A_2, \ldots \) for \( J \) in \( \text{Id} A \). An increasing sequence \( J_1, J_2, \ldots \) of ideals \( J_k \) of \( A_k \), is such a sequence precisely when \( J_r = A_r \cap (\cup_k J_k), r = 1, 2, \ldots \). Let us call such a sequence an inductive sequence of ideals. Then, more precisely, Proposition 2.1 allows us to identify \( \text{Id} A \) with the lattice of increasing inductive sequences of ideals. From the definition of direct limits of lattices, we see that \( \text{Id} A \) is isomorphic to \( \lim_k \text{Id} A_k \).

We have already observed that the limit lattice of a unital direct system of primary lattices is primary. Similar reasoning or direct argument with Proposition 2.1 shows that the ideal lattice \( \text{Id} T((n_k)) \) is meet-irreducible.

**Remark.** Similar reasoning applies in the context of nest subalgebras of AF algebras considered in [8]. For example it is possible to define a natural upper triangular subalgebra, \( \text{TM}((n_k)) \), say, of \( M((n_k)) \), which is the inductive limit algebra \( \lim_k T(n_1 \ldots n_k) \), with respect to certain natural embeddings 'by refinement'. For this algebra we can obtain the identification

\[
\text{Id} \text{TM}((n_k)) = \lim_k \text{Inc}(\delta(n_1 \ldots n_k), 2).
\]
3. Finite primary lattices

We now collect together some elementary facts concerning finite factorizations and finite primary lattices. The arguments here are partly derived from Arveson's paper [1].

**Proposition 3.1.** Let $M$ be a finite unital lattice with unital sublattices $L_1, \ldots, L_n$ which form a factorization of $M$. If each factor $L_k$ is primary then $M$ is primary.

**Proof.** Let $e_1, \ldots, e_n$ be the largest non unis in $L_1, \ldots, L_n$ respectively. We show that $e_1 \lor \ldots \lor e_n$ is the largest non unit in $L$, and hence that $L$ is primary. Note first that $e_1 \lor \ldots \lor e_n$ is strictly less than $1$. For otherwise $1 \land 1 \leq e_1 \lor \ldots \lor e_n$ and so by independence $1 \leq e_2 \lor \ldots \lor e_n$. Continuing with this argument obtain the contradiction $1 \leq e_n$. On the other hand let $a \in L$, $a \neq 1$. Then $a = a_1 \lor \ldots \lor a_r$ with each $a_j$ of the form $x_{ji} \land \ldots \land x_{jm}$, with $x_{ji} \in L_i$, for all $i$. For each $j$ there exists $x_{jk}$ with $x_{jk} \leq e_k$. (For otherwise $a_j = 1$, and $a = 1$). Thus $a_j \leq e_1 \lor \ldots \lor e_n$ and so $a \leq e_1 \lor \ldots \lor e_n$, as desired.

In view of Proposition 1.7 we now deduce that if $L_1, L_2 \ldots$ is a coherent factorization of the approximately finite lattice $L$, then $L$ is primary if each factor $L_k$ is primary.

**Corollary 3.2.** Let $M$ be a finite unital lattice with unital sublattices $L_1, \ldots, L_n$ which form a factorization of $M$. If $p$ is a prime element of $L_i$ for some $i$, and if $M$ is primary, then $p$ is a prime element of $M$.

**Proof.** Let $p$ be a non-zero prime element of $L_i$ and define $N = p \land M$, $N_k = p \land L_k$, for $k = 1, \ldots, n$. We claim that $N_1, \ldots, N_n$ is a factorization of $N$.

Clearly, $N_1, \ldots, N_n$ generate $N$. Fix $r$ and elements $a, a' \in L_r$, $b, b' \in \bigvee_{j \neq r} L_j$, and assume that

$$(p \land a) \land (p \land b) \leq (p \land a') \lor (p \land b').$$

If $r = i$ then $(p \land a) \land b = (p \land a) \land (p \land b) \leq ((p \land a') \lor b') \land (p \leq (p \land a') \lor b').$ Hence $p \land a \leq p \land a'$ or $b \leq b'$. On the other hand if $r \neq i$ then $a \land (p \land b) = (p \land a) \land (p \land b) \leq (p \land a') \lor (p \land b')$, and so $a \leq a'$ or $p \land b \leq p \land b'$. In both cases we have the desired alternative, $p \land a \leq p \land a'$ or $p \land b \leq p \land b'$.

We next show that each of the lattices $N_k$ is primary, and the corollary will follow from Proposition 3.1.

Assume that $p = (p \land a) \lor (p \land b)$ with $a$ and $b$ in $N_k$. If $k = i$ then $p \land a = p$ or $p \land b = p$ because $p$ is prime in $L_i$. On the other hand if $k \neq i$ then $p \land 1 = p = (p \land (a \lor b)) \leq a \lor b = 0 \lor (a \lor b)$ and so, by independence, $p \leq 0$ or $1 \leq a \lor b$. Hence $1 = a \lor b$ and $a = 1$ or $b = 1$ because $M$ is primary. Hence $p \land a = p$ or $p \land b = b$ as required.
Corollary 3.3. Let $M$ be a unital primary lattice with unital primary sublattices $L_1, \ldots, L_n$ which form a factorization of $M$. Let $p$ be an element of the form $p = \bigwedge_{r=1}^{l} p_r$, where each $p_r$ is a prime in $M_r$. Then $p$ is prime in $M$.

Proof. By Proposition 3.1 it suffices to show that each of the sublattices $p \wedge L_i$ is primary. Suppose then that $a, b$ are elements of $L_i$ such that $p = (p \wedge a) \vee (p \wedge b)$, and $p \neq 0$. Let $q_i = \bigwedge_{r \neq i} p_r$ so that $p_i \wedge q_i = p = ((p_i \wedge a) \vee (p_i \wedge b)) \wedge q_i$. Since the lattices $L_i$ and $\bigvee_{j \neq i} L_j$ are independent it follows that $p_i = (p_i \wedge a) \vee (q_i \wedge b)$ and hence $p_i = p_i \wedge a$ or $p_i = p_i \wedge b$, since $p_i$ is prime. Hence $p = p \wedge a$ or $p = p \wedge b$, and $p \wedge L_i$ is primary.

The converse to the last corollary is also valid; every prime element $p$ of the lattice $M$ is of the form $p_1 \wedge \ldots \wedge p_n$ where each $p_k$ is prime in $L_k$. We see this in the next section where we obtain an analogous representation for prime elements in certain approximately finite lattices admitting a factorization $L_1, L_2, \ldots$ by finite primary sublattices.

4. Prime elements and unique factorisation.

Our context in this section concerns approximately finite lattices $L$ which arise as in the statement of Proposition 1.4, that is, $L$ is isomorphic to the approximately finite lattice $\lim_n (L_1 \vee \ldots \vee L_n)$ associated with the sequence $L_1, L_2, \ldots$ which is a factorisation of $L$ by finite lattices. We call such a factorisation a coherent factorisation of the AF lattices $L$.

A factorisation is said to be indecomposable when none of the sublattices $L_k$ admits a nontrivial factorisation. The following theorem is the counterpart of a theorem of Arveson for distributive metric lattices [1, Theorem 3.3.2].

Theorem 4.1. Let $L_1, L_2, \ldots$ and $N_1, N_2, \ldots$ be two indecomposable coherent factorisations of the approximately finite unital primary lattice $L$. Then there is a permutation $\pi$ such that $N_k = L_{\pi(k)}$ for all $k$.

However, we will not be able to use this theorem in the context of the ideal lattice of $T((n_k))$ since this lattice is not primary. It is however meet-irreducible, and we have the following theorem.

Theorem 4.2. Let $L_1, L_2, \ldots$ and $N_1, N_2, \ldots$ be two indecomposable coherent factorisations of the approximately finite meet-irreducible lattice $L$. Then there is a permutation $\pi$ such that $N_k = L_{\pi(k)}$ for all $k$. 
These unique factorisation theorems follow from Theorem 4.3 which is the key result of this section. In fact we only use this theorem in what follows.

**Theorem 4.3.** Let $L_1$, $L_2$, . . . be a coherent factorisation of the approximately finite unital lattice $L$.

(i) $L$ has nonzero prime elements if and only if $L_k$ is primary for almost every $k$. Moreover the nonzero primes are precisely the elements $p$ of the form

$$p = p_1 \wedge p_2 \wedge \ldots \wedge p_m$$

where $p_k$ is a nonzero prime in $L_k$, for $1 \leq k \leq m$, $m = 1, 2, \ldots$

(ii) The nontrivial meet-irreducible elements of $L$ are precisely those of the form

$$p = \lor p_k$$

where $p_k$ is a nontrivial meet-irreducible element in $L_k$, for $k = 1, 2, \ldots$.

**Proof.** We first show that for each prime $p \neq 0$ we have

$$p = \bigwedge \{ a_k : a_k \geq p, a_k \in L_k \}.$$  

(This is the AF version of Theorem 3.1.2 in [2]). Let $p_0$ denote the infimum and let $p_n = \bigwedge \{ a : a \geq p, a \in L_{n+1} \lor L_{n+2} \lor \ldots \}. Then p_0 \land p_n \geq p$, and in fact it will be enough to show that for each $n, p \geq p_0 \land p_n$. To see that this is enough, note that

$$p = p_0 \land p_n = \lor (p_0 \land p_n) = p_0 \land (\lor p_n) = p_0 \land 1.$$  

The last two equalities here follow from Proposition 1.1 and property (iii) of Definition 1.3 respectively.

Suppose then that $x \geq p$. We show that $x \geq p_0 \land p_n$. Let $\beta_1, \ldots, \beta_l$ be an enumeration of the elements of the form $x_1 \land \ldots \land x_{n-1}$ with $x_i$ in $L_i$. Consider the collection $N$ of elements of the form

$$\bigvee_{k=1}^l (\beta_k \land c_k)$$

with $c_k$ in $L_n \lor L_{n+1} \lor \ldots$. Then $N$ is a complete lattice. Hence for some $c_1, \ldots c_l$ we have

$$p \leq x = \bigvee_{k=1}^l (\beta_k \land c_k).$$

Since $p$ is a prime element it follows that $p \leq \beta_k \land c_k$ for some $k$ and so $p \leq \beta_k$ and $p \leq c_k$. We have $p_0 \leq \beta_k$ and $p_n \leq c_k$ and so $p_0 \land p_n \leq \beta_k \land c_k \leq x$ as required. Let

$$p_k = \bigwedge \{ a_k : a_k \geq p, a_k \in L_k \},$$

with $p_k = 1$ if this set is empty. Suppose that $p_k = a \lor b$ with $a, b$ in $L_k$. Let
\( q_k = \bigwedge_{i \neq k} p_i \). Then \( p = p_k \wedge q_k = (a \vee b) \wedge q_k = (a \wedge q_k) \vee (b \wedge q_n) \) and so \( p = a \wedge q_k \) or \( p = b \wedge q_k \). Suppose that \( p = a \wedge q_k \). Then \( p = p_k \wedge q_k = a \wedge q_k \leq a \vee 0 \). By independence \( p_k \leq a \) (since \( q_k \neq 0 \)). Also \( a \leq a \vee b = p_k \), and so \( p_k = a \).
The other case, namely \( p = b \wedge q_n \), leads to \( p_n = b \).

Thus \( p_k \) is prime for all \( k \), and by Proposition 1.6, \( p_k = 1 \) for all but a finite number of \( k \). On the other hand the results in section 3 show that \( p \) is prime in \( L \) if \( p \) has the form given in (i).

(iii) We can argue exactly as above for a meet-irreducible element \( p \neq 0 \) by replacing prime by meet-irreducible, \( \vee \) by \( \wedge \), \( \wedge \) by \( \vee \), and 0 and 1 by 1 and 0, respectively. In this way we see that \( p = \vee p_k \) with \( p_k \) a meet-irreducible element of \( L_k \) for all \( k \). It remains to show that \( p = \vee p_k \) is meet-irreducible for every choice of meet-irreducible elements \( p_k \in L_k \).

First note that \( p_1 \vee \ldots \vee p_n \) is meet-irreducible for each \( n \). This follows by applying Corollary 3.3 to the lattice \( L_1 \vee \ldots \vee L_n \) with the converse partial ordering. The infinite case follows in an elementary way from local arguments.

For suppose that \( p = a \wedge b \) and \( p, a, b \) are represented by the ideals \( \pi, \alpha, \) respectively, of the countable lattice \( L_0 \), where \( L_0 \) is the union of the finite \( L_{(n)} = L_1 \vee \ldots \vee L_n \). Then
\[
\pi \cap L_{(n)} = (\alpha \cap \beta) \cap L_{(n)} = (\alpha \cap L_{(n)}) \cap (\beta \cap L_{(n)}).
\]

However, by independence \( \pi \cap L_{(n)} \) is the principal ideal for the meet-irreducible element \( p_1 \vee \ldots \vee p_n \), and so either \( \alpha \cap L_{(n)} = \pi \cap L_{(n)} \) or \( \beta \cap L_{(n)} = \pi \cap L_{(n)} \). This is true for all \( n \) and so \( \alpha = \pi \) or \( \beta = \pi \), as required.

Notice that we have shown that \( L \) is primary (resp. meet-irreducible) if and only if each factor \( L_{k} \), of the coherent factorisation, is primary (resp. meet-irreducible).

**Proofs of Theorems 4.1 and 4.2.** Let \( L_m, N_n \) be as in Theorem 4.1 or 4.2. Set \( L_{m,n} = L_m \cap N_n \). Then we claim that it follows from Theorem 4.3 that \( L_{m,1}, L_{m,2}, \ldots \) and \( L_{1,n}, L_{2,n}, \ldots \) are coherent factorisations of \( L_m \) and \( N_n \) respectively. In fact the arguments for this are virtually identical to those in the proof of Theorem 3.3.1 in [1] and so we omit the details. Since each \( L_m \) is indecomposable it follows that the factors \( L_{m,k} \) are all trivial except for a single factor, \( L_{m,\pi(m)} \) say. Similarly the factors \( L_{1,\pi(m)}, L_{2,\pi(m)}, \ldots \) are all trivial except for a single factor which must be \( L_{m,\pi(m)} \). Thus \( L_m = L_{m,\pi(m)} = \vee_j L_{j,\pi(m)} = N_{n(m)} \). So, with \( \pi \) replaced by \( \pi^{-1} \), the proofs are complete.

**Remark.** We have obtained the unique factorisation theorem, Theorem 4.1, without recourse to Arveson’s factorisation theorem for distributive metric lattices. It seems logical to make the elementary context independent of the topological one. However it may well be possible to deduce our theorem from Arveson's by constructing normal valuations on AF lattices.
5. Isomorphisms and the automorphism group of $T((n_k))$.

The following theorem characterizes the Banach algebra isomorphisms and epimorphisms between the algebras $T((n_k))$ and $T((m_k))$ where, as usual, $(n_k)$ and $(m_k)$ are sequences of positive integers greater than unity.

**Theorem 5.1.** (i) $T((n_k))$ and $T((m_k))$ are isomorphic if and only if there is a permutation $\pi$ such that $m_k = n_{\pi(k)}$, $k = 1, 2, \ldots$.

(ii) There is an onto unital homomorphism from $T((n_k))$ to $T((m_k))$ if and only if there is an injection $\pi$: $\mathbb{N} \rightarrow \mathbb{N}$ such that $m_k \leq n_{\pi(k)}$ for all $k$.

Let $L((n_k))$ be the approximately finite unital lattice $\lim_k \text{Inc} (\delta(n_1), \ldots, \delta(n_k), 2)$ so that by Proposition 2.3 $L((n_k))$ and $\text{Id} T((n_k))$ are isomorphic. By Proposition 1.7 $L((n_k))$ is meet-irreducible. There are canonical identifications of the lattice $L_j = \text{Inc} (\delta(n_j), 2)$ as a unital sublattice of $L((n_k))$ and, by Proposition 1.4 $L_1, L_2, \ldots$ is a factorization of $L((n_k))$. However the factorization is not indecomposable. Each sublattice $L_j$ admits a factorization $L_j^* \vee L_j^\rho$, where $L_j^*$ and $L_j^\rho$ are copies of the nest lattice $L(n_j)$:

\[
L_j^* = \{ \phi, \in \text{Inc} (\delta(n_j), 2): \phi,((i,j)) = 1 \iff 1 \leq i \leq t \}
\]

\[
L_j^\rho = \{ \psi, \in \text{Inc} (\delta(n_j), 2): \psi,((i,j)) = 0 \iff t \leq j \leq 1 \}
\]

Thus $L((n_k))$ is a meet-irreducible unital approximately finite lattice with indecomposable coherent factorisation $L_1^*, L_2^*, L_2^\rho, L_2^\rho, \ldots$.

The set of non zero meet-irreducible ideals of $T(n)$ is isomorphic to $\delta(n)$. The point $(i,j)$ in $\delta(n)$ corresponds to the ideal spanned by the matrix units $e_{k,1}$ where $k < i$ or $1 > j$. Theorem 4.3 shows that the set of nontrivial meet-irreducible ideals of $T((n_k))$ is canonically isomorphic to the product set

\[
\delta((n_k)) = \delta(n_1) \times \delta(n_2) \times \ldots
\]

with the product partial ordering.

**Lemma.** If $n_k \geq 1$ and $m_k \geq 1$ for all $k$ then there is an order isomorphism $\gamma: \delta((n_k)) \rightarrow \delta((m_k))$ if and only if there is a permutation $\pi$ such that $m_k = n_{\pi(k)}$, for all $k$.

**Proof.** Let $t_k = (1, 1)$ and note that the interval $[(1, n_k), (1, 1)]$ is a chain in $\delta(n_k)$. Moreover, if $t = (0, 0, \ldots, t_k, 0, \ldots)$, where the $j$th zero is the zero element $(n_j, n_j)$ of $\delta(n_j)$, then the interval $[0, t]$ in $\delta((n_k))$ is a chain of length $n_k$. Moreover it is easy to see that every segment $[0, t']$ which is totally ordered, is of this form. Thus $\alpha$ induces a bijection of such segments, and it follows that $m_k = n_{\pi(k)}$ for some permutation $\pi$.

**Proof of Theorem 5.1.** Since a Banach algebra isomorphism between $T((n_k))$
and $T((m_k))$ induces a bijection between the sets of meet-irreducible ideals, part (i) follows from the discussion above.

Suppose now that $m_k \leq n_{\pi(k)}$ for all $k$ for some fixed injection $\pi$. Then there is a contractive unital algebra homomorphism $\sigma_k : T(n_{\pi(k)}) \to T(m_k)$, and the contractive maps $\sigma_1 \otimes \ldots \otimes \sigma_k$, from $T((n_{\pi(k)}))$ to $T((m_k))$, converge pointwise to an epimorphism.

On the other hand let $\sigma$ be an epimorphism from $T((n_k))$ to $T((m_k))$. Then ker $\sigma$ is a meet-irreducible ideal and so by Theorem 4.3 (ii), ker $\sigma = \bar{J}_1 \vee \bar{J}_2 \vee \ldots$, where $\bar{J}_k$ is the ideal in $T((n_k))$ generated by a meet-irreducible ideal $J_k$ of $T(n_k)$. Thus there are integers $1 \leq r_k \leq n_k$ and a natural isomorphism

$$T((n_k))/\bar{J}_1 \vee \ldots \vee \bar{J}_r \to T(r_1) \otimes \ldots \otimes T(r_r) \otimes T(m_{l+1}) \otimes T(m_{l+2}) \otimes \ldots$$

Since ker $\sigma$ is the closed union of the ideals $\bar{J}_1 \vee \ldots \vee \bar{J}_r$, it follows that $T((n_k))/\text{ker } \sigma$ is isometrically isomorphic to $T((r_k))$. Now (ii) follows from (i).

The automorphism group. In the next two lemmas we show that the automorphisms fixing the ideal lattice are precisely the pointwise-inner automorphisms. We write $\gamma_\pi$ for the canonical permutation automorphism of $T((n_\pi))$ associated with a permutation $\pi$ such that $n_k = n_{\pi(k)}$ for all $k$.

**Lemma 5.2.** Let $\gamma \in \text{Aut } T((n_\pi))$. Then $\gamma = \beta \circ \gamma_\pi$ where $\gamma_\pi$ is a permutation automorphism and $\beta$ is an automorphism with $\beta(J) = J$ for every two-sided ideal $J$.

**Proof.** Since $\gamma$ preserves meet-irreducibility of ideals, $\gamma$ induces an isomorphism $\gamma : \delta((n_k)) \to \delta((n_\pi))$. But such an automorphism is a composition of a permutation automorphism $\gamma_\pi$ and an automorphism, $\beta$ say, which acts locally. In fact each $\delta(n_i)$ supports a flip automorphism (exchanging coordinates in $\delta(n_i)$), and $\beta$ must either fix or flip each coordinate. Since $\beta$ derives from the algebra automorphism $\beta = \alpha \circ \alpha_\pi^{-1}$, it is easy to check that in fact $\beta$ has no flip action, and hence that $\beta(J) = J$ for every meet-irreducible ideal, and hence for all ideals.

The hypothesis in the next lemma cannot be relaxed too much as can be seen the following example. Let $\mathcal{A}$ be the subalgebra of $T(4)$ spanned by the matrix units $e_{ij}$ of $T(4)$ other than $e_{12}$ and $e_{34}$. It can be seen that $\mathcal{A}$ admits automorphisms that preserve ideals but which are not inner. For example consider the automorphism $\alpha$ such that $\alpha(e_{14}) = -e_{14}$ and $\alpha(e_{ij}) = e_{ij}$ for all other matrix units in $\mathcal{A}$. This fails to be inner because $\alpha$ fails to preserve the rank of some elements. (See [6] for related matters).

**Lemma 5.3.** Let $\mathcal{A}$ be a subalgebra of the algebra $T(n)$ which contains the matrix units $e_{ii}, e_{in}$ for $1 \leq i \leq n$. If $\alpha$ is an automorphism of $\mathcal{A}$ such that $\alpha(J) = J$ for every two sided ideal $J$ then $\alpha$ is an inner automorphism. Moreover the same holds true for ideal preserving automorphisms if the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{B}$ is a commutative unital C*-algebra.
PROOF. By the ideal invariance of $\alpha$ we see that $\alpha(e_{11}) = e_{11} + \sum_{j=2}^{n} a_{1j}e_{1j}$. Let $a_{1r}$ be a nonzero coefficient with $r \geq 2$ and let $S_{1r}(\lambda) = I + \lambda e_{1r}$. Then $S_{1r}(\lambda)^{-1} = S_{1r}(-\lambda)$ and we see that $S_{1r}(\lambda)$ is an invertible element of $\mathcal{A}$ such that
\[ (S_{1r}(\lambda)^{-1} \alpha(e_{11}) S_{1r}(\lambda))_{1r} = a_{1r} - \lambda. \]

It follows that we may construct an invertible element $S$ in $A$ such that $S^{-1} \alpha(e_{11}) S = e_{11}$.

Since $\alpha$ is an automorphism we observe that for $1 < i \leq j$
\[ (S^{-1} \alpha(e_{ij}) S)_{1r} = (e_{11} S^{-1} \alpha(e_{ij}) S)_{1r} = (S^{-1} \alpha(e_{11} e_{ij}) S)_{1r} = 0. \]

Thus $\alpha$ leaves invariant the subalgebra, $\mathcal{A}_1$ say, spanned by $\{e_{ij}; e_{ij} \in \mathcal{A}, 2 \leq i\}$. In particular, with respect to the associated decomposition $C^n = C \oplus C^{n-1}$, we can assume $\alpha$ has the form
\[ \alpha: \begin{pmatrix} a_{11} & a \\ 0 & A_1 \end{pmatrix} \to \alpha(A) = \begin{pmatrix} a_{11} & \delta(a) \\ 0 & \alpha_1(A_1) \end{pmatrix} \]

where $\alpha_1$ is the restriction of $\alpha$ to $\mathcal{A}_1$, and $\delta$ is a linear map on the linear space of row vectors $a$.

We shall show that $\alpha$ is inner by induction on $n$. By the induction hypothesis $\alpha_1$ is implemented by an invertible element $T_1$ of the algebra $\mathcal{A}_1$. Conjugating by $T = e_{11} \oplus T_1$ obtain a new ideal preserving automorphism which is the identity map on $\mathcal{A}_1$. Without loss then, we assume that $\alpha$ already has this form. In particular $\delta(a A_1) = \delta(a) A_1$ for all operators $A_1$ in $\mathcal{A}_1$, from which it follows that $\delta(e_{1j}) = d_j e_{1j}$ for some scalars $d_j$ (associated with indices $j \geq 2$ for which $e_{1j}$ is in $\mathcal{A}$). Suppose $e_{1j}$ lies in $\mathcal{A}$. Then $d_n e_{1n} = \delta(e_{1n}) = \delta(e_{1j} e_{jn}) = \delta(e_{1j}) \delta(e_{jn}) = d_j e_{1j} e_{jn} = d_j e_{1n}$. Thus all the $d_j$ coincide with a single scalar $d$. Thus $\alpha(\cdot) = D^{-1} \cdot D$ where $D$ is the diagonal matrix with entries $1, d, d, \ldots, d$, and the first assertion is proven.

Note that $\mathcal{A} \otimes B$ can be considered as the algebra of matrices from $\mathcal{A}$ whose entries are operators in $B$. Replacing the role of the scalar field by $B$ in the argument above leads to an almost identical proof for the second assertion of the proposition.

The next lemma characterizes the ideal fixing automorphisms as the pointwise inner automorphisms.

**Lemma 5.4.** Let $\alpha$ be an automorphisms of $T((n_k))$ such that $\alpha(J) = J$ for every closed two sided ideal $J$. Then there exist invertible operators $S_r$, with $S_r$ and $S_r^{-1}$ in
$T((n_k))$, for $r = 1, 2, \ldots$, such that $S_r^{-1} XS_r \to \alpha(X)$ as $r \to \infty$ for every element $X$ of $T((n_k))$.

Proof. Let $A_r = \bigotimes_{k=1}^{r} T(n_k)$, $A' = \bigotimes_{k=r+1}^{\infty} T(n_k)$, $C_r = \bigotimes_{k=1}^{r} C(n_k)$, $C' = \bigotimes_{k=r+1}^{\infty} C(n_k)$, regarded as the usual subalgebras of $T((n_k))$. The Jacobson radical $\text{rad } A'$ of the subalgebra $A'$ is the strictly upper triangular part of $A'$ and we have $A' = C' + \text{rad } A'$. Moreover $J = A_r \otimes \text{rad } A'$ is an ideal such that the quotient $T((n_k))/J$ is canonically isomorphic to $A_r \otimes C'$. To see this observe that $A_r \otimes \text{rad } A'$ is the kernel of the natural contractive homomorphism from $T((n_k))$ to $A_r \otimes C'$. In particular, since $J$ is invariant, $\alpha$ induces an automorphism $\alpha_r$ of $A_r \otimes C'$, and moreover $\alpha_r$ leaves invariant the ideals of $A_r \otimes C'$. The ascending subalgebras $A_r \otimes C'$ have dense union in $T((n_k))$, and so it will be sufficient to show that each automorphism $\alpha_r$ is inner. This follows from the second part of Lemma 5.3, since the algebras $A_r$ are subalgebras of $T(n_1 n_2 \ldots n_r)$ of the required form.

The results above are summarized in the next theorem. We write $\text{Out } T((n_k))$ for the quotient group determined by the normal subgroups of pointwise inner automorphisms.

Theorem 5.5. Let $\Pi((n_k))$ be the discrete group of permutations $\pi$ such that $n_k = n_{n(k)}$, $k = 1, 2, \ldots$. Then each automorphism $\alpha$ in $\text{Aut } T((n_k))$ admits a decomposition $\alpha = \beta \circ \alpha_r \pi$ with $\beta$ a pointwise inner automorphism and $\pi$ in $\Pi((n_k))$. In particular $\text{Out } T((n_k))$ is the discrete group $\Pi((n_k))$.

Remark. The classification in Theorem 5.1(i) has been substantially generalized in the author's paper "Classifications of tensor products of triangular operator algebras" (to appear in Proc. London Math. Soc.).

6. The $K_0$ group.

Let $A$ be the algebra $T((n_k))$ with diagonal subalgebra $C = C((n_k))$, associated as usual with integers $n_k \geq 2$, $k = 1, 2, \ldots$. We show that $K_0(A) = K_0(C)$. In particular $K_0$ does not distinguish the isomorphism type.

Recall that $A$ decomposes as a direct sum $A = C + \text{rad } A$, where $\text{rad } A$ is the Jacobson radical. Let $p = (p_{ij})$ be an idempotent in $M_n(A)$ and let $p_{ij} = c_{ij} + r_{ij}$ in $C$ and $r_{ij}$ in $\text{rad } A$, so that $c = (c_{ij})$ is an idempotent in $M_n(C)$. We show that there is continuous path of idempotents $p^t$, $0 \leq t \leq 1$, in $M_n(A)$, such that $p^2 = p$ and $p^0 = c$. From this it will follow that the natural map $K_0(A) \to K_0(C)$ induced by the quotient mapping, is an isomorphism.
Let $d_{r,k}$ be the invertible element of $C$ given by

$$d_{r,k} = 1 \otimes \ldots \otimes 1 \otimes D_{r,k} \otimes 1 \otimes \ldots,$$

$$D_{r,k} = \begin{bmatrix}
1 & t \\
t & t^2 \\
& \ddots \\
& & \ddots & t^n
\end{bmatrix}, \quad 0 < t < 1.$$

Then the inner automorphism $\alpha_{r,k}: a \to d_{r,k}^{-1} \text{ad}_{r,k}$ is a contractive on $A$. It follows that we can define the pointwise inner homomorphism $\alpha_t$ by

$$\alpha_t(a) = \lim_{k \to \infty} \alpha_{r,1} \circ \alpha_{r,2} \circ \ldots \circ \alpha_{r,k}(a).$$

Indeed, this limit exists on a dense subspace, and the composed automorphisms are contractive. Note that $\alpha_t$ is a homomorphism and $\alpha_t(a), 0 < t \leq 1$, is a continuous path in $A$. A simple approximation argument shows that if $a = c + r$ with $c$ in $C$ and $r$ in $\text{rad } A$, then $\alpha_0(a) = \lim_{t \to 0} \alpha_t(a) = c$. Thus the idempotents $p' = (\alpha_t(p_{ij}))$ form a path with the desired properties.

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