ON THE UPPER SEMICONTINUITY INTERSECTION DEFECT

LEOVIGILDO ALONSO TARRIO and ANTONIO G. RODICIO

Abstract.

Let $X$ be a locally noetherian scheme. We prove that the complete intersection defect function is upper semi-continuous on $X$ in the following two cases: i) $X$ is locally immersible in a locally complete intersection scheme, ii) $X$ is excellent.

Let $(A, m, K)$ be a local noetherian ring. The complete intersection defect of $A$ is defined to be the integer

$$d(A) = \dim(A) - \varepsilon_0(A) + \varepsilon_1(A)$$

where $\dim$ denotes Krull dimension, and $\varepsilon_i(A)$ is the $i$-th deviation of $A$ [2].

The function $d(\quad)$ has the following properties [2]:

i) $d(A) \geq 0$, and equality holds if and only if $A$ is a complete intersection (i.e. the $m$-adic completion $\hat{A}$ of $A$ is isomorphic to a quotient of a regular local ring by an ideal generated by a regular sequence),

ii) $d(\hat{A}) = d(A),$

iii) $d(A_p) \leq d(A)$ for every $p \in \text{Spec}(A)$

iv) If $A = R/I$, where $R$ is a regular local ring, then

$$d(A) = \mu(I) - (\dim(R) - \dim(A)),$$

where $\mu(I)$ is minimum number of generators of $I$.

Using the equality of iv) and previous results of A. Grothendieck, L. L. Avramov has proved in [2, Proposition 3.4] the following result:

If $X$ is a locally noetherian scheme, locally immersible in a regular scheme, then the function $x \mapsto d(O_{X,x})$ is upper semi-continuous.

It is also noted in [2, Remark 3.5] that not every $X$ has this property.

In the present paper we obtain a generalization of Avramov's result, namely:

**Theorem.** Let $X$ be a locally noetherian scheme. Then, the function $x \mapsto d(O_{X,x})$ is upper semi-continuous in the following two cases:

---

Received April 22, 1988; in revised form July 30, 1988.
a) $X$ is locally immersible in a locally complete intersection scheme  
b) $X$ is excellent.

We shall use the André-Quillen homology functors $H_n(A, B, \cdot)$ [1] and some results on the openness of the complete intersection loci due to S. Greco and M. G. Marinari [3]. The ideas used here have been applied by A. Ragusa [5] to similar problems, namely the study of the semicontinuity of André deviations $\delta_n(A) = \dim_K H_n(A, K, K)$.

**Lemma 1.** Let $A$ be a ring, $B$ an $A$-algebra and $n$ an integer number. Assume that $H_i(A, B, B)$ is a flat $B$-module for $0 \leq i \leq n$. Then $H_n(A, B, W) \cong H_n(A, B, B) \otimes_B W$ for every $B$-module $W$.

**Proof.** This follows from the spectral sequence

$$E^2_{p,q} = \Tor^B_p(H^q(A, B, B), W) \Rightarrow H_{p+q}(A, B, W).$$

**Lemma 2.** Let $(A, m, K)$ be a local noetherian ring and let $p \in \Spec(A)$ be such that $H_i(A, A/p, A/p)$ is $A/p$-free, $i = 1, 2$. Then

$$-\varepsilon_0(A) + \varepsilon_1(A) \leq -\varepsilon_0(A/p) + \varepsilon_1(A/p) - \varepsilon_0(A/p) + \varepsilon_1(A/p).$$

**Proof.** Recall that $\varepsilon_i(A) = \dim_k H_i(A, K, K), i = 1, 2$ [2, Remark 1.4]. Let $k(p) = A_p/pA_p$ be the residue field of $A_p$. Since $H_i(A, A/p, A/p)$ is $A/p$-free, $i = 1, 2$, we obtain from lemma 1 and [1, lemme 3.20 and corollaire 5.27]

$$\dim_k H_i(A, A/p, K) = \dim_k H_i(A, A/p, A/p) \otimes_{A/p} K =$$

$$\dim_{k(p)} H_i(A, A/p, A/p) \otimes_{A/p} k(p) = \dim_{k(p)} H_i(A_p, k(p), k(p)) = \varepsilon_i(A_p), i = 1, 2.$$

Consider the Jacobi-Zariski exact sequence [1, theoreme 5.1] associated to the homomorphisms $A \longrightarrow A/p \longrightarrow K$

$$H_2(A, A/p, K) \longrightarrow H_2(A, K, K) \longrightarrow H_2(A/p, K, K) \longrightarrow$$

$$H_1(A, A/p, K) \longrightarrow H_1(A, K, K) \longrightarrow H_1(A/p, K, K) \longrightarrow 0.$$

Let $N = \Ker \phi$. We obtain

$$\dim_k N - \varepsilon_1(A) + \varepsilon_1(A/p) - \dim_k H_1(A, A/p, K) + \varepsilon_0(A) - \varepsilon_0(A/p) = 0.$$  

Moreover $\dim_k N \leq \dim_k H_2(A, A/p, K) = \varepsilon_1(A_p)$. Therefore

$$\varepsilon_1(A_p) - \varepsilon_1(A) + \varepsilon_1(A/p) - \varepsilon_0(A/p) + \varepsilon_0(A) - \varepsilon_0(A/p) \geq 0.$$

**Corollary 3.** Let $(A, m, K)$ be a local noetherian ring and let $p \in \Spec(A)$ be such that $\dim(A) = \dim(A_p) + \dim(A/p)$ (this is true for all $p \in \Spec(A)$ if $A$ is catenary). Assume that $H_i(A, A/p, A/p)$ is $A/p$-free for $i = 1, 2$. Then

$$d(A) \leq d(A_p) + d(A/p).$$
PROPOSITION 4. Let $A$ be a noetherian catenary ring and $p \in \text{Spec}(A)$. Then there exists an open neighbourhood $U$ of $p$ in $\text{Spec}(A)$ such that, for every $q \in U \cap V(p)$, we have

$$d(A_q) \leq d(A_p) + d(A_q/pA_q).$$

PROOF. Since $H_i(A, A/p, A/p)$ is $A/p$-finite for each $i$ [1, proposition 4.55], there exist $f_i \notin p$ such that $H_i(A_{f_i}, A_{f_i}/p_{f_i}, A_{f_i}/p_{f_i}) \simeq H_i(A, A/p, A/p)_{f_i}$ is $A_{f_i}/p_{f_i}$-free [4, 22.A, lemma 1].

Now we can shrink to the open neighbourhood of $p$, $A_{f_1f_2}$, which from now on is denoted by $A$, and then we have $H_i(A, A/p, A/p)$ is $A/p$-free for $i = 1, 2$; furthermore, by localization in every $q \supseteq p$ of such an open set, we can assume $A$ to be a local noetherian ring and we have to prove

$$d(A) \leq d(A_p) + d(A/p).$$

This follows from corollary 3.

PROOF OF THE THEOREM. We only shall prove part a) since part b) is analogous.

The assertion being local on $X$, one can assume that $X$ is a closed subscheme $\text{Spec}(B/I)$ of the scheme $\text{Spec}(B)$, where $B$ is a locally complete intersection ring. Let $A = B/I$. $B$ is catenary and therefore, so is $A$. Let $n$ be an integer. We have to show that the set $U_n(A) = \{p \in \text{Spec}(A) | d(A_p) \leq n\}$ is open in $\text{Spec}(A)$. By [2, Proposition 3.8] $U_n(A)$ is stable under generalization. Hence [4, 22.B, lemma 2] we only have to show that for any $p \in U_n(A)$, $U_n(A) \cap V(p)$ contains a non-empty open subset of $V(p)$.

Take $p \in \text{Spec}(A)$; by proposition 4 we can find a neighbourhood of $p$, $U'$, such that for $q' \in U' \cap V(p)$, we have

$$d(A_{q'}) \leq d(A_p) + d(A_{q'}/pA_{q'}).$$

$A/p$ is a quotient ring of locally complete intersection ring and therefore $U_0(A/p) = U_0(A/p)$ is open [3, corollary 3.4]. Then there exist another neighbourhood of $p$, $U''$, such that for $q'' \in U'' \cap V(p)$, $d(A_{q''}/pA_{q''}) = 0$.

Now for any $q \in U' \cap U'' \cap V(p)$ we have

$$d(A_q) \leq d(A_p) \leq n.$$

REFERENCES


DEPARTEMEMTO DE ALGEBRA
FACULTAD DE MATEMATICAS
15771 SANTIAGO DE COMPOSTELA
SPAIN