THE SIMPLICITY OF THE QUOTIENT ALGEBRA $M(A)/A$
OF A SIMPLE $C^*$-ALGEBRA

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Abstract.

It is shown that the $C^*$-algebra $M(A)/A$, where $A$ is a $\sigma$-unital stably semi-finite $C^*$-algebra and $M(A)$ is the multiplier algebra of $A$, is simple if and only if either $A$ has a continuous dimension scale or $A$ is elementary.

Let $A$ be a $C^*$-algebra, and denote by $A^{**}$ the enveloping von Neumann algebra of $A$. The multiplier algebra $M(A)$ is the idealiser of $A$ in $A^{**}$. We denote by $\mathcal{K}$ the $C^*$-algebra of all compact operators on an infinite dimensional separable Hilbert space, and by $\mathcal{B}(H)$ the $C^*$-algebra of all bounded operators on $H$. It is well known that $M(\mathcal{K}) = \mathcal{B}(H)$ and $M(\mathcal{K})/\mathcal{K}$ is simple. The ideal structure of the $C^*$-algebra $M(A)/A$ for $A$ a simple AF $C^*$-algebra has been studied in [5], [7] and [6], and for $A$ a factorial simple $C^*$-algebra has been studied in [8]. In the present note we shall show that in the case of a $\sigma$-unital, stably semi-finite $C^*$-algebra, $M(A)/A$ is simply if and only if either $A$ has a continuous dimension scale or $A$ is elementary. We shall also show that for every $\sigma$-unital purely infinite $C^*$-algebra $A$, $M(A)/A$ is simple.

1. Preliminaries.

1.1. Let $B$ be a dense hereditary*-subalgebra of a $C^*$-algebra $A$, and $a$, $b$ elements of $B$. Following Cuntz, we write $a \preceq b$ if there are $x$, $y$ in $A$ such that $a = xby$. We write $a \preceq b$ if there is a sequence $\{a_n\}$ in $B$ such that $a_n \preceq b$ and $a_n \to a$. This relation is transitive and reflexive. We write $a \approx b$ if $a \preceq b$ and $b \preceq a$. We say that $a$ is orthogonal to $b$ (or $a \perp b$) if $ab = ba = a^*b = ba^* = 0$.

Let $A$ be a simple $C^*$-algebra, $K(A)$ its Pedersen ideal, $\mathcal{F}$ the algebra of operators of finite rank on an infinite-dimensional separable Hilbert space $H$ and $\mathcal{K}$ the $C^*$-algebra of compact operators on $H$. We denote by $\mathcal{F} \otimes K(A)$ the algebraic tensor product of $\mathcal{F}$ and $K(A)$. We call an element $x$ in $\mathcal{F} \otimes K(A)$ infinite if $y \preceq x$ for every $y$ in $\mathcal{F} \otimes K(A)$.

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There are three possibilities for a simple $C^*$-algebra $A$.

(i) $\mathcal{F} \otimes K(A)$ contains only finite elements. In this case we shall call $A$ stably semi-finite.

(ii) $\mathcal{F} \otimes K(A)$ contains non-zero finite and infinite elements.

(iii) All non-zero elements in $\mathcal{F} \otimes K(A)$ are infinite.

It is not known if case (ii) can appear. If $A$ has a lower semi-continuous semi-finite trace, then $A$ is stably semi-finite.

We call a function $d: \mathcal{F} \otimes K(A) \to \mathbb{R}_+$ a dimension function (on $K(A)$) if

(a) $d(x) = 0$ if and only if $x = 0$

(b) $x \leq y$ implies $d(x) \leq d(y)$

(c) $d(x + y) = d(x) + d(y)$ for all orthogonal $x, y$ in $\mathcal{F} \otimes K(A)$.

A dimension function also satisfies

(d) $d(x + y) \leq d(x) + d(y)$ for all $x, y \in \mathcal{F} \otimes K(A)$.

Given $x \in \mathcal{F} \otimes K(A)$, we denote by $\langle x \rangle$ the $\approx$-equivalence class of $x$ in $\mathcal{F} \otimes K(A)$. Let $F$ be the free abelian group generated by $\{\langle x \rangle \mid x \in \mathcal{F} \otimes K(A)\}$ and let $R$ be the subgroup of $F$ generated by all elements of the form $\langle x \rangle + \langle y \rangle - \langle x_1 + y_1 \rangle$ for $x_1 \in \langle x \rangle$, $y_1 \in \langle y \rangle$, $x_1 \perp y_1$. We denote by $\Delta(A)$ the quotient $F/R$. $\Delta(A)$ is an ordered group with the order induced by "$\leq$". We shall use "$\leq$" for the order. $\Delta(A) \neq \{0\}$ if and only if $A$ is stably semi-finite. Moreover, there is a bijective correspondence between non-zero positive homomorphisms $h: \Delta(A) \to \mathbb{R}$ and dimension functions $d$ on $K(A)$ given by $d(x) = h(\langle x \rangle)$, and if $\Delta(A) \neq \{0\}$, $K(A)$ admits a dimension function. For the details of the relations $\approx$ and "$\leq$", dimension functions and the ordered group, readers are referred to [1], [2], [3] and [7].

1.2. Given $\varepsilon > 0$, let $f_\varepsilon$ be the continuous function on $\mathbb{R}$ defined by

$$f_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in (-\infty, \varepsilon/2] \\ \text{linear} & \text{if } t \in [\varepsilon/2, \varepsilon]. \\ 1 & \text{if } t \in [\varepsilon, \infty) \end{cases}$$

If $a \in A$, set

$$A_a = A_{|a|} = \bigcup_{\varepsilon > 0} f_\varepsilon(|a|)A f_\varepsilon(|a|).$$

1.3. We now identify $p \otimes K(A)$ with $K(A)$ and $p \otimes A$ with $A$ for a fixed one dimensional projection $p$ in $\mathcal{F}$. Suppose that $a$ and $b$ are in $K(A)$, and $\langle a \rangle \leq \langle b \rangle$. So $a \leq b$ in $\mathcal{F} \otimes K(A)$ and $a^*a \leq b^*b$. Since $a \approx a^*a$, $b^*b \approx b$ both in $K(A)$ and $\mathcal{F} \otimes K(A)$, we may assume that $0 \leq a$ and $0 \leq b$. There are $x_n \in \mathcal{F} \otimes K(A)$ such that $x_n \leq b$, $x_n \to a$. We can find a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$f_{\varepsilon_n}(a)x_nf_{\varepsilon_n}(a) \to a$$

since $f_{\varepsilon_n}(a)x_nf_{\varepsilon_n}(a) \leq b$, $a \leq b$ in $K(A)$. \]
1.4. If \( A \) is a stably semi-finite simple C*-algebra and \( u \) is a non-zero positive element in \( K(A) \), then \( \langle u \rangle \) is an order unit (see [2, 4.2]). A positive homomorphism \( h: \Delta(A) \to \mathbb{R} \) is called a state (with respect to \( \langle u \rangle \)) if \( h(\langle u \rangle) = 1 \). The collection \( S = S_u(\Delta(A)) \) of all states on \( \Delta(A) \) is a convex compact subset of the locally convex space \( \mathbb{R}^{\Delta(A)} \) of all functions \( f: \Delta(A) \to \mathbb{R} \) with the product topology. \( S \) is the set of all dimension functions \( d \) on \( K(A) \) such that \( d(u) = 1 \). We define a positive homomorphism \( \hat{g}: \Delta(A) \to \text{Aff} S, g \mapsto \hat{g} \) by setting \( \hat{g}(h) = h(g) \), where \( \text{Aff} S \) is the set of all continuous real affine functions on \( S \).

Let us say that \( g \in \Delta(A) \) is infinitesimal if \( -\varepsilon u \leq g \leq \varepsilon u \) for every positive rational number \( \varepsilon \). (If \( \varepsilon = p/q, p, q \in \mathbb{N} \), then \( g \leq \varepsilon u \) means that \( qg \leq pu \).)

The notation "\( \hat{g} \succ 0 \)" for \( g \in \Delta(A) \) means \( \hat{g}(d) > 0 \) for all \( d \in S \).

1.5. PROPOSITION (Corollary of [4, 4.2]). The homomorphism \( \theta: \Delta(A) \to \text{Aff} S \) determines the order on \( \Delta(A) \) in the sense that \( \Delta(A)^+ = \{ g \in \Delta(A) | \hat{g} \succ 0 \} \cup \{ 0 \} \). Hence we have \( g \in \ker \theta \) if and only if \( g \) is infinitesimal.

PROOF. To apply 4.2 of [4] one need note only that the ordered group \( \Delta(A) \) is unperforated.

1.6. When \( a \) is a positive element in a C*-algebra \( A \), we shall denote by \([a]\) the range projection of \( a \) in the enveloping von Neumann algebra \( A^{**} \). Suppose that \( A \) is \( \sigma \)-unital (and non-unital), and let \( e \) be a strictly positive element of \( A \). By choosing a proper sequence of continuous functions \( h_n \), we can construct an approximate identity \( \{ e_n = h_n(e) \} \) for \( A \) satisfying

(i) \( g_n = e_n - e_{n-1} \neq 0 \) (\( e_0 = 0 \)), and \( g_m g_n = 0 \) if \( |m - n| \geq 2 \),

(ii) There are \( a_n \in A^+, a_n \neq 0 \) such that \( 0 \leq a_n \leq [a_n] \leq g_n, a_n g_n = g_n a_n = a_n \) and \( a_n g_m = g_m a_n = 0 \), if \( n \neq m \).

Any subsequence \( \{ e_{n_k} \} \) of \( \{ e_n \} \) is also an approximate identity satisfying (i) and (ii).

2. The results.

2.1. DEFINITION. Let \( A \) be a simple C*-algebra. Call \( A \) purely infinite if every two non-zero elements are \( \approx \)-equivalent. (This definition is weaker than [8, 2.3]).

It is clear that every simple C*-algebra in the case (1.1) (iii) is purely infinite.

2.2. LEMMA. Let \( f \) be a continuous function on \([-1, 1]\). For every \( \varepsilon > 0 \) there is a constant \( M > 0 \) such that for any two self-adjoint elements \( a \) and \( b \) in the unit ball of a C*-algebra \( A \),

\[
\| f(a) - f(b) \| \leq M \| a - b \| + \varepsilon
\]
Proof. For each integer $k$,
\[ \|a^{k+1} - b^{k+1}\| = \|a(a^k - b^k) + (a - b)b^k\| \leq \|a^k - b^k\| + \|a - b\|.\]
Thus we have
\[ \|a^k - b^k\| \leq k \|a - b\| \]
for all $k$. Therefore for every polynomial $p(t)$,
\[ \|p(a) - p(b)\| \leq M(p) \|a - b\| \]
where $M(p)$ is a constant depending only on $p$. By the Weierstrass approximation theorem, there is a polynomial $p$ such that
\[ \sup \{|f(t) - p(t)| \mid t \in [-1, 1]\} < \varepsilon/2. \]
Thus
\[ \|f(a) - f(b)\| \leq \|f(a) - p(a)\| + \|p(a) - p(b)\| + \|p(b) - f(b)\| < \varepsilon/2 + M(p) \|a - b\| + \varepsilon/2 = M(p) \|a - b\| + \varepsilon. \]

2.3. Theorem. Let $A$ be a $\sigma$-unital simple $C^*$-algebra. If $A$ is purely infinite then $M(A)/A$ is simple.

Proof. Suppose that $J$ is an ideal of $M(A)$ properly containing $A$. Choose a positive element $x$ in $J \setminus A$. Let $\{e_n\}$ and $\{g_n\}$ be as in 1.6. Passing to a subsequence if necessary, we may assume that
\[ \|(1 - e_{n+1})xe_n\| < 1/2^n \text{ and } \]
\[ \|e_nx(1 - e_{n+1})\| < 1/2^n \]
for all $n$. Then the elements
\[ \sum_{n=1}^{\infty} (1 - e_{n-1})xg_n, \sum_{n=1}^{\infty} g_nx(1 - e_{n+1}), \sum_{n=3}^{\infty} g_{n-2}xg_n \text{ and } \sum_{n=3}^{\infty} e_{n-2}xg_n \]
are in $A$. Therefore the element
\[ y = x - \sum_{n=1}^{\infty} (1 - e_{n+1})xg_n \]
is in $J \setminus A$. Since

$$y - \sum_{n=3}^{\infty} e_{n-2} x g_n = e_2 x g_1 + e_3 x g_2$$

$$+ \sum_{n=3}^{\infty} g_{n+1} x g_n + \sum_{n=3}^{\infty} g_n x g_n + \sum_{n=3}^{\infty} g_{n-1} x g_n,$$

one of the last three elements must be in $J \setminus A$. Suppose that $\sum_{n=3}^{\infty} g_{n+1} x g_n$ is in $J \setminus A$.

Since

$$\left[ \sum_{n=3}^{\infty} g_{n+1} x g_n \right] \ast \left[ \sum_{n=3}^{\infty} g_{n+1} x g_n \right]$$

$$= \sum_{n=3}^{\infty} g_n x g_{n+1}^2 x g_n + \sum_{n=3}^{\infty} g_n x g_{n+1} g_{n+2} x g_{n+1}$$

$$+ \sum_{n=4}^{\infty} g_n x g_{n+1} g_n x g_{n-1},$$

and

$$\| g_n x g_{n+1} g_{n+2} x g_n \| = \| g_n x g_{n+2} g_{n+1} x g_{n+1} \| \leq 1/2^n,$$

$$\| g_n x g_{n+1} g_n x g_{n-1} \| = \| g_n x g_{n+1} x g_{n-1} \| \leq 1/2^{n-1},$$

$$\sum_{n=3}^{\infty} g_n x g_{n+1} g_{n+2} x g_{n+1} + \sum_{n=4}^{\infty} g_n x g_{n+1} g_n x g_{n-1} \text{ is in } A.$$

Thus $\sum_{n=3}^{\infty} g_n x g_{n+1}^2 x g_n$ is in $J \setminus A$. Similarly, if $\sum_{n=3}^{\infty} g_{n-1} x g_n$ is in $J \setminus A$,

$$\sum_{n=3}^{\infty} g_n x g_{n-1}^2 x g_n \text{ is in } J \setminus A.$$ In either case, it follows that $\sum_{n=1}^{\infty} g_n x^2 g_n$ is in $J \setminus A$. By changing notation, we may therefore assume that $\sum_{n=1}^{\infty} g_n x g_n$ is in $J \setminus A$.

So $\sum_{k=1}^{\infty} g_{2k} x g_{2k}$ and $\sum_{k=1}^{\infty} g_{2k+1} x g_{2k+1}$ are in $J$ and one of them is in $J \setminus A$. We may assume that $y = \sum_{k=1}^{\infty} g_{2k} x g_{2k}$ is in $J \setminus A$. Hence for a sufficiently small $\delta > 0$,

$$f_\delta(y) \in J \setminus A.$$ Since $g_{2k} x g_{2k} \perp g_{2j} x g_{2j}$ if $k \neq j$,

$$f_\delta(y) = \sum_{k=1}^{\infty} f_\delta(g_{2k} x g_{2k}).$$

Without loss of generality, we may assume that

$$f_\delta(g_{2k} x g_{2k}) \neq 0 \text{ for each } k.$$
Then \( f_\delta(g_{2k}xg_{2k}) \approx g_k \) for each \( k \). Let \( M_k \) be the constant in Lemma 2.2 such that
\[
\|a^{1/2} - b^{1/2}\| \leq M_k \|a - b\| + 1/2^k
\]
for all \( a, b \in A, \|a\| \leq 1, \|b\| \leq 1, k = 1, 2, \ldots \). For every \( \varepsilon > 0 \) and \( k \), there is \( x_k \in K(A) \) such that
\[
x_k \leq f_\delta(g_{2k}xg_{2k}) \text{ and } \|x_k - g_k\| < \frac{\varepsilon}{2^k(M_k + 1)}
\]
We may assume that \( 0 \leq x_k \leq 1 \) for each \( k \). By \([1, 1.7]\) there is \( z_k \in A \) such that \( z_kz_k^* = x_k \) and \( z_k^*z_k \leq [f_\delta(g_{2k} \times g_{2k})] \leq f_\delta/2(g_{2k} \times g_{2k}) \). Hence \( z_kz_j^* = 0, \) if \( k \neq j \). So
\[
\left[ \sum_{k=1}^n z_k \right]\left[ \sum_{k=1}^n z_k^* \right]^* = \sum_{k=1}^n z_kz_k^* \text{ and } \left\| \sum_{k=1}^n z_kz_k^* \right\| \text{ is bounded. Thus } \left\{ \sum_{k=1}^n z_k \right\} \text{ is bounded. It is then easy to see that } \sum_{k=1}^n z_k \text{ converges in the right strict topology to an element } z = \sum_{k=1}^\infty z_k \text{ in the right multipliers } RM(A). \text{ To show that } \sum_{k=1}^n z_k \text{ converges strictly to } z, \text{ it is enough to show that for each } n, g_n \sum_{k=N}^\infty z_k \text{ converges (in norm) to zero as } N \to \infty. \text{ Write } z_k = (z_kz_k^*)^{1/2}u_k. \text{ Then}
\[
\left\| \sum_{k=N}^\infty z_k - \sum_{k=N}^\infty g_ku_k \right\| \leq \sum_{k=N}^\infty \|(z_kz_k^*)^{1/2} - g_k\| < \sum_{k=N}^\infty \varepsilon/2^k + \sum_{k=N}^\infty 1/2^k \to 0,
\]
as \( N \to \infty. \text{ Since } g_n \sum_{k=N}^\infty g_ku_k = 0, \text{ for } N > n + 1, \text{ we conclude that } \left\| g_n \sum_{k=N}^\infty z_k \right\| \to 0 \text{ as } N \to \infty. \text{ So } z \in M(A). \text{ From}
\[
zf_{\delta/2}(y) = \left[ \sum_{k=1}^\infty z_k \right]\left[ \sum_{k=1}^\infty f_{\delta/2}(g_{2k}xg_{2k}) \right] = \sum_{k=1}^\infty z_k,
\]
we conclude that \( z \in J. \text{ On the other hand,}
\[
\|zz^* - 1\| = \left\| \sum_{k=1}^\infty z_kz_k^* - \sum_{k=1}^\infty g_k \right\| < \varepsilon,
\]
so \( 1 \in J. \)

2.4. DEFINITION. Let \( A \) be a \( \sigma \)-unital, stably semi-finite, simple \( C^* \)-algebra. If \( A \) is not unital, fix a strictly positive element \( e \) and choose \( \{e_n\} \) as in 1.6. We define
\[
\hat{I}(d) = \lim_{n \to \infty} d(e_n) \text{ for every } d \in S_u(A(A))
\]
for some fixed \( u \in K(A)^+ \setminus \{0\}. \text{ We shall say that } A \text{ has a continuous dimension}
scale if \( \hat{1} \) is a continuous function on \( S = S_u(A(A)) \) for some strictly positive element \( e \). For convenience, we shall also say that every unital simple \( C^* \)-algebra has a continuous dimension scale. It is clear that the definition does not depend on the choice of \( u \). Later we shall see that \( \lim_{n \to \infty} d(e_n) \) is continuous for every approximate identity \( \{e_n\} \) as described in 1.6 if it is continuous for one of them.

We now fix \( u \in K(A)^+ \setminus \{0\} \), and a strictly positive element \( e \) and an approximate identity \( \{e_n\} \) as in 1.6.

2.5. For every \( a \in M(A)_+ \), we define

\[
\tilde{d}(a) = \sup \{d(b) \mid b \leq a, b \in A_e\},
\]

\( d \in S \). Then \( \hat{1}(d) = \tilde{d}(1) \) for every \( d \in S \). If \( a \in AA_e A \), then \( \langle a^* a \rangle = \langle a \rangle \) and \( a^* a \) has the form \( b^* x^* x b \) with \( b \in A \) and \( x \in A_e \). So there is \( c \in (A_e)^+ \) such that \( \langle c \rangle = \langle a \rangle \). Hence, if \( a \in AA_e A \),

\[
\tilde{d}(a) = d(a) \text{ for each } d \in S.
\]

2.6. Set

\[
I_0 = \{a \in M(A) \mid \exists a_n \in AA_e A \text{ such that } \tilde{d}((a - a_n)^*(a - a_n)) \to 0 \text{ uniformly on } S \}.
\]

Clearly \( I_0 \) is a \(*\)-invariant subset of \( M(A) \). Suppose that \( a, b \in I_0 \), and

\[
\tilde{d}((a - a_n)^*(a - a_n)) \to 0, \quad \tilde{d}((b - b_n)^*(b - b_n)) \to 0 \text{ uniformly on } S,
\]

where \( a_n, b_n \in AA_e A \). Since for each \( k \),

\[
e_k(a + b - a_n - b_n)^*(a + b - a_n - b_n)e_k
\]

\[
= e_k(a - a_n)^*(a - a_n)e_k + e_k(b - b_n)^*(b - b_n)e_k
\]

\[
+ e_k(b - b_n)^*(a - a_n)e_k + e_k(a - a_n)^*(b - b_n)e_k
\]

and

\[
d[e_k(b - b_n)^*(a - a_n)e_k]
\]

\[
\leq d[e_k(b - b_n)^*(b - b_n)e_k],
\]

we conclude that

\[
\tilde{d}((a + b - a_n - b_n)^*(a + b - a_n - b_n))
\]

\[
\leq 2[\tilde{d}((a - a_n)^*(a - a_n)) + \tilde{d}((b - b_n)^*(b - b_n))] \to 0
\]

uniformly on \( S \). Therefore \( I_0 \) is a \(*\)-invariant linear space. Suppose that \( b \in M(A) \), \( a \in I_0 \) and \( a_n \in AA_e A \) are such that

\[
\tilde{d}((a - a_n)^*(a - a_n)) \to 0 \text{ uniformly on } S.
\]
Then $ba_n \in AA_+A$ and
\[
\tilde{d}((ba - ba_n)^* (ba - ba_n)) = \tilde{d}((a - a_n)^* b^* b (a - a_n)) \leq \tilde{d}((a - a_n)^* (a - a_n)) \to 0
\]
uniformly on $S$.

So $I_0$ is an ideal of $M(A)$. We denote by $I$ the closure of $I_0$. Clearly, $I$ is a closed ideal of $M(A)$ containing $A$.

2.7. LEMMA. Let $A$ be a non-elementary, $\sigma$-unital, non-unital, stably semi-finite simple C*-algebra, and let $I$ be as defined in 2.6. Then $I$ contains $A$ properly.

PROOF. Clearly, $I$ contains $A$. Let $\{a_n\}$ be as in 1.6, and fix $n$. For each $k > 0$, as shown in [8, 2.7] there are $\varepsilon > 0$ and $h_1, \ldots, h_k \in K(A)_+$ such that $h_i \perp h_j$ for $i \neq j$,

\[
h_1 \preceq h_2 \preceq \ldots \preceq h_k \quad \text{and} \quad f_{\varepsilon}(a_n) \geq \sum_{i=1}^{k} h_i.
\]

Thus
\[
\langle f_{\varepsilon}(a_n) \rangle \geq \langle h_1 \rangle + \ldots + \langle h_k \rangle \geq k \langle h_k \rangle,
\]
whence $\tilde{d}(h_k) = d(h_k) \leq k^{-1} d[f_{\varepsilon}(a_n)] = k^{-1} \tilde{d}[f_{\varepsilon}(a_n)]$ for $d \in S$. We conclude that for each $n$, there is $x_n \in (A_\varepsilon)^+$ such that $\|x_n\| = 1, x_n \leq g_n$ and $\tilde{d}(x_n) = d(x_n) \leq 1/2^n$ for all $d \in S$. It is clear that $x = \sum_{n=1}^{\infty} x_n \in I \setminus A$.

2.8. THEOREM. Let $A$ be a $\sigma$-unital stably semi-finite simple C*-algebra. Then $M(A)/A$ is simple if and only if either $A$ has a continuous dimension scale or $A$ is elementary.

PROOF. Suppose that $M(A)/A$ is simple and $A$ is neither unital nor elementary. By 2.7, $1 \in I$. Thus there is $a \in I^+$ such that

\[
\|1 - a\| < 1/4.
\]

This implies $\tilde{d}(1) = \tilde{d}(a)$ so $\hat{1}(d) = \tilde{d}(1)$ is continuous on $S$ and $A$ has a continuous dimension scale.

If $A$ is elementary, it is well known that $M(A)/A$ is simple. Now suppose that $\hat{1}(d)$ is continuous on $S$. By Dini's theorem,

\[
\tilde{d}(1 - e_n) \to 0
\]

uniformly on $S$. Passing to a subsequence if necessary, we may assume that

\[
\tilde{d}(1 - e_n) < \frac{1}{2^n}
\]
uniformly on $S$. Therefore both $\sum_{n=1}^{\infty} d(g_{2n})$ and $\sum_{n=1}^{\infty} d(g_{2n+1})$ converge uniformly on $S$.

Suppose that $J$ is a closed ideal properly containing $A$. Choose $w \in J^+ \setminus A$. As in 2.3, we may assume that $y = \sum_{k=1}^{\infty} g_{2k}wg_{2k} \in J^+ \setminus A$. Therefore for a sufficiently small $\delta > 0$, $f_\delta(y) \in J^+ \setminus A$. Since $f_\delta(y) = \sum_{k=1}^{\infty} f_\delta(g_{2k}wg_{2k})$, we may assume that $f_\delta(g_{2k}wg_{2k}) = \pm 0$ for each $k$. Since $S$ is compact, then $\inf \{d(f_\delta(g_{2k}wg_{2k})) \mid d \in S\} > 0$ for each $k$. Choose an integer $n_0$ such that

$$\sum_{k > n_0} d(g_{2k+1}) < \inf \{d(f_\delta(g_{2w}g_{2})) \mid d \in S\}$$

for all $d \in S$. Since $\inf \{d(f_\delta(g_{2k}wg_{2k})) \mid d \in S\} > 0$ for each $k$, we can find a partition of the set $\{n_0 + 1, n_0 + 2, \ldots\}$ into finite subsets $N_1, N_2 \ldots$ (of consecutive integers) such that for each $n = 1, 2, \ldots$,

$$\sum_{k \in N_n} d(g_{2k+1}) < d(f_\delta(g_{2n}wg_{2n}))$$

for all $d \in S$. Then by 1.5 and 1.3,

$$\sum_{k \in N_n} g_{2k+1} \leq f_\delta(g_{2n}wg_{2n}) \text{ in } K(A).$$

For any $\varepsilon > 0$, there are $x_n \in K(A)$ such that

$$x_n \not\leq f_\delta(g_{2n}wg_{2n}) \text{ and } \left\|x_n - \sum_{k \in N_n} g_{2k+1}\right\| < \varepsilon/2^n.$$}

We may assume that $0 \leq x_n \leq 1$. It follows from [1, 1.7] that there are $z_n \in A$ such that

$$z_nx^*_n = x_n \text{ and } z^*_nz_n \leq [f_\delta(g_{2n}wg_{2n})] \leq f_\delta(g_{2n}wg_{2n}).$$

As in 2.3, this implies that $z = \sum_{n=1}^{\infty} z_n$ is in $J$ and

$$zz^* = \sum_{n=1}^{\infty} z_nz^*_n.$$}

Hence

$$\left\|zz^* - \sum_{k > n_0} g_{2k+1}\right\| < \varepsilon.$$
Therefore $\sum_{k > n_0} g_{2k+1}$ is in $J$, and hence so is $\sum_{k = 0}^{\infty} g_{2k+1}$. Similarly $\sum_{k = 1}^{\infty} g_{2k}$ is in $J$. Hence $1 \in J$.

2.9. From 2.8, together with its proof, we see that if $A$ has a continuous dimension scale then for any $e$ and $\{e_n\}$ in 1.6, $\int(d) = \lim_{n \to \infty} d(e_n)$ is continuous.

Choose a separable, algebraically simple AF $C^*$-algebra $A$ such that $M(A)/A$ is not simple, or equivalently $A$ has no continuous scale (see [7]). By 2.8, $\int(d)$ is not continuous. Since $e \in K(A)$, $d(e)$ is continuous on $S$. So

$$\sup\{d(a) \mid a \in A_e\} \neq d(e) \text{ for some } d \in S.$$ 

If $d$ is lower semi-continuous, then it is easy to check that

$$\sup\{d(a) \mid a \in A_e\} = d(e)$$

Thus we conclude the following:

2.10. COROLLARY. If $A$ is a separable algebraically simple AF $C^*$-algebra without continuous scale then there is at least one dimension function $d$ on $A$ which is not lower semi-continuous. Consequently, $d$ is not determined by a trace.

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REFERENCES