QUASIHYPERBOLIC GEODESICS IN JOHN DOMAINS

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1. Introduction.

Suppose that $D$ is a proper subdomain of euclidean $n$-space $\mathbb{R}^n$. The quasihyperbolic length of an arc $\gamma$ in $D$ is defined as

$$k_D(\gamma) = \int_{\gamma} d(x, \partial D)^{-1} \, ds,$$

where $d(x, \partial D)$ denotes the euclidean distance from $x$ to $\partial D$. Next the quasihyperbolic distance between two points $x_1, x_2$ in $D$ is given by

$$k_D(x_1, x_2) = \inf_{\gamma} k_D(\gamma),$$

where the infimum is taken over all rectifiable arcs $\gamma$ joining $x_1$ to $x_2$ in $D$. A quasihyperbolic geodesic is an arc $\gamma$ for which the infimum in (1.2) is attained; see [GO], [GP] and [M].

Suppose that $x_0, x_1 \in D$ and that $b \geq 1$. A rectifiable arc $\gamma$ is said to be a $b$-cone arc from $x_1$ to $x_0$ if $\gamma$ joins $x_1$ to $x_0$ in $D$ and if

$$l(\gamma(x_1, x)) \leq b \, d(x, \partial D)$$

for all $x \in \gamma$; here $\gamma(x_1, x)$ denotes the subarc of $\gamma$ between $x_1$ and $x$ and $l(\alpha)$ the euclidean length of an arc $\alpha$. The domain $D$ is then said to be a $b$-John domain with center $x_0$ if for each $x_1 \in D$ there is a $b$-cone arc from $x_1$ to $x_0$. Inequality (1.3) implies that $D$ contains the (curvilinear) $b$-cone

$$\text{Cone}(\gamma, b; x_0) = \bigcup_{x \in \gamma} B \left( x, \frac{1}{b} \, l(\gamma(x_1, x)) \right),$$

* This research was supported in part by the National Science Foundation, Grant DMS-87-02356.

Received October 7, 1988
with axis $\gamma$, vertex $x_1$ and center $x_0$; here $B(x, r)$ denotes the open $n$-ball with center $x$ and radius $r$. If $\gamma$ is the closed segment $[x_1, x_0]$, then $\text{Cone}(\gamma, b; x_0)$ is the union of a finite euclidean cone with vertex angle $\theta = \arcsin \left( \frac{1}{b} \right)$ at $x_1$ and a ball about $x_0$.

A rectifiable arc $\gamma$ is said to be a double $b$-cone arc from $x_1$ to $x_2$ if $\gamma$ joins $x_1$ to $x_2$ in $D$ and if

$$l(\gamma) \leq b|x_1 - x_2|, \quad \min (l(\gamma(x_1, x)), l(\gamma(x, x_2))) \leq bd(x, \partial D)$$

for all $x \in \gamma$. The domain $D$ is said to be $b$-uniform if for each $x_1, x_2 \in D$ there exists a double $b$-cone arc from $x_1$ to $x_2$. Inequality (1.5) implies that $D$ contains the double cone

$$\text{Cone}(\gamma_1, b; x_0) \cup \text{Cone}(\gamma_2, b; x_0)$$

where $x_0$ denotes the midpoint of $\gamma$ and $\gamma_j = \gamma(x_j, x_0)$ for $j = 1, 2$.

The classes of John and uniform domains described above are closely related. For example, $D$ is a $b$-John domain if and only if all of its points are the vertices of $b$-cones in $D$ with a common center; $D$ is $b$-uniform if an only if each pair of its points are the vertices of two $b$-cones in $D$ with a common center for which the axis length sum does not exceed $b$ times the distance between the vertices. In particular, if $D$ is $b$-uniform, then each pair of its points lie in the closure of a $b$-John subdomain of $D$. Moreover, every bounded uniform domain is a John domain [GM].

If $D$ is $c$-uniform and if $\gamma$ is a quasihyperbolic geodesic which joins $x_1$ and $x_2$ in $D$, then $\gamma$ is a double cone arc with $b = b(c)$ [GO]. It is natural to ask if this result has a counterpart for John domains. In particular, suppose that $D$ is a $c$-John domain with center $x_0$ and that $\gamma$ is a quasihyperbolic geodesic which joins $x_1$ to $x_0$. Is $\gamma$ a $b$-cone arc for some $b = b(c)$? The purpose of this paper is to show that the answer is yes when $n = 2$ and $D$ is simply connected, and in general no when $n > 2$ or $D$ is multiply connected. We establish these assertions in Sections 4 and 5. Section 4 also contains a new characterization for simply connected John domains in $\mathbb{R}^2$. In Section 3 we exhibit two criteria which are necessary and sufficient for a quasihyperbolic geodesic $\gamma$ to satisfy the cone condition (1.3). Section 2 contains estimates for the quasihyperbolic distance and a key lemma on the location of a quasihyperbolic geodesic in a simply connected plane domain.

2. Estimates for the quasihyperbolic distance.

We derive here three estimates for the quasihyperbolic distance in a proper subdomain $D$ of $\mathbb{R}^n$ which will be needed in the remainder of this paper.
2.1 Lemma. Suppose that \( x_1, x_2 \) are points in \( D \) and that \( d_1 = d(x_1, \partial D) \), \( d_2 = d(x_2, \partial D) \), \( t = |x_1 - x_2| \). If \( t < d_1 + d_2 \), then

\[
k_D(x_1, x_2) \leq \log \frac{d_1 + d_2 + t}{d_1 + d_2 - t}.
\]

This bound is sharp. If \( t \leq d_2 \), then

\[
k_D(x_1, x_2) \leq \log \left( 1 + \frac{2t}{d_1} \right).
\]

Proof. Let \( \alpha = [x_1, x_2] \) and \( B_j = B(x_j, d_j) \) for \( j = 1, 2 \). The triangle inequality implies that \( d_1 \leq d_2 + t \) and \( d_2 \leq d_1 + t \). Then by making a preliminary change of variables, we may assume that \( 0, x_1, x_2 \) lie in a line \( \lambda \) and that

\[
d_1^2 - |x_1|^2 = d_2^2 - |x_2|^2 = d^2.
\]

Since \( B_1 \cup B_2 \subset D \),

\[
d(x, \partial D)^2 \geq d(x, \partial (B_1 \cup B_2))^2 = d^2 + |x|^2
\]

for \( x \in \alpha \).

Suppose that \( \lambda \) is parametrized with respect to arclength \( s \) with \( \lambda(0) = 0 \), \( \lambda(s_j) = x_j \) for \( j = 1, 2 \) and \( s_2 > 0 \); by relabeling we may assume that \( s_1 < s_2 \). Then \( t = s_2 - s_1 \) and we obtain

\[
k_D(x_1, x_2) \leq \int_\alpha (d^2 + |x|^2)^{-1/2} ds
\]

\[
= \log \frac{d_2 + s_2}{d_1 + s_1}
\]

\[
= \log \frac{d_1 + d_2 + t}{d_1 + d_2 - t}
\]

from integration and (2.4).

Next if \( D = B_1 \cup B_2 \) and if \( \gamma \) is any arc joining \( x_1 \) and \( x_2 \) in \( D \), then

\[
d(x, \partial D)^2 \leq d^2 + |x|^2
\]

for \( x \in \gamma \) and we obtain equality in (2.2). Finally (2.2) implies (2.3) whenever \( t \leq d_2 \).

2.6. Lemma. Suppose that \( \gamma \) is an arc which joins points \( x_1, x_2 \) in \( D \) and that \( d_1 = d(x_1, \partial D) \), \( d_2 = d(x_2, \partial D) \), \( l = l(\gamma) \). Then

\[
k_D(\gamma) \geq \log \frac{(d_1 + d_2 + l)^2}{4d_1d_2}.
\]
This bound is sharp. In particular,

\[(2.8) \quad k_D(\gamma) \leq \log \left(1 + \frac{l}{d_1}\right).\]

**Proof.** If \( \gamma \) is parametrized by arclength \( s \) with \( \gamma(0) = x_1 \), then

\[d(x, \partial D) \leq d_1 + s, \quad d(x, \partial D) \leq d_2 + l - s\]

for \( x \in \gamma \). Hence \( r = \frac{1}{2}(l + d_2 - d_1) \in [0, l] \) and we obtain (2.7) from

\[k_D(\gamma) = \int_0^r d(x, \partial D)^{-1} ds \]

\[\geq \int_0^r (d_1 + s)^{-1} ds + \int_r^l (d_2 + l - s)^{-1} ds \]

\[= \log \frac{(d_1 + d_2 + l)^2}{4d_1d_2}.\]

Equality holds if \( x_1 \) and \( x_2 \) are points in an open subinterval \( \beta \) of a line \( \lambda \), \( \gamma = [x_1, x_2] \) and \( D = (\mathbb{R}^n \setminus \lambda) \cup \beta \). Finally (2.8) follows from (2.7) and the fact that \( d_2 \leq d_1 + l \).

Our third estimate concerns the location of an arc which is a geodesic for either the quasihyperbolic or hyperbolic metric in a simply connected proper subdomain \( D \) of \( \mathbb{R}^2 \). For each \( x \in \mathbb{R}^2 \) we let \( C(x, r) \) denote the circle with center \( x \) and radius \( r \).

2.9. **Lemma.** Suppose that \( D \) is a simply connected proper subdomain of \( \mathbb{R}^2 \), that \( \gamma \) is a quasihyperbolic or hyperbolic geodesic in \( D \) and that \( x_1, x_0, x_2 \) is an ordered triple of points in \( \gamma \) with \( |x_1 - x_0| = |x_2 - x_0| = r \). If \( D \) contains a component of \( C(x_0, r) \setminus \{x_1, x_2\} \), then

\[(2.10) \quad r \leq a d(x_0, \partial D)\]

where \( a \) is an absolute constant.

**Proof of Lemma 2.9 for the Quasihyperbolic Case.** Suppose that \( \gamma \) is a quasihyperbolic geodesic in \( D \). By performing a preliminary similarity mapping we may assume that \( x_0 = 0 \) and that \( d(0, \partial D) = 1 \). Next by hypothesis, \( C(0, r) \setminus \{x_1, x_2\} \) has a component \( C \) which joins \( x_1 \) and \( x_2 \) in \( D \); by replacing \( \gamma \) and \( C \) by subarcs if necessary, we may assume that \( \gamma \) and \( \bar{C} \) meet just at the points \( x_1 \) and \( x_2 \) and hence bound a Jordan domain \( G \) which lies in \( D \).

Let \( \gamma_j = \gamma(x_j, 0) \) for \( j = 1, 2 \). Then \( C(0, \frac{3r}{4}) \cap G \) has a component \( \bar{C} \) which
joins \( y_1 \in \gamma_1 \) to \( y_2 \in \gamma_2 \) in \( G \). Let
\[
E_1 = \left\{ x \in \bar{C} : d(x, \gamma_1) \leq \min \left( \frac{r}{4}, d(x, \gamma_2) \right) \right\},
\]
\[
E_2 = \left\{ x \in \bar{C} : d(x, \gamma_2) \leq \min \left( \frac{r}{4}, d(x, \gamma_1) \right) \right\}.
\]
(2.11)

Then \( E_1 \) and \( E_2 \) are relatively closed subsets of the open arc \( \bar{C} \) with \( y_1 \in \bar{E}_1 \setminus \bar{E}_2 \) and \( y_2 \in \bar{E}_2 \setminus \bar{E}_1 \). Suppose that \( x \in E_1 \cap E_2 \). Then (2.11) implies that
\[
d = d(x, \gamma_1) = d(x, \gamma_2) \leq \frac{r}{4}
\]
and since \( |x| = \frac{3r}{4} \), the disk \( \bar{B}(x, d) \) lies in \( D \), meets both \( \gamma_1 \) and \( \gamma_2 \) but does not contain 0. Hence \( \bar{B}(x, d) \cap \gamma \) is not connected and we have a contradiction to Theorem 2.2 in [M]. Thus \( E_1 \cap E_2 = \emptyset \) and it follows that \( \bar{C} \setminus (E_1 \cup E_2) \) contains an open subarc \( \alpha \) with endpoints \( z_1 \in E_1 \) and \( z_2 \in E_2 \). Moreover, we see from (2.11) that
\[
d(x, \gamma_1 \cup \gamma_2) \geq \frac{r}{4}, \quad d(x, \partial D) \geq d(x, \partial G) \geq \frac{r}{4}
\]
(2.12)
for \( x \in \bar{C} \) and that \( d(z_1, \gamma_1) = d(z_2, \gamma_2) = \frac{r}{4} \). Thus we can choose points \( w_1 \in \gamma_1 \) and \( w_2 \in \gamma_2 \) such that
\[
|z_1 - w_1| = |z_2 - w_2| = \frac{r}{4}.
\]
(2.13)

We now apply Lemmas 2.1 and 2.6 to obtain upper and lower bounds for \( k_D(w_1, w_2) \) involving \( r \). Let \( d_j = d(w_j, \partial D) \) for \( j = 1, 2 \). Since
\[
d(z_j, \partial D) \geq \frac{r}{4},
\]
(2.13) and Lemma 2.1 imply that
\[
k_D(w_j, z_j) \leq \log \left( 1 + \frac{r}{2d_j} \right)
\]
and hence with (2.12) that
\[
k_D(w_1, w_2) \leq k_D(w_1, z_1) + k_D(w_2, z_2) + k_D(z_1, z_2)
\]
(2.14)
\[
\leq \log \left( 1 + \frac{r}{2d_1} \right) + \log \left( 1 + \frac{r}{2d_2} \right) + 6\pi.
\]
Next \( d(0, \partial D) = 1 \) and
\[
l_j = l(\gamma(w_j, 0)) \geq |w_j| \geq |z_j| - |w_j - z_j| = \frac{r}{2}
\]
for \( j = 1, 2 \). Since \( \gamma \) is a quasihyperbolic geodesic,
\[
k_D(w_j, 0) \geq \log \left( \frac{(d_j + 1 + l_j)^2}{4d_j} \right) \geq \log \left( 1 + \frac{r}{2d_j} \right) + \log \frac{r}{8}
\]
by Lemma 2.6 and we obtain
\[
k_D(w_1, w_2) = k_D(w_1, 0) + k_D(w_2, 0)
\]
(2.15)
\[
\geq \log \left( 1 + \frac{r}{2d_1} \right) + \log \left( 1 + \frac{r}{2d_2} \right) + 2 \log \frac{r}{8}.
\]
Inequalities (2.14) and (2.15) then imply (2.10) with \( a = 8e^{3\pi} \) completing the proof for the quasihyperbolic case.

The proof for the hyperbolic case follows directly from the following result.

2.16. Lemma. Suppose that \( D \) is a simply connected proper subdomain of \( \mathbb{R}^2 \) and that \( \gamma \) is a hyperbolic geodesic joining \( x_1 \) and \( x_2 \) in \( D \). For each \( x_0 \in \gamma \setminus \{x_1, x_2\} \) there exists a crosscut \( \alpha \) of \( D \) containing \( x_0 \) which separates the components of \( \gamma \setminus \{x_0\} \) in \( D \) and satisfies
(2.17) \[
l(\alpha) \leq c \, d(x_0, \partial D)
\]
where \( c \) is an absolute constant.

Proof of Lemma 2.16. Let \( f \) be a conformal mapping of the unit disk \( B \) onto \( D \) normalized so that \( y_j = f^{-1}(x_j) \) are points of the real axis \( L \) and \( y_0 = 0 \). Next let \( C_1 \) and \( C_2 \) denote the components of \( \partial B \setminus L \). By Corollary 10.3 in [P1] we can choose for \( j = 1, 2 \) an open segment \( \beta_j \) joining 0 to \( C_j \) such that
\[
l(f(\beta_j)) \leq \frac{c}{2} \, d(f(0), \partial D) = \frac{c}{2} \, d(x_0, \partial D),
\]
where \( c \) is an absolute constant. Then \( \alpha = f(\beta_1 \cup \{0\} \cup \beta_2) \) is a crosscut of \( D \) with the desired properties.

Proof of Lemma 2.9 for the Hyperbolic Case. Suppose now that \( \gamma \) is a hyperbolic geodesic in \( D \), let \( C \) denote the component of \( C(x_0, r) \setminus \{x_1, x_2\} \) which joins \( x_1 \) and \( x_2 \) in \( D \) and let \( \alpha \) be the crosscut described in Lemma 2.16. Since \( \alpha \) separates \( x_1 \) and \( x_2 \), \( \alpha \) must join \( x_0 \) and \( C \) in \( D \). Hence
(2.18) \[
r \leq l(\alpha)
\]
and we obtain (2.10) with \( a = c \) from (2.17) and (2.18).
3. Quasihyperbolic geodesics as cone arcs.

Suppose that $D$ is a proper subdomain of $\mathbb{R}^n$. We derive in this section two criteria for a quasihyperbolic geodesic $\gamma$ in $D$ to be a cone arc. We begin with the following preliminary result.

3.1. LEMMA. Suppose that $\gamma$ is a rectifiable arc which joins $x_1$ to $x_0$ in $D$ and that $c \geq 1$. If

$$k_D(\gamma(y_1, y_2)) \leq c \log \left(1 + \frac{|y_1 - y_2|}{d(y_1, \partial D)}\right)$$

for all $y_1, y_2$ in $\gamma$ with $y_1$ between $x_1$ and $y_2$, then $\gamma$ is a $b$-cone arc where $b$ depends only on $c$ and $a$,

$$a = \sup_{y \in \gamma} \frac{d(y, \partial D)}{d(x_0, \partial D)} < \infty.$$

PROOF. We define inductively a sequence of points $y_1, \ldots, y_{m+1}$ in $\gamma$ as follows. Set $y_1 = x_1$, suppose that $y_j$ has been defined for some $j \geq 1$ and set $d_j = d(y_j, \partial D)$. If $d(x_0, \partial D) \geq 2d_j$, let $y_{j+1}$ denote the first point of $\gamma(y_j, x_0)$ for which

$$d_{j+1} = d(y_{j+1}, \partial D) = 2d_j$$

as we traverse $\gamma$ from $y_j$ towards $x_0$; otherwise set $y_{j+1} = x_0$ and $m = j$. Next let $\gamma_j = \gamma(y_j, y_{j+1})$ and $l_j = l(\gamma_j)$. If $x \in \gamma_j$, then

$$d(x, \partial D) \leq 2d_j$$

if $j = 1, \ldots, m - 1$ and

$$d(x, \partial D) \leq a d(x_0, \partial D) \leq 2ad_m$$

if $j = m$; hence

$$\frac{l_j}{d_j} \leq 2a \int_{\gamma_j} d(x, \partial D)^{-1} \, ds = 2a k_D(\gamma_j)$$

for $j = 1, \ldots, m$. Next (3.2) implies that

$$k_D(\gamma_j) \leq c \log \left(1 + \frac{l_j}{d_j}\right) \leq c \left(\frac{l_j}{d_j}\right)^{1/2}$$

and we conclude that

$$l_j \leq (2ac)^2 d_j$$
for all $j$.

Now fix $x \in \gamma$. Then $x \in \gamma_j$ for some $j \leq m$ and

$$\log \frac{d_j}{d(x, \partial D)} \leq k_D(y_j, x) \leq k_D(\gamma_j) \leq 2ac^2$$

by Lemma 2.6 or Lemma 2.1 of [GP], (3.6) and (3.7). Hence by (3.7), (3.4) and (3.8),

$$k(\gamma(x_1, x_2)) \leq \sum_{i=1}^{j} l_i \leq (2ac)^2 \sum_{i=1}^{j} d_i \leq (2ac)^2 \sum_{i=1}^{j} 2^{i-j} d_j \leq 8(ac)^2 d_j \leq b(d(x, \partial D))$$

where $b = 8(ac)^2 e^{2ac^2}$. This is the desired inequality (1.3).

Condition (3.2) allows us to characterize the quasihyperbolic geodesics which are cone arcs.

3.9 THEOREM. Suppose that $\gamma$ is a quasihyperbolic geodesic joining $x_1$ to $x_0$ in $D$. If $\gamma$ satisfies (3.2), then $\gamma$ is a $b$-cone arc where $b$ depends only on $c$ in (3.2) and $a$ in (3.3). Conversely, if $\gamma$ is a $b$-cone arc, then $\gamma$ satisfies (3.2) where $c$ depends only on $b$.

PROOF. The sufficiency is an immediate consequence of Lemma 3.1. For the necessity, since $\gamma$ is a quasihyperbolic geodesic, it suffices to show there exists a constant $c$ such that

$$k_D(y_1, y_2) \leq c \log \left(1 + \frac{|y_1 - y_2|}{d(y_1, \partial D)}\right)$$

for all $y_1, y_2 \in \gamma$ with $y_1 \in \gamma(x_1, y_2)$.

Fix $y_1, y_2 \in \gamma$ and let $d = d(y_1, \partial D), t = |y_1 - y_2|, l = l(\gamma(y_1, y_2))$. If $t \leq \frac{d}{2}$, then $d(y_2, \partial D) \geq t$ and

$$k_D(y_1, y_2) \leq \log \left(1 + \frac{2t}{d}\right) \leq 2 \log \left(1 + \frac{t}{d}\right)$$

by Lemma 2.1; this is the required inequality (3.10) with $c = 2$. If $t > \frac{d}{2}$, choose $y \in \gamma$ so that $l(\gamma(y_1, y)) = \frac{d}{2}$. Then $|y_1 - y| \leq \frac{d}{2}$ and

$$k_D(y_1, y) \leq \log 2$$
by (3.11). Next if \( \gamma \) is parametrized by arclength \( s \) with \( \gamma(0) = y_1 \), then for each \( x \in \gamma(y_1, y_2) \)

\[
s \leq l(\gamma(x_1, x)) \leq b d(x, \partial D)
\]

whence

\[
(3.13) \quad k_D(y, y_2) = \int_{y(y, y_2)} d(x, \partial D)^{-1} ds \leq b \int_{d/2}^{l} s^{-1} ds = b \log \frac{2l}{d}
\]

by (1.3). Finally

\[
l \leq l(\gamma(x_1, y_2)) \leq b d(y_2, \partial D) \leq b(d(y_1, \partial D) + |y_1 - y_2|) = b(t + d)
\]

by (1.3), and since \( b > 1 \),

\[
k_D(y_1, y_2) \leq \log 2 + b \log (2b) + b \log \left(1 + \frac{t}{d}\right)
\]

\[
\leq 2b \log (2b) + b \log \left(1 + \frac{t}{d}\right)
\]

\[
\leq \left(\frac{2b \log (2b)}{\log (3/2)} + b\right) \log \left(1 + \frac{t}{d}\right)
\]

by (3.12) and (3.13). Thus again we obtain inequality (3.10) with \( c = c(b) \) and the proof for Theorem 3.9 is complete.

We derive next a second criterion for a quasihyperbolic geodesic \( \gamma \) joining \( x_1 \) to \( x_0 \) in \( D \) to be a cone arc. In this case, inequality (3.2) is replaced by an engulfing condition, namely that for some constant \( c \geq 1 \),

\[
(3.14) \quad \gamma(x_1, x) \subset \overline{B}(x, c d(x, \partial D))
\]

for all \( x \in \gamma \).

3.15. Remark. It follows from [MS, pp. 385–386] that \( D \) is a John domain with center \( x_0 \) if and only if for each \( x_1 \in D \) there exists an arc \( \gamma \) from \( x_1 \) to \( x_0 \) which satisfies (3.14) for some constant \( c = c(D) \). Thus condition (3.14) characterizes John domains. However, an arbitrary arc \( \gamma \) which satisfies (3.14) need not be a \( b \)-cone arc with \( b = b(c) \).

3.16 Theorem. Suppose that \( \gamma \) is a quasihyperbolic geodesic joining \( x_1 \) to \( x_0 \) in \( D \). If \( \gamma \) satisfies (3.14), then \( \gamma \) is a \( b \)-cone arc where \( b \) depends only on \( c \) and \( n \). Conversely, if \( \gamma \) is a \( b \)-cone arc, then \( \gamma \) satisfies (3.14) where \( c = b \).

Proof. The necessity is an immediate consequence of inequality (1.3). For the sufficiency we again define inductively a sequence of points \( y_1, \ldots, y_{m+1} \) in \( \gamma \). Set
\( y_1 = x_1, \) suppose that \( y_j \) has been defined for some \( j \geq 1 \) and set \( d_j = d(y_j, \partial D). \) If

\[
|x_0 - y_j| \geq \frac{1}{2} d_j,
\]

let \( y_{j+1} \) denote the last point of \( \gamma(y_j, x_0) \) for which

\[
|y_{j+1} - y_j| = \frac{1}{2} d_j
\]
as we traverse \( \gamma \) from \( y_j \) towards \( x_0; \) otherwise let \( y_{j+1} = x_0 \) and \( m = j. \)

Now set \( \gamma_j = \gamma(y_j, y_{j+1}) \) and \( l_j = l(\gamma_j). \) If \( B \) is any ball with \( \overline{B} \subset D, \) then \( \overline{B} \cap \gamma \) is connected by Theorem 2.2 in [M] because \( \gamma \) is a quasihyperbolic. Hence it follows that

\[
\gamma_j \subset \overline{B}(y_j, \frac{1}{4} d_j)
\]
for \( j = 1, \ldots, m \) and that

\[
|y_k - y_j| \geq \frac{1}{2} d_j
\]
for \( 1 \leq j < k \leq m. \)

Since \( |y_j - y_{j+1}| \leq \frac{1}{2} d_j, \)

\[
t_d(y_i, y_{i+1}) \leq \log 2
\]
by Lemma 2.1 while

\[
\log \left( 1 + \frac{l_j}{d_j} \right) \leq k_d(\gamma_j)
\]
by Lemma 2.6. Because \( \gamma_j \) is a quasihyperbolic geodesic, these inequalities imply that \( l_j \leq d_j, \) and with (3.14) we conclude that

\[
l_j \leq d_j \leq (c + 1)d_k
\]
for \( 1 \leq j \leq k \leq m. \)

Choose an integer \( p = p(c, n) \) so that \( 8^{-m}p > (c + 1)^n. \) Observe that if \( m > p, \) then for each \( j \in (p, m] \) there exists an integer \( j \) such that

\[
1 \leq j - \tilde{j} \leq p, \quad d_j \leq \frac{1}{2} d_j.
\]
For if this were not the case we would have

\[
d_k > \frac{1}{4} d_j
\]
for \( j - p \leq k < j. \) Then the balls \( B_k = B(y_k, \frac{1}{8} d_j) \) would be disjoint by (3.18) and (3.23), they would lie in \( B = B(y_j, (c + 1)d_j) \) by (3.14), and we would obtain

\[
p\Omega_n(\frac{1}{8} d_j)^n = \sum m(B_k) \leq m(B) = \Omega_n((c + 1)d_j)^n
\]
contradicting our choice of the integer \( p. \)

Now fix \( x \in \gamma. \) Then \( x \in \gamma_j \) for some integer \( j \leq m. \) Next we can use inequality
(3.22) to define inductively a decreasing sequence of integers \( j_1, \ldots, j_{q+1} \) with \( j_1 = j \) and \( j_{q+1} = 0 \) such that

\[
1 \leq j_k - j_{k+1} \leq p, \quad d_{j_k} \leq 2^{1-k}d_j
\]

for \( k = 1, \ldots, q \). Then

\[
\ell(\gamma(x_1, x)) \leq \sum_{k=1}^{q} (l_{j_k} + \ldots + l_{j_{k+1}+1})
\]

(3.25)

\[
\leq \sum_{k=1}^{q} (j_k - j_{k+1})(c + 1)d_{j_k}
\]

\[
\leq 2p(c + 1)d_j
\]

by (3.21) and (3.24). Finally \( x \in \bar{B}(y_j, \frac{1}{4}d_j) \) by (3.17). Hence

(3.26)

\[
d(x, \partial D) \geq \frac{1}{4}d_j
\]

and we obtain (1.3) with \( b = 4p(c + 1) \) from (3.25) and (3.26). This completes the proof of Theorem 3.16.

We require the following hyperbolic analogue of Theorem 3.16 in what follows.

3.27. THEOREM. Suppose that \( D \) is a simply connected domain in \( \mathbb{R}^2 \) and that \( \gamma \) is a hyperbolic geodesic joining \( x_1 \) to \( x_0 \) in \( D \). If \( \gamma \) satisfies (3.14), then \( \gamma \) is a \( b \)-cone arc where \( b \) depends only on \( c \). Conversely, if \( \gamma \) is a \( b \)-cone arc, then \( \gamma \) satisfies (3.14) where \( c = b \).

PROOF. The necessity is clear. For the sufficiency we define the points \( y_1, \ldots, y_{m+1} \) in \( \gamma \) as in the proof for Theorem 3.16. If \( B \) is any disk with \( \bar{B} \subset D \), then \( \bar{B} \cap \gamma \) is connected by Theorem 2 in [J]; hence (3.17) and (3.18) hold as above. Next since \( D \) is simply connected, the Schwarz lemma and Koebe distortion theorem imply that

(3.28)

\[
\frac{1}{4}d(x, \partial D)^{-1} \leq \rho_D(x) \leq d(x, \partial D)^{-1}
\]

where \( \rho_D \) is the hyperbolic density in \( D \). Thus for \( 1 \leq j \leq m \),

\[
h_D(y_j, y_{j+1}) \leq k_D(y_j, y_{j+1}) \leq \log 2
\]

and

\[
\frac{1}{4} \log \left( 1 + \frac{l_j}{d_j} \right) \leq \frac{1}{4} k_D(\gamma_j) \leq h_D(\gamma_j)
\]

by (3.19), (3.20) and (3.28). Hence \( l_j \leq 15d_j \),

(3.29)

\[
l_j \leq 15d_j \leq 15(c + 1)d_k
\]

for \( 1 \leq j \leq k \leq m \) and the proof concludes as above with (3.29) in place of (3.21).
4. Simply connected John domains in $\mathbb{R}^2$.

We show next that quasihyperbolic and hyperbolic geodesics in a simply connected John domain $D$ in $\mathbb{R}^2$ satisfy the cone condition (1.3).

4.1. Theorem. Suppose that $D$ is a simply connected c-John domain in $\mathbb{R}^2$ with center $x_0$ and that $x_1$ is a point in $D$. If $\gamma$ is either a quasihyperbolic or hyperbolic geodesic from $x_1$ to $x_0$ in $D$, then $\gamma$ is a $b$-cone arc where $b$ depends only on $c$.

Proof. Let $a$ denote the absolute constant in Lemma 2.9. By Theorems 3.16 and 3.27, it is sufficient to show that $\gamma$ satisfies the engulfing condition

$$\gamma(x_1, x) \subset B(x, (a + 2)(2c + 1)d(x, \partial D))$$

for all $x \in \gamma$.

Suppose that (4.2) does not hold for some $x \in \gamma$ and let $d = d(x, \partial D)$ and $r = (a + 1)d$. Then there exists a point $z_1 \in \gamma(x_1, x)$ such that

$$\frac{(a + 2)(2c + 1)d}{2c} < |z_1 - x| \leq \text{dia}(D),$$

and since $D$ is a c-John domain with center $x_0$, we see that

$$|x_0 - x| \geq d(x_0, \partial D) - d(x, \partial D) \geq \frac{\text{dia}(D)}{2c} - d > (a + 1)d = r.$$  

Thus $x_0$ and $x$ are separated by $C(x, r)$. Then since $d < r$ and since $D$ is simply connected, $C(x, r) \setminus D \neq \emptyset$ and there exists an open subarc $C$ of $C(x, r) \cap D$ which separates $x_0$ and $x$ in $D$. (See, for example, Theorem VI.7.1 in [N]). In particular, there exists a point $y_0 \in \gamma(x_0, x) \cap C$.

Suppose next that $\gamma(x_1, x) \cap C = \emptyset$ and let $z_1$ be as in (4.3). By hypothesis there exists a $c$-cone arc $\beta$ joining $z_1$ to $x_0$ in $D$ which must intersect $C$ at some point $z$. With (4.3) we obtain

$$\text{dia}(C) \geq d(z, \partial D) \geq \frac{1}{c} \ell(\beta(z_1, z)) \geq \frac{1}{c} |z_1 - z| \geq \frac{1}{c} (|z_1 - x| - |z - x|) > 2r,$$

contradicting the fact that $C$ is a subarc of $C(x, r)$. We conclude that there exists a point $y_1 \in \gamma(x_1, x) \cap C$.

Now $y_0, x, y_1$ is an ordered triple of points on $\gamma$, $|y_0 - x| = |y_1 - x| = r$ and $C(x, r)$ contains a subarc which joins $y_0$ and $y_1$ in $D$. Hence Lemma 2.9 implies that

$$(a + 1)d = r \leq a d(x, \partial D) = ad$$

and we have a contradiction. Thus (4.2) holds for each $x \in \gamma$ and the proof for Theorem 4.1 is complete.
There are many ways to describe the class of simply connected John domains in $\mathbb{R}^2$. The following characterization, reminiscent of Ahlfors’ beautiful criterion for quasicircles, follows from results in Sections 2 and 3. It arose in the course of a conversation with C. Pommerenke; see [P2].

4.4. THEOREM. Suppose that $D$ is a simply connected bounded domain in $\mathbb{R}^2$. Then $D$ is a John domain if and only if there exists a constant $a$ such that for each crosscut $\alpha$ of $D$,

$$\min (\text{dia} (D_1), \text{dia} (D_2)) \leq a \text{ dia} (\alpha)$$

where $D_1$ and $D_2$ are the components of $D \setminus \alpha$.

PROOF. Suppose that $D$ is a John domain with center $x_0$, let $\alpha$ be a crosscut of $D$ and let $D_1$ be a component of $D \setminus \alpha$ which does not contain $x_0$. If $x_1, x_2 \in D_1$, then for $j = 1, 2$ there exists a $b$-cone arc $\gamma_j$ which joins $x_j$ to $x_0$ and meets $\alpha$ in a point $y_j$; obviously

$$|y_1 - y_2| \leq \text{dia} (\alpha).$$

Then (1.3) and the fact that $\alpha$ joins $y_j$ to $\partial D$ imply that

$$|x_j - y_j| \leq l(\gamma_j (x_j, y_j)) \leq b d(y_j, \partial D) \leq b \text{ dia} (\alpha)$$

for $j = 1, 2$. Thus

$$|x_1 - x_2| \leq |x_1 - y_1| + |y_1 - y_2| + |x_2 - y_2| \leq (2b + 1) \text{ dia} (\alpha)$$

and we obtain (4.5) with $a = 2b + 1$.

Suppose next that $D$ satisfies condition (4.5) for some constant $a$. We show first there exists a point $x_0 \in D$ such that

$$\text{dia} (D) \leq 4ac d(x_0, \partial D),$$

where $c$ is the absolute constant in Lemma 2.16. For this choose $y_1, y_2 \in D$ so that

$$\text{dia} (D) \leq 2|y_1 - y_2|,$$

let $\gamma$ be the hyperbolic geodesic joining $y_1$ and $y_2$ in $D$ and choose $x_0 \in \gamma$ so that $|y_1 - x_0| = |y_2 - x_0|$. Then by Lemma 2.16 there exists a crosscut $\alpha$ of $D$ containing $x_0$ which separates $y_1$ and $y_2$ and satisfies

$$l(\alpha) \leq c d(x_0, \partial D).$$

If $D_1, D_2$ denote the components of $D \setminus \alpha$, then (4.5) implies that

$$\begin{cases} 
\text{dia} (D) \leq 2|y_1 - y_2| \leq 4|y_j - x_0| \\
\leq 4 \min (\text{dia} (D_1), \text{dia} (D_2)) \leq 4a l(\alpha)
\end{cases}$$

and we obtain (4.6) from (4.7) and (4.8).
Now fix $x_1 \in D$, let $\gamma$ be the hyperbolic geodesic which joins $x_1$ to $x_0$ in $D$ and choose $x \in \gamma \setminus \{x_0, x_1\}$. Again by Lemma 2.16 there exists a crosscut $\alpha$ of $D$ containing $x$ which separates the components of $\gamma \setminus \{x\}$ and satisfies

$$l(\alpha) \leq c \ d(x, \partial D).$$

Let $D_0$ and $D_1$ denote the components of $D \setminus \alpha$ which contain $x_0$ and $x_1$, respectively, and set $r = ac \ d(x, \partial D)$. If $d(x_0, \partial D) \leq 3r$, then

$$\text{dia} (D_1) \leq \text{dia} (D) \leq 12acr$$

by (4.6). Otherwise since $a \geq 1$ and $c \geq 1$,

$$|x - x_0| \geq d(x_0, \partial D) - d(x, \partial D) > 2n$$

and with (4.9) and (4.11) we obtain

$$B(x_0, r) \subset D \setminus \alpha, \quad \text{dia} (D_0) > 2r.$$ 

Then (4.5) and (4.9) imply that

$$\min (\text{dia} (D_0), \text{dia} (D_1)) \leq r$$

and hence that

$$\text{dia} (D_1) \leq r.$$ 

Since $\gamma(x_1, x) \subset D_1 \cup \{x\}$, we conclude from (4.10) and (4.12) that

$$\gamma(x_1, x) \subset B(x, 12(ac)^2d(x, \partial D))$$

and thus by Theorem 3.27 that $\gamma$ is a $b$-cone arc where $b = b(a)$. This completes the proof of Theorem 4.4.

5. Examples.

We conclude this paper with examples which show that a quasihyperbolic geodesic in a $c$-John domain need not be a $b$-cone arc with $b = b(c)$ unless $n = 2$ and $D$ is simply connected. Thus these hypotheses on $D$ in Theorem 4.1 are necessary.

5.1. Example. For each $b \geq 1$ there exists a doubly connected 10-John domain $D_1$ in $\mathbb{R}^2$ with center $x_0$ and a point $x_1$ in $D_1$ such that any $b$-cone arc from $x_1$ to $x_0$ is not a quasihyperbolic geodesic.

5.2. Example. There exists an infinitely connected 10-John domain $D_2$ in $\mathbb{R}^2$ with center $x_0$ and, for each $b \geq 1$, a point $x_1$ in $D_2$ such that any $b$-cone arc from $x_1$ to $x_0$ is not a quasihyperbolic geodesic.
5.3. BASIC CONSTRUCTION. For each $\sigma \in (0, \frac{1}{4}]$ and $\tau \in [0, \frac{1}{4}]$ set

$$S_1 = \{ z = u + iv : \sigma^4 \leq u \leq \sigma, v = \tau + u \tan \theta \},$$
$$S_2 = \{ z = u + iv : \sigma^4 \leq u \leq \sigma, v = \tau - u \tan \theta \}$$

where $\theta = \arcsin(1/10)$, and let

$$D_0 = B(0, 2) \setminus (S_1 \cup S_2), \quad x_0 = -1, \quad x_1 = \sigma^3 + i\tau.$$

5.6. LEMMA. $D_0$ is a 10-John domain with center $x_0$.

**Proof.** Fix $x = u + iv \in D_0$ and let

$$y = \begin{cases} \frac{x}{|x|} & \text{if } |x| \geq 1, \\ (1 - v^2)^{1/2} + iv & \text{if } |x| < 1 \text{ and } |v - \tau| \leq u \tan \theta, \\ -(1 - v^2)^{1/2} + iv & \text{if } |x| < 1 \text{ and } |v - \tau| > u \tan \theta. \end{cases}$$

Then it is easy to check that $\alpha = [x, y]$ is a 10-cone arc joining $x$ to $y$ in $D_0$. Next the unit circle contains an arc $\beta$ joining $y$ to $x_0$ with $l(\beta) \leq \pi$ and $d(z, \partial D) \geq \frac{\sigma}{3}$ for $z \in \beta$. Hence $\gamma = \alpha \cup \beta$ is a 10-cone arc from $x$ to $x_0$ in $D_0$.

5.8. LEMMA. If $b < \frac{6}{\sigma}$ and if $\gamma$ is a $b$-cone arc from $x_1$ to $x_0$ in $D_0$, then $\gamma$ is not a quasihyperbolic geodesic.

**Proof.** Fix $b < \frac{6}{\sigma}$, suppose that $\gamma$ is a $b$-cone arc joining $x_1$ to $x_0$ in $D_0$ and set

$$T_1 = \{ z = \sigma^4 + i(\tau + t) : |t| \leq \sigma^4 \tan \theta \}, \quad T_2 = \{ z = \sigma + i(\tau + t) : |t| \leq \sigma \tan \theta \}.$$

Then $\gamma \cap T_2 \neq \emptyset$ since otherwise we could find a point $w \in \gamma \cap T_1$ such that

$$\frac{1}{3} \sigma^3 \leq \sigma^4 - \sigma^4 \leq l(\gamma(x_1, w)) \leq b d(w, \partial D) \leq b \sigma^4 \tan \theta < \frac{b \sigma^4}{9}$$

contradicting our choice of $b$.

Next set $y_1 = \sigma^4 + i\tau, z_1 = -\frac{1}{3} + i\tau$ and let $w_1$ be the first point in $\gamma \cap T_2$ as we traverse $\gamma$ from $x_1$ towards $x_0$. If $x \in \gamma(x_1, w_1)$, then

$$d(x, \partial D) \leq \Re(x) \tan \theta < \frac{\Re(x)}{9}$$

and we obtain

$$k_{D_0}(\gamma) = \int_{\gamma} d(x, \partial D_0)^{-1} \, ds > 9 \log \left( \frac{\Re(w_1)}{\Re(x_1)} \right) = 18 \log \frac{1}{\sigma}.$$
Similarly if \( x \in \alpha = [x_1, y_1] \), then
\[
d(x, \partial D_0) \geq \Re(x) \sin \theta = \frac{\Re(x)}{10}\]
and hence
\[(5.10) \quad k_{D_0}(x_1, y_1) \leq \int_{\alpha} d(w, \partial D_0)^{-1} ds \leq 10 \log \left( \frac{\Re(x_1)}{\Re(y_1)} \right) = 10 \log \frac{1}{\sigma}.
\]
Next
\[
d(y_1, \partial D_0) = \sigma^4 \tan \theta, \quad d(z_1, \partial D_0) \geq \frac{1}{2} + \sigma^4, \quad |y_1 - z_1| = \frac{1}{2} + \sigma^4
\]
and thus
\[(5.11) \quad k_{D_0}(y_1, z_1) \leq \log (1 + (2 + \sigma^{-4}) \cot \theta) < 6 \log \frac{1}{\sigma}
\]
by Lemma 2.1. Finally \( d(x, \partial D_0) \geq \frac{1}{2} \) for \( x \in \beta = [z_1, x_0] \) and hence
\[(5.12) \quad k_{D_0}(z_1, x_0) \leq 2l(\beta) < 2 \log \frac{1}{\sigma}.
\]
Then (5.9), (5.10), (5.11) and (5.12) imply that
\[(5.13) \quad k_{D_0}(x_1, x_0) < 18 \log \frac{1}{\sigma} < k_{D_0}(\gamma)
\]
and hence that \( \gamma \) is not a quasihyperbolic geodesic in \( D_0 \).

5.14 Proof for Example 5.1. Fix \( b \geq 1 \), let \( \theta = \arcsin(1/10) \) and choose \( \sigma \in (0, \frac{1}{4}) \) so that \( b < \frac{6}{\sigma} \). Next set
\[
S_1 = \{ z = u + iv : \sigma^4 \leq u \leq \sigma, \quad v = \tau + u \tan \theta \}, \\
\bar{S}_2 = \{ z = u + iv : \sigma^4 \leq u \leq 2, \quad v = \tau - u \tan \theta \}
\]
and let
\[
D_1 = B(0, 2) \setminus (S_1 \cup \bar{S}_2).
\]
Suppose that \( x = u + iv \in D_1 \). If \( |x| < 1 \), let \( y \) and \( \alpha \) be as in the proof of Lemma 5.6. Then again there exists a subarc \( \beta \) of the unit circle such that \( y = \alpha \cup \beta \) is a 10-cone arc from \( x \) to \( y \) in \( D_1 \). If \( |x| \geq 1 \), choose \( \phi \in [-\pi, \pi] \) so that \( x = |x|e^{i\phi} \) and let \( \gamma \) denote the arc defined by
\[
x(t) = \begin{cases} 
|x|^{1-t}e^{i((1-t)\phi + \tau \phi)} & \text{if } \phi > -\theta, \\
|x|^{1-t}e^{i((1-t)\phi - \tau \phi)} & \text{if } \phi < -\theta, 
\end{cases} \quad t \in [0, 1].
\]
Then an elementary calculation shows that $\gamma$ is again a 10-cone arc from $x$ to $x_0$ in $D_1$. Thus $D_1$ is a 10-John domain.

Next suppose that $\gamma$ is a $b$-cone arc from $x_1 = \sigma^3$ to $x_0 = -1$ in $D_1$. Then the proof of Lemma 5.8 with $\tau = 0$ implies that (5.9), (5.10), (5.11), (5.12) and (5.13) hold with $D_1$ in place of $D_0$. Hence $\gamma$ is not a quasihyperbolic geodesic in $D_1$.

5.15. **Proof for Example 5.2.** Let $\theta = \arcsin(1/10)$, let

$$S_{1,j} = \{z = u + iv: \sigma_j^4 \leq u \leq \sigma_j, \; v = \tau_j + u \tan \theta\},$$

$$S_{2,j} = \{z = u + iv: \sigma_j^4 \leq u \leq \sigma_j, \; v = \tau_j - u \tan \theta\}$$

for $j = 1, 2, \ldots$, where $\sigma_j = \tau_j = 4^{-j}$, and set

$$D_2 = B(0, 2) \setminus \bigcup_{j=1}^{\infty} (S_{1,j} \cup S_{2,j}).$$

Next fix $x = u + iv \in D_2$, let

$$y = \begin{cases} 
\frac{x}{|x|} & \text{if } |x| \geq 1, \\
(1 - v^2)^{1/2} + iv & \text{if } |x| < 1 \text{ and } |v - \tau_j| \leq u \tan \theta \text{ for some } j, \\
-(1 - v^2)^{1/2} + iv & \text{if } |x| < 1 \text{ and } |v - \tau_j| > u \tan \theta \text{ for all } j,
\end{cases}$$

and set

$$C_k = \{z = u + iv: 0 \leq u < \infty, \; |v - \tau_k| \leq u \tan \theta\}$$

for $k = 1, 2, \ldots$. Then

$$(S_{1,k} \cup S_{2,k}) \subset \partial C_k, \quad (S_{1,j} \cup S_{2,j}) \cap C_k = \emptyset \quad \text{for } j \neq k,$$

and again it is easy to show that $\alpha = [x, y]$ is a 10-cone arc from $x$ to $y$. Hence $D_2$ is a 10-John domain as in the proof of Lemma 5.6.

Finally fix $b \geq 1$, choose $j$ so that $b\sigma_j < 6$ and let $\gamma$ be a $b$-cone curve which joins $x_1 = \sigma_j^3 + i\tau_j$ to $x_0 = -1$ in $D_2$. Then again the proof of Lemma 5.8 with $\sigma = \tau = \sigma_j = \tau_j$ shows that $\gamma$ is not a quasihyperbolic geodesic in $D_2$.

5.17. **Remark.** Similar examples exist in $\mathbb{R}^n$ for each $n \geq 2$. For example, in the $n$-dimensional analogue of the domain $D_2$ we replace each set $S_{1,j} \cup S_{2,j}$ by the lateral surface $\sum_{j}$ of a frustum of an $n$-cone with vertex angle $\theta$. Then when $n > 2$, the frustums $\sum_{j}$ can be joined by segments so that the resulting domain has a connected boundary.
REFERENCES


