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SELF-PARALLEL CURVES

F. J. CRAVEIRO DE CARVALHO AND S. A. ROBERTSON

Introduction.

The notion of self-parallelism [3] applies to immersions of an n-dimensional manifold M in Euclidean m-space E^m, for any $m = n + k > n > 0$, and is closely related to the notion of transnormality [10]. In this note we shall be mainly concerned with self-parallelism for embeddings f: $S^1 \to E^3$.

We start by showing that for plane curves transnormality is equivalent to self-parallelism. If the curve is not plane the situation is quite different and although it is true that transnormality implies self-parallelism [12] the converse does not hold. Since every transnormal embedding f: $S^1 \to E^3$ is 2-transnormal [5] one might however think that any self-parallel embedding f: $S^1 \to E^3$ must have Z_2 as self-parallelism group. We show this is not so by constructing for some $m > 2$ self-parallel embeddings of S^1 in E^3 whose self-parallelism group contains elements of order m. As a by-product of our construction we find that there are self-parallel injective immersions of R in E^3, albeit with non-closed image, with infinite self-parallelism group. We recall that the only transnormal embeddings of R in E^3 with closed image are 1-transnormal [13, 15].

In § 3 we consider spherical immersions and characterize those smooth maps f: $S^n \to E^m$ whose graphs are self-parallel.

We end with some results on the length and the total torsion of self-parallel curves.

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1. Definitions.

Let M denote a smooth ($= C^\infty$) connected boundaryless n-manifold. If $f\colon M \to E^m$ is a smooth immersion with codimension k we shall denote by $N_f(x)$ the affine normal k-plane to f at x.

DEFINITION 1. Let $f\colon M \to E^m$ be an immersion. We say that f is *transnormal* if, for every $x \in M$, $N_f(x) = N_f(y)$, whenever $f(y) \in N_f(x)$.

DEFINITION 2. Let $f, g\colon M \to E^m$ be immersions. We say that f is *parallel* to g, written $f \parallel g$, if, for every $x \in M$, $N_f(x) = N_g(x)$.

If $f \parallel g$ then $\|f(x) - g(x)\|$ does not depend on $x \in M$. A diffeomorphism $\delta\colon M \to M$ is a *self-parallelism* of M with respect to the immersion $f\colon M \to E^m$ if $f \parallel f \circ \delta$. The set $G(f)$ of such diffeomorphisms is a subgroup of the group of self-diffeomorphisms of M. We call $G(f)$ the *self-parallelism group* of f. In this note we shall use *self-parallel* to mean that $G(f)$ is non-trivial. If M is compact and $f\colon M \to E^{n+1}$ is a self-parallel embedding then $G(f) \approx Z_2$ [3].

By the *graph* of a smooth map $f\colon S^n \to E^m$ we mean the smooth embedding $F\colon S^n \to E^{n+1} \times E^m \equiv E^{n+m+1}$ given by $F(x) = (x, f(x))$.

DEFINITION 3. A map $f\colon S^n \to E^m$ is *spherical* if $f(S^n)$ is contained in a round $(m-1)$-sphere.

In definition 3 we allow the radius of the receiving sphere to be zero. We remark that if $f\colon S^n \to E^m, m \leqq n$, is spherical and preserves antipodal points then it must be constant by the Borsuk-Ulam theorem.

DEFINITION 4. Let $f \cdot M \to E^{n+1}$ be an embedding, where M is compact. We say that f is convex if, for every $x \in M$, there exists an affine n-plane $T(x)$ such that $T(x) \cap f(M) = \{f(x)\}$.

This notion of convexity is stronger than the usual one and was already used in [1]. Of course for each $x \in M$ $T(x)$ is unique and is the affine tangent n-plane to f at x. Also by Reeb's theorem [7] M must be homeomorphic (and, in fact, diffeomorphic) to S^n. Examples of convex embeddings occur when the Gaussian curvature of f never vanishes. The important remark to make about such embeddings is that no straight line in E^{n+1} intersects $f(M)$ in more than two points [1].

Although sometimes we give more specific references the general reference for the theory of transnormal embeddings is [10]. For the theory of parallel immersions see [3].

2. Plane curves.

PROPOSITION 1. *Any convex self-parallel embedding $f\colon S^n \to E^{n+1}$ is transnormal.*

PROOF. Let δ be the non-trivial self-parallelism of S^n with respect to f and $x \in S^n$. The normal line L to f at x intersects $f(S^n)$ only in $f(x)$ and $f(\delta(x))$. Since $f \| f \circ \delta$ L is also normal to f at $\delta(x)$.

THEOREM 1. *Let $f\colon S^1 \to E^2$ be an embedding. Then f is transnormal iff it is self-parallel.*

PROOF. It is well known from the theory of transnormal manifolds that if f is transnormal then it is also self-parallel [9]. Suppose now that f is self-parallel. Then $G(f) \approx Z_2$ and there is a non-trivial diffeomorphism $\delta\colon S^1 \to S^1$ such that $\delta^2 = i$, where i is the identity map. Since δ is non-trivial there is $c > 0$ such that, for every $x \in S^1$, $\|f(x) - f(\delta(x))\| = c$. Moreover δ has no fixed points and therefore must be an orientation preserving diffeomorphism.

Let us consider the map exp: $R \to S^1$ given by $\exp(t) = (\cos 2\pi t, \sin 2\pi t)$. There is a diffeomorphism $\bar{\delta}\colon R \to R$, with $\bar{\delta}' > 0$, such that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\ \bar{\delta}\ } & R \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ S^1 & \xrightarrow{\ \delta\ } & S^1 \end{array}$$

where h is either exp or the composition of exp with a change of parameter so that $f \circ h$ is parametrized by arc-length. To simplify notations we write just f instead of $f \circ h$ (a similar procedure will be adopted in § 4, 5). We now proceed to show that the curvature K_f of f never vanishes.

For every $t \in R$, $(f'(t), f(t) - f(\bar{\delta}(t)))$ always determines the same orientation for E^2. At t and $\bar{\delta}(t)$ f has parallel tangents and in fact $f'(t) = \pm f'(\bar{\delta}(t))$. However we cannot have $f'(t) = f'(\bar{\delta}(t))$ because in that case $(f'(\bar{\delta}(t)), f(\bar{\delta}(t)) - f(\bar{\delta}(\bar{\delta}(t)))) = (f'(t), f(\bar{\delta}(t)) - f(t))$ would determine the opposite orientation for E^2. If we now take $g\colon R \to E^2$ given by $g(t) = f(t) - f(\bar{\delta}(t))$ it is easy to check that $K_g(t) = K_f(t)/(1 + \bar{\delta}'(t))$. Since g is spherical K_g and consequently K_f never vanish. The result now follows from proposition 1.

We point out that if $f\colon S^{2n} \to E^{2n+1}$ is an embedding then transnormality is also equivalent to self-parallelism. In fact it can be deduced from [3] that the Gaussian curvature of a self-parallel embedding $f\colon S^{2n} \to E^{2n+1}$ never vanishes.

Since there are self-parallel embeddings $f\colon M \to E^{n+1}$ of codimension one which are not transnormal [3] the following result may be of some interest.

THEOREM 2. *An embedding $f\colon M \to E^{n+1}$, with M compact, is transnormal iff* *there is a diffeomorphism* δ: $M \to M$ *such that, for every* $x \in M$,

$$\|f(x) - f(\delta(x))\| = \text{diameter}\, f(M).$$

PROOF. Again it is well known that the condition is necessary [9]. To prove sufficiency, observe first that any such f is convex. In fact, for every $x \in M$, the affine n-plane containing $f(x)$ and normal to $f(x) - f(\delta(x))$ intersects $f(M)$ in exactly one point. Also, the normal line $N_f(x)$ is normal at $\delta(x)$, as a consequence of the fact that $\|f(x) - f(\delta(x))\|$ is maximal.

We shall give below (see figure 1) an example which shows that theorem 2 does not hold for codimensions greater than one.

3. Spherical immersions.

The aim of this section is to introduce and deal with a particular type of self-parallel immersions. Consideration of such immersions allows us to characterize those smooth maps f: $S^n \to E^m$ whose graphs are self-parallel.

THEOREM 3. *Let* f: $S^n \to E^m$ *be an immersion. If* f *is spherical and preserves antipodal points then it is self-parallel.*

PROOF. Suppose that $f(S^n)$ is contained in a round $(m - 1)$-sphere with centre x_0. Let A: $S^n \to S^n$ and A_1: $E^m \to E^m$ denote the reflections in the origin and in x_0 respectively. That f preserves antipodal points means that $A_1 \circ f = f \circ A$. Since f is spherical, to show that $f \| f \circ A$ we only need to prove that, for every $x \in S^n$, $(f \circ A)_{*x}(T_x S^n) = f_{*x}(T_x S^n)$, where $(f \circ A)_{*x}$ and f_{*x} are the linear homomorphisms between the tangent spaces and we identify the tangent spaces to E^m with E^m itself in the obvious way. But that follows at once since $f \circ A = A_1 \circ f$ and $A_{1*f(x)}$ regarded as a map from E^m into itself is just reflection in the origin.

It is perhaps worth pointing out and clear from the above argument that if

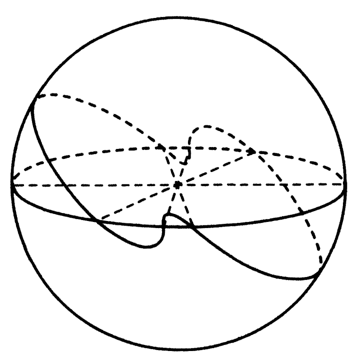

Figure 1

δ: $S^n \to S^n$ is a diffeomorphism and f: $S^n \to E^m$ is a spherical immersion such that $A_1 \circ f = f \circ \delta$ then δ is also a self-parallelism.

It is known [12] that if f: $S^1 \to E^m$ is transnormal then there is a self-parallelism δ: $S^1 \to S^1$ such that $\|f(x) - f(\delta(x))\| = \text{diameter } f(S^1)$. The converse is not true as figure 1 shows.

In figure 1 we have the image of an embedding f: $S^1 \to E^3$ with $Z_2 = \{i, \delta\}$ as self-parallelism group and $\|f(x) - f(\delta(x))\| = \text{diameter } f(S^1)$. However f is not transnormal: for instance, the equatorial plane is normal to f at only two of the six points whose image under f lies in that plane.

THEOREM 4. *Let f: $S^n \to E^m$ be a smooth map. Then its graph F is self-parallel iff f is spherical and preserves antipodal points. If F is self-parallel then $G(F) \approx Z_2$.*

PROOF. In [2] it is shown that the conditions we state above are necessary for the graph to be transnormal. The proofs which were given also work, with minor modifications, if we assume that F is self-parallel. At the same time one sees that if F is self-parallel then its self-parallelism group $G(F)$ is $\{i, A\}$, where i is the identity and A is the reflection in the origin.

Conversely, if f is spherical and preserves antipodal points the same happens with F. Therefore, by theorem 3, F is self-parallel.

Although, as we observed above, the conditions stated in theorem 4 are necessary for the graph to be transnormal they are not sufficient. Take f: $S^1 \to E^2$ given by $f(z) = z^3$. Then F is not transnormal, for otherwise it would be 2-transnormal [2], that is to say, we would have $\# F^{-1}(N_F(x)) = 2$, $x \in S^1$. However a straightforward calculation shows that, for instance, the normal 3-plane to F at (1,0) intersects $F(S^1)$ in more than two points.

4. Space curves.

If one compares the theories of transnormal and self-parallel embeddings one sees that the order of transnormality corresponds to the order of the self-parallelism group [3]. Irwin showed in [5] that any transnormal embedding f: $S^1 \to E^3$ is 2-transnormal. One might perhaps guess that any self-parallel embedding f: $S^1 \to E^3$ must have Z_2 as self-parallelism group. As we shall see below, this is not so.

Let f: $S^1 \to E^3$ be a smooth embedding which we shall treat as a map f: $R \to E^3$ of period l assuming that it is parametrized by arc-length and has length l. We shall also suppose that the curvature K_f never vanishes so that there is a well defined Serret-Frenet frame at each point $x \in R$.

Let now g: $R \to E^3$ be an immersion given by $g(x) = f(x) + \alpha(x)n(x) + \beta(x)b(x)$, where $n(x)$, $b(x)$ denote the principal normal and binormal vectors of f at x. The immersion g is parallel to f iff $\alpha'(x) - \beta(x)\tau(x) = 0$, $\beta'(x) + \alpha(x)\tau(x) = 0$, where

τ denotes the torsion of f. Therefore to obtain an immersion g: $R \to E^3$ parallel to f one has only to look for the solutions of the differential equation $(\alpha, \beta)' = \tau(\beta, -\alpha)$ and make sure that g is an immersion. The differential equation has global solutions defined in R and we point out that if (α, β) is a solution then $\|(\alpha(x), \beta(x))\|$ does not depend on x. Also if $\|(\alpha(x), \beta(x))\| = \varepsilon > 0$, with ε small enough then g will be an immersion. The choice of ε is related to the fact that, for ε small enough, the end-point map embeds in E^3 the submanifold $\{(z, v) \mid z \in f(S^1), \|v\| = \varepsilon\}$ of the total space of the normal bundle of $f(S^1)$.

Suppose now that we have such a g and that we write $(\alpha(x), \beta(x)) = \varepsilon(\cos 2\pi\theta(x), \sin 2\pi\theta(x))$. Then $2\pi\theta'(x) = -\tau(x), x \in R$ and $\int_x^{x+l} 2\pi\theta'(u)du = -\int_x^{x+l} \tau(u)du = -\bar{\tau}$, where $\bar{\tau}$ denotes the total torsion of f. That is to say $g(x + l)$ is obtained from $g(x)$ by a rotation of angle $-\bar{\tau}$ around $f(x)$ in the affine normal plane to f at x. Two cases may occur:

(1) The total torsion $\bar{\tau}$ is $\pm(m/n)2\pi$, with m and n coprime integers. In this case g is periodic of period nl and gives rise to an embedding g: $S^1 \to E^3$. The self-parallelism group of g contains a subgroup isomorphic to Z_n. The generator of such a subgroup is the diffeomorphism δ which, regarded as a map δ: $g(R) \to g(R)$, maps $g(x)$ to $g(x + l)$.

(2) The total torsion is not $\pm(m/n)2\pi$. In this case we obtain an injective immersion g: $R \to E^3$ of which the self-parallelism group contains a subgroup isomorphic to Z, with generator as in (1).

One question which arises naturally is whether embeddings f with total torsion as in (1) and (2) exist: we shall indicate very briefly how to construct examples. For full details we refer the reader to [8].

Suppose we start with an embedding f with non-zero total torsion and such that when projecting in the (x, y)-plane we have a curve with non-vanishing curvature and, of course, zero total torsion. Then the curve we started with can be non-degenerately deformed into its projection, that is to say at every stage of the deformation the corresponding curve has non-vanishing curvature. Moreover at every stage except possibly the last we have an embedding. Since the total torsion varies continuously during the deformation it follows that embeddings with total torsion of the type mentioned in (1) and (2) can be obtained.

The very simple device indicated in Millman and Parker [6] which produces closed space curves for which the integral of the torsion attains any prescribed value appears to require the existence of points where the torsion is not defined and therefore their approach cannot be applied here.

We can now state

THEOREM 5. *There are self-parallel embeddings* f: $S^1 \to E^3$ *with self-parallelism group not isomorphic to* Z_2.

THEOREM 6. *There are injective immersions $f: R \to E^3$ with infinite selfparallelism group.*

Theorem 6 is of interest in the context of the theory of transnormal manifolds. We recall that B. Wegner showed that the order of any transnormal embedding with closed image is finite [13, 15]. In particular in the case of embeddings of R^m it is equal to 1.

5. Length and total torsion.

As before (§ 2, § 4) we shall regard an embedding g: $S^1 \to E^3$ (or E^n) as a periodic map g: $R \to E^3$ (or E^n). Whenever possible and convenient, parametrization by arc-length will be assumed. Diffeomorphisms of S^1 will be lifted to diffeomorphisms of R as in § 2.

PROPOSITION 2. *If the curvature of g: $S^1 \to E^3$ is nowhere vanishing and $f \| g$ then*

(1) $$t_f = \pm t_g, \; n_f = \pm n_g, \; b_f = b_g$$

(2) $$\text{total curvature of } f = \text{total curvature of } g$$

$$\text{total torsion of } f = \pm \text{total torsion of } g$$

where t, n, b denote the unit tangent, principal normal and binormal vector fields respectively and the negative sign is to be taken if $f'(x) = \lambda(x)g'(x)$, with $\lambda(x) < 0$.

PROOF. Write $f(x) = g(x) + \alpha(x)n_g(x) + \beta(x)b_g(x)$ and use the definitions.

THEOREM 7. *Let f: $S^1 \to E^3$ be a self-parallel embedding with nowhere vanishing curvature and let δ be a self-parallelism. If $f'(\delta(x))$ and $f'(x)$ point in opposite directions, then the total torsion of f is zero.*

PROOF. If $f'(\delta(x)) = -f'(x)$ then $(f \circ \delta)'(x) = -\delta'(x)f'(x)$. Since δ is orientation preserving $\delta'(x) > 0$. Therefore by proposition 2 we conclude that the total torsions of f and $f \circ \delta$ are symmetrical. But on the other hand they are also equal. Consequently the total torsion of f must be zero.

On the other hand for instance the self-parallel closed curves in § 4 have total torsion equal to $\pm 2\pi m$.

Since the curvature of a transnormal embedding f: $S^1 \to E^3$ never vanishes it makes sense to speak of the total torsion of a transnormal curve.

COROLLARY 1. *If f: $S^1 \to E^3$ is transnormal then its total torsion is zero.*

PROOF. Let δ be the antipodal involution of S^1. Since f is 2-transnormal then $f'(\delta(x)) = -f'(x)$.

Corollary 1 already appears in [14] where it is proved using the fact that a spherical embedding of S^1 in E^3 has zero total torsion [4].

Our final result generalizes a theorem of Bückner [14] and is an immediate corollary to a theorem proved in [11].

THEOREM 8. *Let* $f: S^1 \to E^n$ *be an embedding. If* $\delta \in G(f)$ *has order* 2 *and* $\|f(x) - f(\delta(x))\| = a$ *then* length $f \geqq \pi a$.

PROOF. Take $g(x) = f(x) - f(\delta(x))$. Then $\|g'(x)\| \leqq 1 + \delta'(x)$ and therefore length $g \leqq 2$ length f. Since g is spherical and it is not contained in an open hemisphere then length $g \geqq 2\pi a$ [11]. The result then follows.

REFERENCES

1. S. Carter, *A class of compressible embeddings*, Proc. Cambridge Philos Soc. 65 (1969), 23–26.
2. F. J. Craveiro de Carvalho, *Transnormal graphs*, Portugal. Math. 39 (1980), 285–287.
3. H. R. Farran and S. A. Robertson, *Parallel immersions in euclidean space*, J. London Math. Soc. 35 (1987), 527–538.
4. W. Fenchel, *Über einen Jacobischen Satz der Kurventheorie*, Tôhoku Math. J. 39 (1934), 95–97.
5. M. C. Irwin, *Transnormal circles*, J. London Math. Soc. 42 (1967), 545–552.
6. R. S. Millman and G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall, Englewood Cliffs, 1977.
7. J. W. Milnor, *Morse Theory*, Ann. of Math. Studies 51, Princeton, 1963.
8. W. Pohl, *The self-linking number of a closed space curve*, J. Mathematics and Mechanics 10 (1968), 975–985.
9. S. A. Robertson, *On transnormal manifolds*, Topology 6 (1967), 117–123.
10. S. A. Robertson, *Smooth curves of constant width and transnormality*, Bull. London Math. Soc. 16 (1984), 264–274.
11. H. Rutishauser and H. Samelson, *Sur le rayon d'une sphère dont la surface contient une courbe fermée*, C.R. Acad. Sci. Paris Ser. I Math. 227 (1948), 755–757.
12. B. Wegner, *Decktransformationen transnormaler Mannigfaltigkeiten*, Manuscripta Math. 4 (1971), 179–199.
13. B. Wegner, *Transnormale Isotopien und transnormale Kurven*, Manuscripta Math. 4 (1971), 361–372.
14. B. Wegner, *Globale Sätze über Raumkurven konstanter Breite*, Math. Nachr, 53 (1972), 337–344.
15. B. Wegner, *Einige Bemerkungen zur Geometrie transnormaler Mannigfaltigkeiten*, J. Differential Geom. 16 (1981), 93–100.

DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE DE COIMBRA
3000 COIMBRA
PORTUGAL

FACULTY OF MATHEMATICAL STUDIES
UNIVERSITY OF SOUTHAMPTON
SOUTHAMPTON S09 5NH
U.K.