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RADICAL OF SPLITTING RING EXTENSIONS

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Abstract.

The ring extension $R \subseteqq S$ of associative rings with the same identity 1 is said to be right splitting if there exists a homomorphism $p: S_R \to R_R$ of the right R-modules such that $p(1) = 1$. Estimates for projective and injective modules, and the Jacobson radical of right splitting extensions are given.

The objective of this paper is to lay elementary foundations to the study of one-sided splitting ring extensions, which appear to be natural generalizations of classical Everrett extensions, semitrivial extensions [8], monoid rings, and skew-polynomials.

All rings considered are associative unless specified otherwise and in what follows R and S stand for rings with identity $1 \neq 0$.

The most general form of a right splitting extension is given by a ring monomorphism $i: R \to S$ preserving the identity 1 and $p \in \mathrm{Hom}_R(S_R, R_R)$ such that $p(1) = 1$. Here, the right R-module structure of S_R is given by $s \cdot r = si(r)$. Without loss of generality we may redefine the extension by requiring that i is a subring inclusion. Then p is an epimorphism and $K_R = \mathrm{kernel}(p)$ is a direct summand of S_R.

The class of all splitting extensions of R is closed under compositions (i.e., if $R_1 \subseteqq R_2 \subseteqq R_3$ is a chain of extensions and $p_1: R_2 \to R_1$ and $p_2: R_3 \to R_2$ are the corresponding epimorphisms then $R_1 \subseteqq R_3$ with the epimorphism $p_1 p_2$ is again a splitting extension of R_1). Notice, that by using compositions we can build up generalized triangular and full matrix rings ([6], [8]).

Given a ring automorphism σ of R and a σ-derivation δ (δ is an abelian group endomorphism of R such that $(rs)^\sigma = r^\delta s^\sigma + rs^\delta$, for $r, s \in R$) we denote the general skew-polynomial ring over R by $R[x; \sigma, \delta]$, where the commutation is subject to $rx = xr^\sigma + r^\delta$ ([4], p. 34).

Define $R[x; \sigma] = R[x; \sigma, 0]$ and $R[x; \delta] = R[x; 1, \delta]$.

Since every field extension, or more generally, an extension of an artinian semisimple ring is splitting, a meaningful complete classification of splitting extensions of a given ring does not seem to be feasible. (An extension S of

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a commutative ring R such that S_R is projective is also splitting, [3]). However, by imposing conditions which emulate particular classes of splitting extensions, e.g., monoid rings or matrices, we can obtain results valid for families of ring extensions larger than those we wanted to emulate. Advantage is a unified treatment allowing us to see interdependence of particular classes of splitting extensions.

For the used definitions and notation the reader is referred to [1] or [5]. Let us recall some of the less frequently used terminology. Let T be also a ring and ${}_TA_R$ and ${}_SB_R$ be bimodules. The set of all homomorphisms $\mathrm{Hom}_R({}_TA_R, {}_SB_R)$ is equipped with S-T bimodule structure is given by $(sft)(a) = s(f(ta)$, for $s \in S$, $t \in T$, and $a \in A$. Consequently, when we consider right homomorphisms we apply the arguments to the right of the homomorphisms f and the composition of two homomorphisms f, g is given by $(fg)(a) = f(g(a))$. For the left homomorphisms, change it mutatis mutandis.

A subset L of a ring is said to be right (left) T-nilpotent if for every sequence $a_1, a_2, \ldots$ in L there is an n such that $a_n \ldots a_2 a_1 = 0$ $(a_1 a_2 \ldots a_n = 0)$.

1. Structure of splitting extensions.

1.1. Theorem. *Let ${}_RA_R$ be a bimodule endowed with a binary operation making it a ring (possibly non-associative and without identity), $\alpha \in \mathrm{Hom}_R(R \otimes_Z A_R, R_R)$, $\beta \in \mathrm{Hom}_R(A \otimes_R A_R, R_R)$ be such that*

(i) $(rs \otimes a)^\alpha - (r \otimes sa)^\alpha = r(s \otimes a)^\alpha$

(ii) $r(ab) - (ra)b = (r \otimes a)^\alpha b$

(iii) $a(br) = (ab)r$

(iv) $r(a \otimes b)^\beta - (ra \otimes b)^\beta = ((r \otimes a)^\alpha \otimes b)^\alpha - (r \otimes ab)^\alpha$

(v) $(ar)b - a(rb) = a(r \otimes b)^\alpha$

(vi) $a(bc) - (ab)c = (a \otimes b)^\beta c - a(b \otimes c)^\beta$

(vii) $(a \otimes bc)^\beta - (ab \otimes c)^\beta = ((a \otimes b)^\beta \otimes c))^\alpha$,

for every choice of $r, s \in R$ and $a, b, c \in A$.

Then $S = R X A$ as the abelian group with multiplication given by

$$(r, a)(s, b) = (rs + (r \otimes b)^\alpha + (a \otimes b)^\beta, rb + as + ab)$$

is a right splitting extension of R, and it will be denoted by $R\,{}^\alpha\nabla^\beta A$ (tensor representation of S). Conversely, every right splitting extension of R arises in this way, up to a ring-isomorphism.

Proof. Let $S = R \times A$ be the abelian group with the described multiplication. Then the distributivity of tensor product implies the left and right distributivity of the multiplicaton, and (1,0), obviously, serves as the identity. It takes a tedious checking to verify that the conditions (i)–(vii) and $(1 \otimes a)^\alpha = 0$, for every $a \in A$, are equivalent to the associativity of the multiplication. However,

$(1 \otimes a)^{\alpha} = 0$, for every $a \in A$, is a direct consequence of the condition (i). Now, the projection map p: $S \to R$ is clearly a right R-homomorphism and $p(1,0) = 1$. Hence S with the defined multiplication is a right splitting extension.

Conversely, if $R \subseteq S$ and $p: S_R \to R_R$ is a right splitting extension then we can define $A_R = \ker(p)$, $(r \otimes a)^{\alpha} = p(ra)$, $(a \otimes b)^{\beta} = p(ab)$, $a \cdot b = ab - p(ab)$, and the left R-module structure of A by $r \cdot a = ra - p(ra)$, for every $r \in R$, and $a, b \in A$. Since p is an epimorphism and R_R is projective, A_R is a direct summand and $S = R_R \oplus A_R$. Also, the associativity of S implies the conditions (i) through (vii).

Obviously, the correspondence between right splitting extensions of R and their tensor representations is 1-1 up to a ring-isomorphism which is stable on R.

1.2. Example. Let R be a ring with a derivation D (with respect to the identity automorphism) such that $D^2 = 0$ and the ideal $2(R)^D R \neq$ R. Put $T = R/((2(R)^D R)$ and define $S = T^{\alpha}\nabla^{\beta} A$, where $\beta = 0$, ${}_T A_T = {}_T T_T$ with the multiplication given by $t * s = (t^{\delta})s$, and $(t \otimes a)^{\alpha} = (t^{\delta})a$ (δ is the derivation of T induced by D).

Then S is a right splitting extension of T isomorphic to a factor ring of $R[x; D]/(x^2)$.

Notice that $SA = (T^{\delta})T \oplus A$ is nilpotent if and only if T^{δ} is so (c.f., Theorem 1.4.).

The interdependence between splitting extensions and their tensor representations is illustrated on the following proposition which is left to the reader to verify. Notice that $\alpha = \beta = 0$ corresponds to classical Everret extensions and $\alpha = 0$ together with A being the zero-ring corresponds to semitrivial extensions [8].

1.3. Propositon. *Let $R \subseteq S$ and $p: S_R \to R_R$ be a right splitting extension and $S = R^{\alpha}\nabla^{\beta} A$ be its tensor representation. Then*

(i) *The following are equivalent*

a) $\alpha = 0$,

b) $\alpha \in \mathrm{Hom}_R(R \otimes_R A_R, R_R)$,

c) $p \in \mathrm{Hom}_R({}_R S, {}_R R)$,

d) ${}_R(\ker(p)) \subseteq {}_R S$.

(ii) *The following are equivalent*

a) $\alpha = \beta = 0$,

b) $\ker(p)$ *is a left ideal of* S,

c) $\ker(p)$ *is an ideal of* S,

d) p *is a ring-epimorphism.*

(iii) $\beta = 0$ *if and only if* $\ker(p)$ *is a right ideal of* S.

1.4. Theorem. *Let $R \subseteq S$ and $p: S_R \to R_R$ be a right splitting extension,* $K = \ker(p)$, and $I = SK \cap R$. Suppose that M_S is an S-module and N_R is an

R-module. Then the following assertions hold:

i) If M_S is projective then $(M/MK)_R$ is (R/I)-projective. Conversely, if N_R is projective then $(N \otimes_R S)_S$ *is projective.*

ii) *If SK is right T-nilpotent and P_R is a projective cover of $(M/MK)_R$ then $(P \otimes_R S)_S$ is a projective cover of M_S.*

iii) *Suppose that SK is right T-nilpotent and $(R/I)_R$ is projective. Then, M_S is projective if and only if $(M/MK)_R$ is projective and $M_S \simeq (M/MK) \otimes_R S_S$. In such a case, $(MK)_R$ is a direct summand of M_R.*

iv) *If M_S is injective then $(M:K) = \{m \in M;\ m(SK) = 0\}$ is (R/I)-injective. Conversely, if N_R is injective then $(\mathrm{Hom}_R({}_SS_R, N_R))_S$ is injective.*

v) *If SK is left T-nilpotent and E_R is an injective hull of $(M:K)_R$ then $(\mathrm{Hom}_R({}_SS_R, E_R))_S$ is an injective hull of M_S.*

vi) *Suppose that SK is left T-nilpotent and ${}_R(R/I)$ is flat. Then M_S is injective if and only if $(M:K)_R$ is injective and*

$$M_S \simeq (\mathrm{Hom}_R({}_SS_R, (M:K)_R))_S.$$

In such a case $(M:K)_R$ is a direct summand of M_R.

Proof. (i) Suppose that M_S is projective. Without loss of generality we may assume that M_S is a direct summand of $(S^{(\omega)})_S = M_S \otimes N_S$, for some cardinal ω. Then $(SK)^{(\omega)} = (S^{(\omega)})(SK) = M(SK) \oplus N(SK) = MK \oplus NK$ and $(R/I)^{(\omega)} \simeq (S/SK)^{(\omega)} \simeq S^{(\omega)}/(S^{(\omega)}(SK)) \simeq M/MK \oplus N/NK$, and consequently M/MK is (R/I) – projective. (Take into account the fact that $SK = I \oplus K$ and SK is an ideal of S). The converse statement follows from the Hom-tensor product adjoint duality ([5], p. 430).

(ii) Consider the following commutative diagram

$$\begin{array}{ccc}
P_R & \longleftrightarrow & P_R \\
\big\downarrow & & \big\downarrow \pi \\
(P/PI)_R & \xrightarrow[\pi']{} & (M/MK)_R \\
\big\updownarrow & & \uparrow \\
(P \otimes_R (S/SK))_S & & \vdots \\
1 \otimes w \big\uparrow & & \uparrow \\
(P \otimes_R S)_S & \xrightarrow[\alpha]{} & M_S,
\end{array}$$

where

π is the assumed projective cover,

π' is the induced epic following from the fact that $SK = I \otimes K$, and that in turn implies $PI \subseteq \ker(\pi)$,

w is the projection ${}_RS \to {}_R(S/SK)$,
$\longleftrightarrow$ is the isomorphism induced by $S/SK \simeq R/I$, and
$\twoheadrightarrow$ denotes natural projections.

Since both (P/PI) and (M/MK) have the trivial structure of right S-modules and in that structure π' is an S-homomorphism, the existence of α now follows from the projectivity of $P \otimes_R S_S$. Furthermore, since π is a superfluous epic, π' is a superfluous epic, too, and SK being right T-nilpotent implies that $((P \otimes_R S)SK)_S = \ker(1 \otimes \omega)_S$ is small ([1], p. 314). Hence the composition $\pi'(\leftrightarrow)(1_p \otimes \omega)$ is a superfluous epic. Similarly, the natural projection $M_S \to (M/MK)_S$ has the small kernel $M(SK)$, and thus it is a superfluous epic. Therefore, α must be a superfluous epic, too ([1], 5.15, p. 74).

(iii) Suppose that M_S is projective. Then, by using (i), $(M/MK)_R$ is (R/I)-projective, and since $(R/I)_R$ is projective, the hom-tensor product adjoint duality yields $\mathrm{Hom}_R((M/MK)_R, (\cdot)_R) \simeq \mathrm{Hom}_R(((M/MK) \otimes_{(R/I)} (R/I))_R, (\cdot)_R) \simeq \mathrm{Hom}_R((M/MK)_R, \mathrm{Hom}_R({}_{(R/I)}(R/I_R, (\cdot)_R))$. Hence $(M/MK)_R$ is projective. In particular, $(MK)_R$ is a direct summand of M_R. Consider the map $g: ((M/MK) \otimes_R S)_S \to (M/MK)_S$ given by $g(m \otimes s)^\wedge = (ms)^\wedge$, where $\hat{m}$ stands for the equivalence class of $m \in M$ modulo MK. Obviously, g is a well defined S-epimorphism and since $((M/MK) \otimes {}_RS)_S$ is projective there exists $g' \in \mathrm{Hom}_S(((M/MK) \otimes_R S)_S, M_S)$ such that $\tau g' = g$, where $\tau: M_S \to (M/M(SK))_S$ is the projection. Furthermore, SK being right T-nilpotent implies that $(MK)_S = (M(SK))_S$ is small in M_S and therefore g' is an epimorphism. Let $\Sigma((\hat{m})_i \otimes s_i) \in \ker(g)$, where $s_i = r_i + k_i$, $r_i \in K$ and $k_i \in k$. Then $\Sigma((\hat{m})_i \otimes s_i) = ((\Sigma \hat{m}_i r_i) \otimes 1) + \Sigma((\hat{m})_i \otimes k_i)$ and since $\Sigma((\hat{m})_i \otimes k_i) \in \ker(g)$ we obtain $(\Sigma(\hat{m})_i r_i) = 0$. Thus $\ker(g) = ((M/MK) \otimes_R S)(SK)$ and that, thanks to SK being right T-nilpotent, implies $\ker(g') \subseteq \ker(g)$ is small, too. Hence g' is a projective cover and since M_S is projective, $M_S \simeq ((M/MK) \otimes_R S)_S$.

The converse statement follows directly from (i).

(iv), (), and (vi) are dual statements to (i), (ii), and (iii), respectively, and the proofs can be run along the same lines as above with slight modifications.

1.5. Example. Ler R be a skew-field and ${}_RA_R$ be the set of all the countably infinite square upper triangular matrices over R with zeroes on the main diagonal and only finitely many non zero entries off the diagonal. Put $\alpha = \beta = 0$. Then A is a right T-nilpotent ideal of $S = R^\alpha \nabla^\beta A$, $J(S) = A$ (c.f., theorem 2.3), and S-projectives are free.

2. Jacobson radical.

The Jacobson radical $J_R(M_R)$ of the right R-module M_R is the intersection at all maximal submodules of M_R. Precisely, $J_R(M_R) = \cap \ker(f), f \in \mathrm{Hom}_R(M_R, T_R)$, where the intersection runs through all choices of f and simple modules T_R.

Denote $J_R(R_R) = J(R)$ which can be characterized as the largest right (left) ideal consisiting of right (left) quasi-invertible elements ([7], p. 196). In the following, $R \subseteqq S$ and $p \in \mathrm{Hom}_R(S_R, R_R)$ is a given right splitting extension, $K_R = \ker(p)$, and $W = \{r \in R | Kr \subseteqq J(S)\}$.

2.1. Theorem. $SW \cap J(S) = (W \cap J(R)) \oplus KW$.

Proof. Obviously, $(W \cap J(R)) \oplus KW \subseteqq SW$ and $KW \subseteqq J(S)$. Let $r \in W \cap J(R)$ and $s = \tau + k \in S$, for some $\tau \in R$ and $k \in K$. Put $\beta = sr = \tau r + kr$. Since $\tau r \in J(R)$ there exists a left quasi-inverse $b \in R$ such that $b * (\tau r) = \tau r + b + b\tau r = 0$ and consequently $b * \beta = kr + bkr = \gamma \in J(S)$. That, in turn, yields the existence of $d \in S$ such that $d * \gamma = 0$ and since the "star" composition $*$ is associative, $(d * b) * \beta = 0$. Therefore, $r \in J(S)$ and we obtain $(W \cap J(R) \oplus KW \subseteqq SW \cap J(S)$. Conversely, since $W \cap J(S) \subseteqq R \cap J(S) \subseteqq J(R)$, we obtain $SW \cap J(S) = (W \oplus KW) \cap J(S) = (W \cap J(S)) \oplus KW \subseteqq (W \cap J(R)) \oplus KW$.

2.2. Example. If $S = R[x; \alpha]$, where α is an automorphism of R, then $W = \{r \in R; xr \in J(S)\}$ and $K = \{\Sigma x^i r_i,\ r_i \in R, \text{ and } i \geqq 1\}$. Furthermore, $J(S) = (W \cap J(R)) \oplus KW$, ([2]).

2.3. Theorem. *Suppose that $J(R)S \subseteqq SJ(R)$. Then either of the following conditions implies $J(R) \subseteqq J(S)$.*

(i) *$J(R)$ is right T-nilpotent,*

(ii) *S_R is finitely generated,*

(iii) *Simple right S-modules are R-projective.*

Proof. Let M_S be a simple S-module. Thanks to $J(R)K \subseteq SJ(R)$, $MJ(R)$ is an S-submodule of M_S. Since either of the three conditions (i), (ii), or (iii) implies that $MJ(R) \neq M$ ([1], p. 198, 314]) we obtain $MJ(R) = 0$. *Hence* $J(R) \subseteq J(S)$. (Notice that the Theorem holds for arbitrary extensions with the same identity).

2.4. Theorem. *Suppose $M + K$ is a right S-ideal for each maximal right ideal $M \subseteqq R$ (e.g., $MK \subseteqq M + K$ and $K^2 \subseteqq J(R) + K$). Then $J(S) \subseteqq J(R) \oplus K$. Moreover, if K is nil modulo $J(R)$ then $J(S) = J(R) \oplus K$.*

Proof. If $M \oplus K$ is a right S-ideal then $J(S) \subseteqq \cap (M \oplus K) = J(R) \oplus K$, where the intersection runs through all maximal $M_R \subseteqq R_R$. Now, assume that K is nil modulo $J(R)$. i.e., for each $k \in K$ there exists a natural number n such that $k^n \in J(R)$. Let $j + k \in J(R) \oplus K$ and $s \in S$. According to the hypothesis $(j + k)s = j' + k'$, where $j' \in J(R)$, i.e., $(1 - (j + k)s) = (1 - j') - k$.

Since $j' \in J(R)$ there exists $r \in R$ such that $(1 - j')r = 1$, i.e., $(1 - (j + k)s)(1 - (j + k)s)r = 1 - k'r$, and $k'r \in K$. Now, $(k'r)^n \in J(R)$, for some n, and hence $(1 - (k'r)^n)r' = 1$, for som $r' \in R$. However, $1 = (1 - (k'r)^n)r' = (1 - k'r) \left(\sum_{i=0}^{n-1} (k'r)^i\right) r'$ yields that $(1 - k'r)$ has a right inverse. Thus $(j + k) \in J(S)$.

The following two theorems provide a generalization and an improvement of the normalizing basis theorem ([9], p. 276).

2.5. THEOREM. *Suppose K_R is projective. Then*

(i) *If* $\mathrm{Hom}_R({}_RS_R, M_R)_R$ *is of finite length n_M for every simple M_R then $J^w(S) \subseteqq SJ(R)$, where $w = \sup \{n_M\}$.*

(ii) *If K_R is finitely generated free then either of the following conditions implies that* $\mathrm{Hom}_R({}_RS_R, M_R)_R$ *is of finite length for every simple M_R.*

a) *Each maximal $N_R \subseteqq R_R$ is an ideal of R and $NS = SN$,*

b) *$R/J(R)$ is artinian and $SJ(R) = J(R)S$,*

c) *$S_R = \sum_{i=1}^{n} x_iR$ is a free module with basis $\{x_i; i = 1,\ldots,n\}; x_1 = 1$; and for every $r \in R$, there are $r_i \in R, 1 \leqq i < n, r_1 = r$, such that $rx_j = \left(\sum_{i=1}^{j-1} x_ir_i\right) + x_j\sigma_j(r), 1 < j \leqq n$, where the σ_j's are ring automorphisms of R (triangular matrix commutation).*

PROOF. (i) If $U_R = \mathrm{Hom}_R({}_RS_R, M_R)_R$ is of finite length then $U_S = \mathrm{Hom}_R({}_SS_R, M_R)_S$ is of finite length, too, and length $(U_S) \leq n_M$. Therefore, $U(J^k(S)) = 0$, where $k = n_M$. Since K_R is projective, S_R is projective, too, and $J_R(S_R) = S(J(R))$([1], *p.* 196). On the other hand, $U(J^k(S)) = 0$ yields that $\mathrm{Hom}_R({}_SS_R, M_R)_S(J^w(S)) = 0$, for every simple M_R, and that in turn implies $J^w(S) \subseteqq J_R(S_R)$ (define $J^w(S) = \cap J^n(S)$, where the intersection runs through $n \in \{n_M\}$).

(ii) (a) Let $M_R \simeq R/N$, where $N_R \subseteqq R_R$ is maximal.

Then $M \otimes_R S_R \simeq (S/NS)_R = (S/SN)_R$ is a homogeneous semisimple R-module of finite length. Since $(\mathrm{Hom}_R({}_RS_R, M_R)_R)N = 0$, $\mathrm{Hom}_R({}_RS_R, M_R)_R$ is semisimple and homogeneous. Now, the natural homomorphism ${}_E\mathrm{Hom}_R(M \otimes_R S_R, {}_EM_R) \simeq {}_E\mathrm{Hom}_R(M_R, {}_E\mathrm{Hom}_R({}_RS_R, {}_EM_R)_R)$, where $E = \mathrm{End}_R(M_R)$ is a skewfield, implies that $\mathrm{Hom}_R({}_RS_R, M_R)_R$ is of finite length, too.

(b) Again, let $M_R \simeq R/N$, when $N_R \subseteqq R_R$ is maximal. Then $M \otimes_R S_R \simeq (S/NS)_R$ is isomorphic to a factor module of $(S/(J(R)S))_R = (S/(SJ(R)))_R$ that is semisimple of finite length. Similarly, since $(\mathrm{Hom}_R({}_RS_R, M_R)_R)J(R) = 0$ and $R/J(R)$ is artinian semisimple we obtain that $\mathrm{Hom}_R({}_RS_R, M_R)_R$ is semisimple, too. Now, using the natural transformation introduced in the proof above again we obtain that $\mathrm{Hom}_R({}_FM'_R, \mathrm{Hom}_R({}_RS_R, M_R)_R)_F$ is finite dimensional over $F = \mathrm{End}_R(M'_R)$, for every simple M'_R. Since the representative set of simple right R-modules is finite (thanks to $R/J(R)$ being artinian), $\mathrm{Hom}_R({}_RS_R, M_R)_R$ is necessarily of finite length.

c) Let $\sum_{i=1}^{n} x_iu_i \in S_R$. Then $r\left(\sum_{i=1}^{n} x_iu_i\right) = \sum_{j=1}^{n} x_jt_j$ and the relationship between

$u_i's$ and $t_j's$ can be expressed in an upper-triangular matrix form by

$$\begin{bmatrix} t_1 \\ \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} r & x & x \dots & x & x \\ 0 & \sigma_2(r) & x \dots & x & x \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & \sigma_n(r) \end{bmatrix} \begin{bmatrix} u_1 \\ \\ \vdots \\ u_n \end{bmatrix}$$

Consequently, there is a ring monomorphism $\Psi: R \to UT_n(R)$, where $UT_n(R)$ is the ring of upper triangular matrices of the size n with entries from R, such that the diagonal entries of $\Psi(r)$ are given by $(\Psi(r))_{ii} = \sigma_i(r)$, $i = 1,\dots,n$, (define $\sigma_1 =$ identity). Now, let $f \in \mathrm{Hom}_R({}_RS_R, M_R)_R$ and $f(x_i) = m_i$, $i = 1,\dots,n$. Then $(fr)\left(\sum_{i=1}^{n} x_i r_i\right) = f\left(\sum_{i=1}^{n} m_j t_j\right)$ and we can view $\mathrm{Hom}_R({}_RS_R, M_R)_R \simeq (M_R^n)_R$ as direct sum $(M_1 \oplus \dots \oplus M_n)_R$, where $M = M_i$, $i = 1,\dots,n$, with the scalar right R-multiplication being accomplished by right matrix multiplication with elements of Ψ(R). In particular, if $1 \leqq \kappa \leqq n$, then for every $m_k \in M_k$, $(0,\dots,m_k,0,\dots 0)\, r = (0,\dots,0,m_k\sigma_k(r),m'_{k+1},\dots,m'_n)$, for some $m'_i \in M_i$, $i = k+1,\dots,n$. Hence $(M_k \oplus \dots \oplus M_n)$ is an R-submodule of $(M_1 \oplus \dots \oplus M_n)_R$ for every $1 \leqq k \leqq n$, and since M_R is simple,

$$(M_k \oplus \dots \oplus M_n)R = (0,\dots,0,m_k,0,\dots,0)R + (M_{k+1} \oplus \dots \oplus M_n)R$$

for each $0 \neq m_k \in M$. Furthermore, σ_k being a ring automorphism implies that $\{r \in R;\, m_k r \in (M_{k+1} \oplus \dots \oplus M_n)\} = \sigma_k^{-1}\{r \in R;\, m_k r = 0\}$, a maximal right ideal of R. Thus $(M_k \oplus \dots \oplus M_n)/(M_{k+1} \oplus \dots \oplus M_n))_R$ is simple for each $1 \leq k \leq n$, and consequently $(M_1 \oplus \dots \oplus M_n)_R$ is of finite length n.

2.6. THEOREM. *If* $\mathrm{Hom}_R({}_RS_R, M_R)_R$ *is semisimple for each simple* M_R *and every S-submodule that is an R-direct summand of an S-module is also an S-direct summand, then* $J(S) \subseteqq J(R) \oplus J_R(K_R)$. *Either of the following conditions implies that* $\mathrm{Hom}_R({}_RS_R, M_R)_R$ *is semisimple, for every simple* M_R.

(i) *Each maximal* $N_R \subseteqq R_R$ *is an ideal of* R *and* $NS \subseteqq SN$

(ii) $R/J(R)$ *is artinian and* $J(R)S \subseteqq S\,J(R)$

(iii) $S_R = \Sigma x_i R$ *is a free module with basis* $\{x_i; i \in \Lambda\}$ *and for every* $r \in R$, $rx_i = x_i\sigma_i(r)$, $i \in \Lambda$, *where* $\sigma_i's$ *are ring automorphisms of* R; *and either* K_R *is finitely generated or* $R/J(R)$ *is artinian. (Diagonal matrix commutation).*

PROOF. Let M_R be simple. The hypothesis implies that $\mathrm{Hom}_R({}_SS_R, M_R)_S$ is semisimple, too, and $0 = \mathrm{Hom}_R(S_R, M_R)J(S)$. Therefore, $J(S)S = J(S) \subseteqq J_R(S_R) = J(R) \oplus J_R(K_R)$. For the proofs of (i) and (ii) we can use the same methods as we did in proving Theorem 2.5 ii) a) and b). (Notice that we don't require that $\mathrm{Hom}_R({}_RS_R, M_K)_R$ be of finite length here).

(iii) If K_R is finitely generated then similarly as in the proof of Theorem 2.5.

ii)c), R is acting on $\mathrm{Hom}_R({}_RS_R, M_R)_R$ as diagonal matrices

$$\begin{bmatrix} r & 0 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2(r) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & \dots & 0 & \sigma_n(r) \end{bmatrix}$$

(we set $\sigma_1(r) = r$, for convenience), $\mathrm{Hom}_R({}_RS_R, M_R)_R \simeq (M_1 \oplus \dots \oplus M_n)_R$, where $(M_k)_R \simeq M_R$, for each $k = 1, \dots, n$, and $(0, \dots 0, M_k, 0, \dots, 0)_R \simeq M_R$, i.e., $\mathrm{Hom}_R({}_RS_R, M_R)_R$ is semisimple. In general, the "diagonal" commutation hypotheses implies that $J(R)S = S\,J(R)$ (since $J(R)$ is stable under ring automorphisms of R). Therefore $\mathrm{Hom}_R({}_RS_R, M_R)_R J(R) = 0$ and $R/J(R)$ being artinian implies that $\mathrm{Hom}_R({}_RS_R, M_R)_R$ is semisimple.

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