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THE FIRST TERM IN A MINIMAL PURE INJECTIVE RESOLUTION

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Abstract.

The zero-the term of the minimal pure injective resolution of a commutative noetherian ring R over itself is well understood and determines (in a sense to be made precise) the Jacobson radical of R. In this paper we will study the first term of this resolution and will show that it determines another radical of R which is related to the completeness of R. If R is a coordinate ring over the real or complex numbers, a complete description of the first term will be given. This term will be used to prove a generalization of the approximation theorem for Dedekind domains to rings of Krull dimension one.

1. Terminology and the Statement of the Theorem.

We will use the notation, terminology and results of Enochs [1] and [2]. R will always denote a commutative noetherian ring. For any R-module M, we let

$$0 \to M \to \mathrm{PE}^0(M) \to \mathrm{PE}^1(M) \to \dots$$

denote a minimal pure injective resolution of M. If $M = F$ is flat, then each $\mathrm{PE}^k(F)$ is flat and in fact is uniquely up to isomorphism a product $\prod T_{\mathfrak{P}}$ (over all $\mathfrak{P} \in \mathrm{Spec}(R)$) where $T_{\mathfrak{P}}$ is the completion of a free $R_{\mathfrak{P}}$-module (see [1], pg. 183, Theorem and [2], pg. 352, Lemma 1.1). The cardinality of the base of the free module is denoted $\pi_k(\mathfrak{P}, F)$. If $\pi_k(\mathfrak{P}, F) > 0$, or equivalently $T_{\mathfrak{P}} \neq 0$, we say that $\mathfrak{P}$ appears in $\mathrm{PE}^k(F)$. In this paper, $T_{\mathfrak{P}}$ will always denote such a completion. We note that by Griffith [17] (Proposition 2.10) or, more generally, by Bartijn and Strooker [18] (Corollary 3.15), every $T_{\mathfrak{P}}$ is a flat $R_{\mathfrak{P}}$-module (and so a flat R-module).

Warfield in [3] proved that if $F = R$, then $\mathrm{PE}^0\, R) = \prod \hat{R}_{\mathfrak{M}}$ (over all maximal ideals $\mathfrak{M}$ of R), so only maximal ideals appear in $\mathrm{PE}^0(R)$. Hence the intersection of the prime ideals appearing in $\mathrm{PE}^0(R)$ is the Jacobson radical, $\mathrm{rad}(R)$, of R. In this paper we will consider the intersection of all the prime ideals $\mathfrak{P}$ that appear in $\mathrm{PE}^0(R)$ or in $\mathrm{PE}^1(R)$. Our object is to prove the following:

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THEOREM 1.1. *Suppose* dim R *is finite. If* X *is the set of prime ideals that appears in* $\mathrm{PE}^1(R)$ *and* I *is an ideal of* R *then*

a) *if* $I \subset \mathrm{rad}(R)$, *then* R *is complete with respect to the* I-*adic topology if and only if* $I \subset \mathfrak{P}$ *for all* $\mathfrak{P} \in X$

b) R/I *is a complete semilocal ring if and if* $I \not\subset \mathfrak{P}$ *for all* $\mathfrak{P} \in X$.

2. Preliminaries.

Inthis section we will again use the terminology and results of [1] and [2].

LEMMA 2.1 *If* F *is a flat and pure injective (or equivalently, flat and cotorsion)* R-*module and* F *is separated with respect to the* I-*adic topology for some ideal* I *of* R, *then*

$$F \to F \otimes R/I = F/IF$$

is a flat cover.

PROOF. By Lemma 4.1 of [2], IF is pure injective, so $\mathrm{Ext}^1(G, IF) = 0$ when G is flat. This means that the canonical surjection ϕ: $F \to F/IF$ is a flat precover, and so there is a decomposition $F = F_1 \oplus F_2$ with $F_1 \subset \ker(\phi) = IF$ so that $F_2 \to F/IF$ is a flat cover. But then $IF_1 = F_1$. Since F is separated with the I-adic topology this means $F_1 = 0$ and so $F \to F/IF$ is a flat cover.

We note that the condition that $F = \prod T_{\mathfrak{P}}$ be separated with the I-adic topology just means that $I \subset \mathfrak{P}$ whenever $T_{\mathfrak{P}} \neq 0$. If $I \subset R$ is an ideal and R^* is the completion of R with respect to the I-adic topology, then $R/I \cong R^*/I^*$ where $I^* = IR^*$. Then the prime ideals $\mathfrak{P}^* \supset I^*$ of R^* are in a one-to-one correspondence with the prime ideals $\mathfrak{P} \supset I$ of R. The correspondence is such that $\mathfrak{P}^* = \mathfrak{P}R^*$ corresponds to $\mathfrak{P}$. Note then that $\hat{R}_{\mathfrak{P}} \cong \hat{R}^*_{\mathfrak{P}^*}$. This means that each $T_{\mathfrak{P}^*}$. Hence flat and pure injective R^*-modules $F^* = \prod T_{\mathfrak{P}^*}$ with $T_{\mathfrak{P}^*} = 0$ unless $\mathfrak{P}^* \supset I^*$ are also flat and pure injective R-modules, and flat and pure injective R-modules $F = \prod T_{\mathfrak{P}}$ with $T_{\mathfrak{P}} = 0$ unless $\mathfrak{P} \supset I$ are flat and pure injective R^*-modules. We also note that when $\mathfrak{P} \supset I$, $T_{\mathfrak{P}} \otimes R/I \cong T_{\mathfrak{P}} \otimes R^*/I^*$.

If $F \to M$ and $F' \to M'$ are flat covers of R-modules with $M \cong M'$, then any isomorphism $M \to M'$ can only be lifted by isomorphisms $F \to F'$. This follows easily from the definition of a cover.

In the proof of the theorem to follow, and also in the following section, we will appeal several times to the change of ring theorem (Theorem 4.2, pg. 363 of [2]). In part the theorem says that if a ring homomorphism $R \to R'$ makes R' into a finite R-module, then for any flat R-module F, $\mathrm{PE}^k(F) \otimes R' \cong \mathrm{PE}^k(F \otimes R')$ for all $k \geqq 0$. This implies that if $\mathfrak{P}' \subset R'$ is a prime ideal lying over $\mathfrak{P} \subset R$ then $\pi_k(\mathfrak{P}', R') = \pi_k(\mathfrak{P}, R)$. Hence $\mathfrak{P}$ appears in $\mathrm{PE}^k(R)$ if and only if $\mathfrak{P}'$ appears in $\mathrm{PE}^k(R')$. We note that $\mathfrak{P}$ appears in $\mathrm{PE}^k(R)$ if and only if $\hat{R}_{\mathfrak{P}}$ is a summand of $\mathrm{PE}^k(R)$. We have

PROPOSITION 2.2. *If a prime ideal* $\mathfrak{P}$ *appears in* $\mathrm{PE}^{k+1}(F)$ *for any flat module* F, *then there is a prime ideal* $\mathfrak{Q} \supsetneqq \mathfrak{P}$ *which appears in* $\mathrm{PE}^k(F)$.

PROOF. If a prime ideal $\mathfrak{Q} \supseteqq \mathfrak{P}$ appears in $\mathrm{PE}^k(R)$, then by Theorem 2.1 of [2] there is a prime ideal $\mathfrak{Q} \supsetneqq \mathfrak{P}$ which appars in $\mathrm{PE}^k(R)$.

Hence suppose no prime ideal $\mathfrak{Q} \supseteqq \mathfrak{P}$ appears in $\mathrm{PE}^k(F) = \prod T_{\mathfrak{Q}}$. Then if $T_{\mathfrak{Q}} \neq 0$, $\mathfrak{Q} \not\supseteq \mathfrak{P}$ and so $T_{\mathfrak{Q}} \otimes R/\mathfrak{P} = 0$ (since if $r \in \mathfrak{P}$, $r \notin \mathfrak{Q}$, $T_{\mathfrak{Q}} \xrightarrow{r} T_{\mathfrak{Q}}$ is an isomorphism and $R/\mathfrak{P} \xrightarrow{r} R/\mathfrak{P}$ is 0). But then $\mathrm{PE}^k(F) \otimes R/\mathfrak{P} = 0$. By the change of ring theorem

$$0 \to F \otimes R/\mathfrak{P} \to \mathrm{PE}^0(F) \otimes R/\mathfrak{P} \to \dots$$

is a minimal pure injective resolution of $F \otimes R/\mathfrak{P}$ over $R/\mathfrak{P}$, so by minimality, if $\mathrm{PE}^k(F) \otimes R/\mathfrak{P} = 0$ then $\mathrm{PE}^{k+1}(F) \otimes R/\mathfrak{P} = 0$. The latter is not possible if $\mathfrak{P}$ appears in $\mathrm{PE}^{k+1}(F)$.

COROLLARY 2.3. *If* $\mathfrak{P}$ *appears in* $\mathrm{PE}^k(F)$ *then* $\operatorname{coht} \mathfrak{P} \geqq k$.

PROOF. Immediate.

We note this was a comment in [2], pg. 356 (but without sufficient justification).

The Corollary immediately gives Gruson and Jensen's result [7], Proposition 7.6) that $I\,\mathrm{PE}^k(F) = \mathrm{PE}^k(F)$ whenever $\dim I \leqq k-1$ (for $k \geqq 1$).

PROOF OF THE THEOREM. By the Theorem of [4], if R is complete with the I-adic topology then $I \subset \mathfrak{P}$ for all $\mathfrak{P}$ that appear in any $\mathrm{PE}^k(R)$ (i.e. for all $k \geqq 0$). This gives the "only if" part of a).

Now we prove the "if" part of a). Let R^* be the completion of R with the I-adic topology and let $I^* = IR^*$. By the remarks of the previous section, each $\mathrm{PE}^k(R^*)$ is pure injective as an R-module. This means that there is a commutative diagram

$$(1)\qquad \begin{array}{ccccccc} 0 \to R & \to & \mathrm{PE}^0(R) & \to & \mathrm{PE}^1(R) & \to & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to R^* & \to & \mathrm{PE}^0(R^*) & \to & \mathrm{PE}^1(R^*) & \to & \dots \end{array}$$

with $R \to R^*$ the natural map.

Now by the change of ring Theorem [2], if we apply $R/I \otimes$ - to the minimal resolution of R we get a minimal resolution of R/I as a module over itself. We do the same with $R^*/I^* \otimes$ - applied to the resolution of R^*. Then diagram (1) gives us a diagram

$$\begin{array}{ccccccc} 0 \to R/I & \to & \mathrm{PE}^0(R) \otimes R/I & \to & \mathrm{PE}^1(R) \otimes R/I & \to & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to R^*/I^* & \to & \mathrm{PE}^0(R^*) \otimes R^*/I^* & \to & \mathrm{PE}^1(R^*) \otimes R^*/I^* & \to & \dots \end{array}$$

with both rows minimal pure injective resolutions of R/I and R^*/I^* (respectively) over themselves. Since $R/I \to R^*/I^*$ is an isomorphism, minimality implies that all the vertical maps are isomorphisms.

Since $I \subset \operatorname{rad}(R)$, $I \subset \mathfrak{M}$ for all maximal ideals, so $\mathrm{PE}^0(R)$ is separated with the I-adic topology. Then using the remarks and the Lemma 2.1 of the previous section,

$$\mathrm{PE}^0(R) \to \mathrm{PE}^0(R) \otimes R/I \cong \mathrm{PE}^0(R)/I\,\mathrm{PE}^0(R)$$

is a flat cover as R-modules, or as R^*-modules.

By hypothesis, $I \subset \mathfrak{P}$ for all $\mathfrak{P}$ that appear in $\mathrm{PE}^1(R)$, so similarly we get

$$\mathrm{PE}^1(R) \to \mathrm{PE}^1(R) \otimes R/I$$

is a flat cover over R and over R^*. Another appeal to the Theorem of [4] says that $I^* \subset \mathfrak{P}^*$ for all prime ideals $\mathfrak{P}^*$ of R^* that appear in any $\mathrm{PE}^k(R^*)$, so in particular in $\mathrm{PE}^0(R^*)$ and in $\mathrm{PE}^1(R^*)$. Then we get that

$$\mathrm{PE}^k(R^*) \to \mathrm{PE}^k(R^*) \otimes R^*/I^*$$

are flat covers as R^*-modules for $k = 0,1$. Then the isomorphisms

$$\mathrm{PE}^k(R) \otimes R/I \to \mathrm{PE}^k(R^*) \otimes R^*/I^*, \quad k = 0,1$$

can be lifted only by isomorphisms $\mathrm{PE}^k(R) \to \mathrm{PE}^k(R^*)$, $k = 0,1$, guaranteeing that each are isomorphisms. An appeal to the diagram (1) then gives that $R \to R^*$ is an isomorphism, and shows that R is already complete. Note that $R \to R^*$ an isomorphism guarantees that in fact each $\mathrm{PE}^k(R) \to \mathrm{PE}^k(R^*)$ is an isomorphism.

For b) we first note that using Warfield's result we see that $R \to \mathrm{PE}^0(R) = \prod \hat{R}_{\mathfrak{M}}$ ($\mathfrak{M}$ a maximal ideal of R) is an isomorphism if and only if R is a complete semilocal ring. But $R \to \mathrm{PE}^0(R)$ is an isomorphism if and only if $\mathrm{PE}^1(R) = 0$. We now use the change of ring theorem. If $\mathrm{PE}^1(R) = \prod T_{\mathfrak{P}}$, then $\mathrm{PE}^1(R/I) \cong \mathrm{PE}^1(R) \otimes R/I$. We see that $\mathrm{PE}^1(R/I) = 0$ if and only if $(\prod T_{\mathfrak{P}}) \otimes R/I \cong \prod (T_{\mathfrak{P}}\, R/I) = 0$. But $T_{\mathfrak{P}} \otimes R/I = 0$ only if $T_{\mathfrak{P}} = 0$ or $I \not\subset \mathfrak{P}$.

In sum, R/I is complete semilocal if and only if $\mathrm{PE}^1(R/I) = 0$ and $\mathrm{PE}^1(R/I) = 0$ if and only if $I \not\subset \mathfrak{P}$ for all $\mathfrak{P} \in X$. This completes the proof of the theorem.

Remark 1. The intersection I of all prime ideals $\mathfrak{P}$ appearing in $\mathrm{PE}^0(R)$ or in $\mathrm{PE}^1(R)$ is the largest ideal such that R is complete with the I-adic topology (cf. Matsumura [5], exercise 8.1, pg. 63). This coincides with what Eakin and Sathaye [6] denote $I_c(R)$.

Remark 2. By the theorems of [4] and the theorem of this paper, we get no further radicals by generalizing the procedure used above, i.e. if $k \geqq 1$, the intersection of all the prime ideals appearing in one of $\mathrm{PE}^0(R)$, $\mathrm{PE}^1(R), \ldots$ $\mathrm{PE}^k(P)$ coincides with the intersection of those appearing in one of $\mathrm{PE}^0(R)$, $\mathrm{PE}^1(R)$.

REMARK 3. The prime ideals $\mathfrak{P}$ that appear in $\mathrm{PE}^1(R)$ all have coht $\mathfrak{P} \geqq 1$. If coht $\mathfrak{P} = 1$, $\mathfrak{P}$ may fail to appear but if and only if $\mathfrak{P}$ is contained in a unique maximal ideal $\mathfrak{M}$ of R and if the formal fibre of $R_{\mathfrak{M}}$ over $\mathfrak{P}$ is trivial, i.e. $\hat{R}_{\mathfrak{M}} \otimes k(\mathfrak{P}) \cong k(\mathfrak{P})$ over R (cf. Proposition 2.1 of [4]).

REMARK 4. If $I = 0$ in the Theorem 1.2, then b) coincides with Gruson and Jensen's theorem 9.1 of [7].

3. $\mathrm{Ext}^i(K, R) = 0$ and $\mathrm{PE}^1(R)$ for coordinate rings.

Gruson ([8], Proposition 3.2) proved that if k is an uncountable field then $\mathrm{Ext}^1(k(x, y), k[x, y]) = 0$. In this section we prove that if R is an inegral coordinate ring over the real or complex numbers and K is its field of fractions then $\mathrm{Ext}^1(K, R) = 0$ whenever $\dim R \geqq 2$ (If $\dim R = 1$ for any domain R then $\mathrm{Ext}^1(K, R) = 0$ if and only if R is a complete local domain). We then use these results to show that the set X of the theorem of the previous section for such coordinate rings (whether integral or not) consists of the prime ideals of coheight 1.

LEMMA 3.1. *If M is an R-module and $E \subset \mathrm{PE}(M)$ is an injective submodule and $E \cap M = 0$, then $E = 0$.*

PROOF. Let $\mathrm{PE}(M) = S \oplus E$ and $x = (\phi_1(x), \phi_1(x))$ for $x \in M$. If $M \cap E = 0$ then $\phi_1(x) \to \phi_2(x)$ is a well-defined linear map so can be extended to g: $S \to E$. Then ψ: $(y_1, y_2) \to (y_1, y_2 - g(y_1))$ is an automorphism of $\mathrm{PE}(M)$, so $\mathrm{PE}(M)$ is a pure injective envelope of $\psi(M)$. Since $\psi(M) \cap E = 0$, we can assume $\psi(M) \subset S$ (let $S \supset \psi(M)$ be maximal with $S \cap E = 0$). Then $\psi(M) \to S$ is a pure injection so $\psi(M) \to \mathrm{PE}(M)/E \cong S$ is too. Hence by minimality, $E = 0$.

LEMMA 3.2. *Let R be a domain and K its field of fractions and*

$$0 \to F \to \mathrm{PE}^0(F) \to \mathrm{PE}^1(F) \to \dots$$

be the minimal pure injective resolution of a flat module F. Then the complex $0 \to \mathrm{Hom}(K, \mathrm{PE}^0(F)) \to \mathrm{Hom}(K, \mathrm{PE}^1(F)) \to \dots$ has all its maps 0.

PROOF. If σ: $K \to \mathrm{PE}^i(F)$ and $\sigma \neq 0$ then σ is an injection since $\mathrm{PE}^1(F)$ is flat and so torsion free. Then we want to argue that $K \to \mathrm{PE}^i(F) \to \mathrm{PE}^{i+1}(F)$ is 0. If not, then it is an injection and so $K \cap \ker(\mathrm{PE}^i(F) \to \mathrm{PE}^{i+1}(F)) = 0$. This contradicts Lemma 3.1.

As an immediate consequence we get

Corollary 3.3. $\mathrm{Ext}^i(K, F) \neq 0$ *if and only if* $K \subset \mathrm{PE}^i(F)$ *(as a submodule)*, i.e. *if and only if* (0) *appears in* $PE^i(R)$.

PROOF. As noted in Raynaud and Gruson [9], since K is flat $\operatorname{Ext}^i(K, F)$ can be computed using pure injective resolutions of F (this uses the fact that $\operatorname{Hom}(-,-)$ is balanced by Flat $\times$ Pure Inj, Enochs and Jenda [10]). The result then follows immediately from the preceding Lemma.

DEFINITION. For a ring R, by the curve-adic topology on a module M, we mean that topology with all IM as neighborhoods of 0 where $I \subset R$ is any ideal with $\dim R/I \leqq 1$.

PROPOSITION 3.4. *When* $n \geqq 2$,

$$\mathrm{PE}(\mathbf{C}[x_1,\ldots,x_n])/\mathbf{C}[x_1,\ldots,x_n]$$

is separated with the curve-adic topology ($\mathbf{C}$ *the complex numbers*).

PROOF. Let $\mathfrak{P} \subset \mathbf{C}[x_1,\ldots,x_n]$ be a homogeneous prime ideal of coheight 1. Then the projective variety $V(\mathfrak{P}) \subset \mathbf{P}^{n-1}(\mathbf{C})$ is a point, say with homogeneous coordinates $(a_1, a_2, \ldots, a_n)$. Suppose $S \in \mathbf{C}[[x_1,\ldots,x_n]]$ is in the closure of $\mathbf{C}[x_1,\ldots,x_n]$ with the curve-adic topology on $\mathbf{C}[[x_1,\ldots,x_n]]$. Then $S - f \in \mathfrak{P}\,\mathbf{C}[[x_1,\ldots,x_n]]$ for some $f \in \mathbf{C}[x_1,\ldots,x_n]$. If $S = S_0 + S_1 + \ldots$ with S_i homogeneous of degree i, then for large i, $S_i \in \mathfrak{P}$ and so $S_i(a_1,\ldots,a_n) = 0$. If S is not a polynomial, then infinitely many $S_i \neq 0$ and by the above $\cup V(S_i)$ (over $S_i \neq 0$) is $\mathfrak{P}^{n-1}(\mathbf{C})$. This is impossible by the Baire category theorem. Now note that $\mathrm{PE}(\mathbf{C}[x_1,\ldots,x_n]) = \prod \mathbf{C}[[x_1 - a_1,\ldots,x_n - a_n]]$ over all $a = (a_1,\ldots,a_n) \in \mathbf{C}^n$ ([3], Theorem 3). Suppose $(S_a) \in \mathrm{PE}(\mathbf{C}[x_1,\ldots,x_n])$ is in the closure of $\mathbf{C}[x_1,\ldots,x_n]$. Then by the above, each S_a is a polynomial. Now given any prime ideal $\mathfrak{P} \subset \mathbf{C}[x_1,\ldots,x_n]$ of coheight 1 and any $k \geqq 1$, there is an $f \in \mathbf{C}[x_1,\ldots,x_n]$ so that $S_a - f \in \mathfrak{P}^k\,\mathbf{C}[[x_1 - a_1,\ldots,x_n - a_n]]$ for all $a \in \mathbf{C}^n$. But $\mathbf{C}[x_1,\ldots,x_n]_{(x_1 - a_1,\ldots,a_n - a_n)}$ is pure in $\mathbf{C}[[x_1 - a_1,\ldots,x_n - a_n]]$ (Serre [11], Proposition 2.7) so we get

$$S_a - f \in \mathfrak{P}^k\,\mathbf{C}[x_1,\ldots,x_n]_{(x_1 - a_1,\ldots,a_n - a_n)}.$$

But then if $b \in \mathbf{C}^n$,

$$S_a - S_b \in \mathfrak{P}^k(\mathbf{C}[x_1,\ldots,x_n]_{(x_1 - a_1,\ldots,a_n - a_n)} + \mathbf{C}[x_1,\ldots,x_n]_{(x_1 - a_1,\ldots,a_n - a_n)})$$

If $a, b \in V(\mathfrak{P}) \subset \mathbf{C}^n$ (for example, suppose $V(\mathfrak{P})$ is the line through a and b) then $S_a - S_b \in \mathfrak{P}^k(\mathbf{C}[x_1,\ldots,x_n]_{\mathfrak{P}})$. Since $k \geqq 1$ was arbitrary, we get $S_a = S_b$. This completes the proof.

THEOREM 3.5. *If R is a coordinate ring over the real or complex numbers, then no prime ideal* $\mathfrak{P} \subset R$ *with* $\operatorname{coht} \mathfrak{P} \geqq 2$ *can appear in* $\mathrm{PE}^1(R)$.

PROOF. By the previous Proposition, if $n \geqq 2$, then $(\mathbf{C}[x_1,\ldots,x_n])/\mathbf{C}[x_1,\ldots,x_n]$ is separated in the curve-adic topology, hence it clearly cannot contain $\mathbf{C}(x_1,\ldots,x_n)$

as a submodule, hence by Lemma 3.2 the prime ideal (0) doesn't appear in $PE^1(\mathbf{C}[x_1,\ldots,x_n])$. If $\mathbf{C}[x_1,\ldots,x_n] \subset R$ is a finite integral extension, then by the change of ring theorem, no minimal prime ideal $\mathfrak{P} \subset R$ can appear in $PE^1(R)$. Now let R be any coordinate ring over $\mathbf{C}$ and suppose $\mathfrak{P}$ with coht $\mathfrak{P} \geqq 2$ appears in $PE^1(R)$. Then by the change of ring theorem, (0) appears in $PE^1(R/\mathfrak{P})$. But (0) is minimal in $R/\mathfrak{P}$ and $\dim R/\mathfrak{P} \geqq 2$, so $R/\mathfrak{P} \supset \mathbf{C}[x_1,\ldots,x_s]$ is finite integral for some $s \geqq 2$. This contradicts the above and completes the proof. The proof for a coordinate ring R over the reals in proved similarly. We only need note that $R \otimes \mathbf{C}$ (over the reals) is a coordinate ring over $\mathbf{C}$ and that $R \to R \otimes \mathbf{C}$ is a finite integral extension and then appeal to the change of ring theorem.

We note that by ([2], Proposition 1.2) when coht $\mathfrak{P} = 1$ in the above, it is not hard to argue that $\pi_1(\mathfrak{P}, R)$ has the cardinality of the continuum. This gives then a complete description of $PE^1(R)$.

Corollary 3.6. *If R is a coordinate ring over the real or complex numbers and $\mathfrak{Q} \subset R$ is a prime ideal with* coht $\mathfrak{Q} \geqq 2$ *then* $\mathrm{Ext}^1(\hat{R}_{\mathfrak{Q}}, R) = 0$.

PROOF. By the theorem $PE^1(R) = \Pi\, T_{\mathfrak{P}}$ (over $\mathfrak{P}$ with coht $\mathfrak{P} = 1$). By [2] (Corollaries 1 and 2, pg. 353), $\mathrm{Hom}(\hat{R}_{\mathfrak{Q}}, PE^1(R)) = 0$ and so $\mathrm{Ext}^1(\hat{R}_{\mathfrak{Q}}, R) = 0$.

COROLLARY 3.7. *If R is an integral coordinate ring over the real or complex numbers and K is its field of fractions, then if* $\dim R \geqq 2$, $\mathrm{Ext}^1(K, R) = 0$.

PROOF. This is a special case of the previous corollary with $\mathfrak{Q} = 0$, for then $\hat{R}_{(0)} = R_{(0)} = K$.

We remark that if R is any domain and $\dim R = 1$, then $\mathrm{Ext}^1(K, R) = 0$ if and only if R is a complete local ring. For the only prime that can appear in $PE^1(R)$ is (0) and if $\mathrm{Ext}^1(K, R) = 0$ it doesn't, so $PE^1(R) = 0$. Hence $R = PE^0(R) = \prod \hat{R}_{\mathfrak{M}}$ (over maximal $\mathfrak{M}$). Since R is a domain, there is only one maximal ideal $\mathfrak{M}$ and $R = \hat{R}_{\mathfrak{M}}$. This is a slight addition to Matlis' Theorem 4 in [12].

In connection with the above we have:

Proposition 3.8. *If R is a domain and R is complete with the I-adic topology for some ideal $I \neq 0$, then* $\mathrm{Ext}^i(K, R) = 0$ *for all i (with K the field of fractions).*

By ([4], Theorem) if (0) appears in $PE^i(R)$ then (0) would have to contain I which is impossible. This means we cannot have $K \subset PE^i(R)$. As noted earlier, if $\mathrm{Hom}(K, PE^i(R)) \neq 0$ then $K \subset PE^i(R)$. This shows $\mathrm{Ext}^i(K, R) = 0$.

Remark 1. If R is a coordinate ring over the real or complex numbers we conjecture that a prime ideal $\mathfrak{P} \subset R$ appears in $PE^i(R)$ if and only if coht $\mathfrak{P} = i$.

By the methods used in the proof of Theorem 3.5 this is equivalent to the following:

For $n \geqq 1$, $\mathrm{Ext}^i(\mathrm{C}(x_1,\ldots,x_n), \mathrm{C}[x_1,\ldots,x_n]) = 0$ for $i < n$ and $\mathrm{Ext}^n(\mathrm{C}(x_1,\ldots,x_n), \mathrm{C}[x_1,\ldots,x_n]) \neq 0$. Gruson in [8] proved this form of the conjecture in case $n = 2$ (for any uncountable field).

There seems to be no hope of proving this conjecture without additional set theory hypotheses (cf. [7], Theorem 7.10).

We note that for even regular local rings, the corresponding form of the conjecture is not true.

Remark 2. There is an interesting analogy between results on the injective resolution of a ring R over itself and its minimal pure injective resolution.

1) In the first primes go up in some sense (see Bass [13], Lemma 3.1) and in the second they go down (Proposition 2.2 above and [2], Theorem 2.1).

2) In this first, if R is local with maximal ideal $\mathfrak{M}$, $\mathrm{Hom}(R/\mathfrak{M}, -)$ applied to the injective resolution gives a trivial complex ([13], pg. 12) in the second $\mathrm{Hom}(K, -)$ gives a trivial complex when R is a domain K its field of fractions (Lemma 3.2).

3) In $E^0(R)$ only the minimal primes appear ([14], Matlis) and in $\mathrm{PE}^0(R)$ only the maximal ideals appear.

REMARK 3. If R is a coordinate ring over any field k, then the change of ring theorem and the relations between chains of prime ideals of R and those of $k[x_1,\ldots,x_n] \subset R$ (where the extension $k[x_1,\ldots,x_n] \subset R$ is integral and $x_1,\ldots,x_n$ are indeterminants over k, cf. Serre [16], Theorem 2, Chapter 3) show that if $\mathfrak{P}, \mathfrak{Q} \subset R$ are prime ideals and $\operatorname{coht} \mathfrak{P} = \operatorname{coht} \mathfrak{Q}$, then $\pi_i(\mathfrak{P}, R) = \pi_i(\mathfrak{Q}, R)$ for all i. In particular, $\mathfrak{P}$ appears in $\mathrm{PE}^i(R)$ if and only if $\mathfrak{Q}$ does.

4. A Generalized Approximation Theorem.

Let R be a Dedekind domain and K its field of fractions. If A is the ring of restricted adeles, then $A \supset \prod \hat{R}_{\mathfrak{M}}$ (over all maximal ideals $\mathfrak{M}$). The approximation theorems says that the natural map $K \to A/\prod \hat{R}_{\mathfrak{M}}$ is a surjection (cf. Bourbaki [15], Proposition 2, pg. 497). A can be identified with $K \otimes \prod \hat{R}_{\mathfrak{M}}$ and so $\prod \hat{R}_{\mathfrak{M}}$ is identified with $A \otimes \prod \hat{R}_{\mathfrak{M}}$. So by right exactness of the tensor product, $K \otimes \hat{R}_{\mathfrak{M}}/A \otimes \prod \hat{R}_{\mathfrak{M}} \cong \mathrm{K/R} \otimes \hat{R}_{\mathfrak{M}}$. Then the image of K in $K/R \otimes \prod \hat{R}_{\mathfrak{M}}$ is $K/R \otimes R$. Hence the theorem says the quotient of $K/R \otimes \prod \hat{R}_{\mathfrak{M}}$ by $K/R \otimes R$ is 0, or equivalently that $K/R \otimes \prod \hat{R}_{\mathfrak{M}}/R$ is 0.

Now let R be any ring of Krull dimension 1 and let K be its total quotient ring. Then since $\mathrm{PE}^0(R) = \prod \hat{R}_{\mathfrak{M}}$ and since $\mathrm{PE}^2(R) = 0$ ([7], Theorem 7.1), $\mathrm{PE}^1(R) = \prod \hat{R}_{\mathfrak{M}}/R$, so the above says $K/R \otimes \mathrm{PE}^1(R) = 0$ when R is Dedekind. We claim

PROPOSITION 4.1. *If R is of Krull dimension 1 and K is its total quotient ring, then $K/R \otimes \mathrm{PE}^1(R) = 0$.*

PROOF. By Proposition 2.2, $\mathrm{PE}^1(R) = \prod T_{\mathfrak{P}}$ with the product over minimal primes $\mathfrak{P}$ of R. If $r \in R$ is regular, then $r \notin \mathfrak{P}$ for any such $\mathfrak{P}$, so $\mathrm{PE}^1(R) \xrightarrow{r} \mathrm{PE}^1(R)$ is an isomorphism so for each $y \in \mathrm{PE}^1(R)$, $y = rz$ for some $z \in \mathrm{PE}^1(R)$. But for $x \in K/R$, $rx = 0$ for some such r. Then $x \otimes y = x \otimes rz = xr \otimes z = 0$.

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