LOCAL MODULI FOR PLANE CURVE SINGULARITIES, THE DIMENSION OF THE $\tau$-CONSTANT STRATUM

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1. Introduction and Generalities

Consider the plane curve singularity defined by $f = x_1^p + x_2^q$, and the set of $\mu$-constant deformations of $f$ with minimal Tjurina number. The set $T_{p,q}$ of isomorphism classes of such deformations, has a natural scheme structure, see [L-M-P]. Zariski, [Z], gave a formula for the dimension of $T_{p,q}$ when $q = p + 1$, and in [D], Delorme proves a formula for the case $\gcd(p,q) = 1$. In the general case there are recursion formulas, see [L-M-P], best to my knowledge, no other closed formulas are known.

The aim of this paper is to give such a closed formula for the dimension of $T_{p,q}$ when $2 | \gcd(p,q)$.

Let $k$ be any field, and consider a polynomial $f \in k[x_1, x_2]$. Put $x^g := x_1^{a_1} x_2^{a_2}$ for $g = (a_1, a_2)$, and let $\{x^g\}_{g \in I}$ be a monomial basis for $H^1(f) := k[x_1, x_2]/(f, \partial f/\partial x_1, \partial f/\partial x_2)$. Put

$$\tau(f) = \dim_k H^1(f),$$
$$\mu(f) = \dim_k k[x_1, x_2]/(\partial f/\partial x_1, \partial f/\partial x_2).$$

When $f = x_1^p + x_2^q$, $I = \{(a_1, a_2) | 0 \leq a_1 \leq p - 2, 0 \leq a_2 \leq q - 2\}$. Moreover, putting $I_\mu = \{(a_1, a_2) \in I | a_1/p + a_2/q \geq 1\}$, one knows that any $\mu$-constant deformation of $f$ is isomorphic to one in the family $F_\mu = x_1^p + x_2^q + \sum_{g \in I_\mu} t_g x_1^{a_1} x_2^{a_2}$. Put $H_\mu = k[t_g]_{g \in I_\mu}, H_\mu = \text{Spec}(H_\mu)$.

The moduli space $T_{p,q}$, parametrizing isomorphism classes of $\mu$-constant deformations of the singularity $f$ with minimal $\tau$, is a quotient scheme $(S/V_\mu)/G$, where $S$ is an open subscheme of $\text{Spec}(H_\mu)$ and $V_\mu$ is the kernel of the Kodaira-Spencer map associated to the family $F_\mu$. Recall, see [L-M-P], that $V_\mu$ is

* This paper contains the results of my cand. scient. thesis at the University of Oslo 1985, see [HOH].
Received May 3, 1988
a graded Lie-algebra generated as $H_n$-module by a finite dimensional Lie-algebra $V_0$, acting rationally on $S$, such that $S/V_\mu = S/\exp V_0$. Finally $G$ is a finite group acting rationally on $(S/V_\mu)$.

Let $H_\mu(f)$ be the subspace of $H^1(f)$ generated by $\{x^a\}_{a \in I_\mu}$. Then $H_\mu^1(f)$ is the tangent space of $H_\mu$ at $0$. In the paper [L-P], Laudal and Pfister consider the action $\sigma$ of $\text{Der}_k(k[x]/(f))$ on $H^1(f)$ defined as follows. Let $D$ be a derivation of $k[x]$ representing the derivation $D \in \text{Der}_k(k[x]/(f))$. Then $D(f) = q : f$. Let $\xi \in k[x]$ represent the element $\xi \in H^1(f) = k[x]/(f, \partial f/\partial x_1, \partial f/\partial x_2)$, then $\sigma(D)$ is the class of $\bar{D}(\xi) - q \cdot \xi$ in $H^1(f)$.

It is easy to see that $H_\mu^1(f)$ is invariant under $\sigma$. Let for $\xi \in H_\mu^1(f)$, $o(\xi) \subseteq H^1(f)$ be the orbit of $\xi$ under $\text{Der}_k(k[x]/(f))$, i.e. $o(\xi) = \{\sigma(D) \cdot \xi \mid D \in \text{Der}_k(k[x]/(f))\}$, then it follows from [L-P], that we have the following results.

**Proposition 1.** Let $f = x_1^p + x_2^q$, then $\dim T_{p,q} = \dim_k H_\mu^1(f) - \max_{\xi \in H_\mu^1(f)} \dim o(\xi)$.

**Proof.** see [L-P] (4.6) (ii), (4.7) and remarks following (4.7) together with (5.7). See also remarks preceding (5.12).

**Proposition 2.** (i) Let $\xi \in H_\mu^1(f)$ be represented by $x^a$, then $o(\xi)$ is the subspace of $H_\mu^1(f)$ generated by the classes of

$$\{(a_1/p + a_2/q - 1)x^a \cdot s \mid s \in k[x]\}$$

(ii) Let $H_\mu^1(f)$ be the subspace of $H^1(f)$ generated by

$$I_+ = \{x^a \mid a \in I_\mu, a_1/p + a_2/q > 1\},$$

then

$$\max_{\xi \in H_\mu^1(f)} \dim o(\xi) = \max_{\xi \in H_\mu^1(f)} \dim_k O(\xi),$$

where $O(\xi)$ is the subspace of $H_\mu^1(f)$ generated by $\{\xi \cdot s \mid s \in k[x]\}$.

**Proof.** See [L-P] (4.6) (iii).

2. Dimension of the generic Component.

The aim of this part is the calculation of the dimension of the maximal orbit of the action $\sigma$ on $H_\mu^1(f)$, which according to proposition 2 above enables us to calculate $\dim T_{p,q}$. The main result, which will be proved at the end of this paper, is

**Theorem 1.** Let $f = x_1^p + x_2^q$ and suppose $2|\gcd(p, q)$. The maximal orbit dimension of the action $\sigma$ on $H_\mu^1(f)$ is then

$$\text{maxorbdim} = \left(\frac{p}{2} - 1\right)\left(\frac{q}{2} - 1\right) - \gcd\left(\frac{p}{2}, \frac{q}{2}\right) + \begin{cases} 1 & \text{if } p|q \text{ or } q|p \\ 0 & \text{otherwise} \end{cases}$$
i) Let $h_{\text{gen}} = \sum_{g \in I_+} t_g x^g$ where the $t_g$ are variables over the field $k$, i.e. $h_{\text{gen}} \in k [t_g] [x] / (x_1^{p-1}, x_2^{q-1})$

ii) Let $h \in H^1_+(f)$, then $h = \sum_{g \in I_+} c_g x^g, c_g \in k$. Define $\text{Support}(h) = S(h) = \{ g \in I_+ | c_g \neq 0 \}$, $S(x) = S(x^{h_{\text{gen}}})$. Then

**Lemma 2.** $S(x) = \{ g' \in I_+ | x_1'/p + x_2'/q > 1 + x_1/p + x_2/q \}$

**Proof.** Follows directly from i) and ii).

iii) Lemma 2 shows that the set $\{ \text{Support}(x) | 0 \leq \alpha_1 \leq p - 2, 0 \leq \alpha_2 \leq q - 2 \}$ is linearly ordered under inclusion.

Let $S(x_1) \subseteq \ldots \subseteq S(x_0) = I_+$ be a maximal chain of proper inclusions. We define a subdivision of $I_+$ as follows:

$I_M = S(x_M), I_m = S(x_m) \setminus S(x_{m+1})$ for $0 \leq m < M$.

Set $I^b_a = \bigcup_{m=a}^b I_m$.

iv) Define $\text{Set}(m) = \{ x^g h_{\text{gen}} | S(x) \subseteq I^M_m, S(x) \not\subseteq I^M_{m+1} \}$. We observe that for fixed $m$, every element of $\text{Set}(m)$ has the same support and that for any $h \in H^1_+(f)$ a basis for the orbit of $h$ can be injectively embedded in $\bigcup_{m=0}^M \text{Set}(m)$.

v) For every finite set $X$, let $\#X$ denote the number of elements of $X$.

We are going to show that there exists a $w \in \mathbb{N}$, such that

**Proposition 3.** $\# \text{Set}(m) \leq \# I_m$,

$\# \text{Set}(w - 1) = \# I_{w-1} + 1$

$\# \text{Set}(w) = \# I_w - 1$

$\# \text{Set}(m) \geq \# I_m$, $w + 1 \leq m \leq M$

Accepting this, we can prove

**Proposition 4.** The dimension of the maximal orbit of the action $\sigma$ on $H^1_+(f)$ is

$$\max_{\xi \in H^1_+(f)} \dim o(\xi) = \sum_{m=0}^M \min(\# I_m, \# \text{Set}(m)) + 1.$$ 

**Proof.** The inequality $\leq$ follows using Proposition 3:

$$\sum_{m=0}^M \min(\# I_m, \# \text{Set}(m)) + 1 = \sum_{m=0}^{w-2} \# \text{Set}(m) + \# I^M_{w-1} \geq \max_{\xi \in H^1_+(f)} \dim o(\xi).$$
The other direction \( \geq \) can be proved as follows:

i) Define a new subdivision of \( I_+ \) by fixing an element \( g \in I_w \), and putting

\[
J_{w-1} = I_{w-1} \cup \{g\} \\
J_w = I_w \setminus \{g\} \\
J_m = I_m \quad \text{otherwise.}
\]

ii) For \( \#\text{Set}(m) \leq \#J_m \) choose \( \#\text{Set}(m) \) points from \( J_m \). Enumerate the polynomials of \( \text{Set}(m) \) and the chosen points of \( J_m \) from 1 to \( \#\text{Set}(m) \). For \( \#\text{Set}(m) > \#J_m \) choose \( \#J_m \) polynomials from \( \text{Set}(m) \). Enumerate the points of \( J_m \) and the chosen polynomials of \( \text{Set}(m) \) from 1 to \( \#J_m \).

iii) We then construct the square matrices \( C_m, m = 0, \ldots, M \) by setting \( c_i^j \) in \( C_m \) equal to the coefficient of the monomial of polynomial \( i \) in \( \text{Set}(m) \) corresponding to point \( j \) in \( J_m \), (i.e. the monomial \( x^a \) corresponds to the point \( g \)). We obtain

\[
0 \geq \det C_m \in k[t_{g_i}]_{g_i \in I_m}, \ m = 0, \ldots, M,
\]

immediately from the fact that no column contains the same \( t_{g_i} \) twice. Setting \( \{t_{g_i}\}_{g_i \in I_m} \) equal to a closed point of \( \text{Spec}(k[t_{g_i}]_{g_i \in I_m}/(1 - \prod_{m=0}^M \det C_m)) \) and counting, (using i), ii)) gives the wanted inequality.

We shall now prove that there is a duality between \( I_{M-m} \) and \( \text{Set}(m) \) and later on we shall actually compute \( I_m \) and therefore \( \text{Set}(m) \).

**Proposition 5.** \( \#\text{Set}(m) = \#I_{M-m} \).

**Proof.** The 1-1 pairing of the two sets is given by associating to \( x_1^{\alpha_1}x_2^{\alpha_2}h_{\text{gen}} \in \text{Set}(m) \) the element \( (p - 2 - \alpha_1, q - 2 - \alpha_2) \in I_{M-m} \).

i) Different \( \text{Set}(i) \) are sent into different \( I_j \): Let \( x^a h_{\text{gen}} \in \text{Set}(m) \) and \( x^a h_{\text{gen}} \in \text{Set}(m') \) where \( m < m' \). Choose an element \( g'' \) of \( I_m \). Then, using Lemma 2, \( 1 + \alpha_1/p + \alpha_2/q \geq \alpha'_1/p + \alpha'_2/q > 1 + \alpha_1/p + \alpha_2/q \) or rearranging \( (p - 2 - \alpha_1)/p + (q - 2 - \alpha_2)/q > 1 + (p - 2 - \alpha_1)/p + (q - 2 - \alpha_2)/q \geq (p - 2 - \alpha'_1)/p + (q - 2 - \alpha'_2)/q \) which means that (see Lemma 2 and iv) above) \( (p - 2 - \alpha_1, q - 2 - \alpha_2) \) and \( (p - 2 - \alpha'_1, q - 2 - \alpha'_2) \) belong to different \( I_j \).

ii) Different \( I_j \) are sent into different \( \text{Set}(i) \): Let \( g \in I_m, g' \in I_m, \) where \( m < m' \). Then there exists \( g'' \), such that (see Lemma 2 and iv) above) \( \alpha_1/p + \alpha_2/q > 1 + \alpha'_1/p + \alpha'_2/q \geq \alpha_1/p + \alpha_2/q \). Rearranging as in i) and using once more Lemma 2 and (v) above, we reach the desired conclusion.

Applying Proposition 3 we get

**Corollary 6.** \( \max_{\xi \in H^1_J(f)} \dim o(\xi) = 2\#I^M_w - 1 \)

**Proof.** Proposition 4 gives

\[
\max_{\xi \in H^1_J(f)} \dim o(\xi) = \sum_{m=0}^{M} \min(\#I_m, \#\text{Set}(m)) + 1
\]
\[ = \sum_{m=0}^{w-1} \#I_m - m + \sum_{m=w}^{M} I_m - 1, \] using Propositions 3 and 5. Now these imply \( M = 2w - 1 \), i.e.

\[
\max_{\xi \in H^1(f)} \dim \alpha(\xi) = 2 \sum_{m=w}^{M} \#I_m - 1.
\]

Set \( r = p/\gcd(p, q), s = q/\gcd(p, q) \) and assume, as we may, \( r \geq s \). We call \( \{(x, y) \in I_m^M \mid y = n\} \) a line in \( I_m^M \).

**Proposition 7.** Suppose \( x^g h_{\text{gen}} \in \text{Set}(m) \). Then

\[
\#I_m \geq \#\{\text{lines in } I_m^M \mid r(y - \alpha_2) \equiv 1(\text{mod } s)\}
\]

with equality if there exists an \( g' \) with

\[ s\alpha_1' + r\alpha_2' = 1 \text{ and } \alpha_1 + \alpha_1' \geq 0, \alpha_2 + \alpha_2' \geq 0. \]

**Proof.** Lines with \( r(y - \alpha_2) \equiv 1(\text{mod } s) \) represent the points in \( I_m^M \) minimizing the expression \( x/p + y/q \). \( (*) \) implies the existence of an \( g'' \) with support \( g'' \) excluding exactly these points of \( I_m^M \).

We say that \( m \) satisfies (C1) if there exists \( x^g h_{\text{gen}} \in \text{Set}(m) \) and \( g' \), such that \( (*) \) holds. Denote by \( [x] \max\{z \in \mathbb{Z} \mid z \leq x\} \). Lemma 2 implies that one can find an element \( x^g h_{\text{gen}} \in \text{Set}(m) \) with \( g = (x + \delta, \alpha_2) \) where \( 0 \leq r, 0 \leq \alpha_2 < s \).

**Corollary 8.** Let \( x^g h_{\text{gen}} \in \text{Set}(m) \) be of the above mentioned type and suppose that \( m \) satisfies (C1). Then

\[
\#I_m = \gcd(p, q) - \alpha = \begin{cases} 
2 & \text{if } \delta \geq \left\lfloor \frac{r^{-1}r}{s} \right\rfloor - 1 \text{ and } \alpha_2 \geq s - (r^{-1} + 1) \\
1 & \text{if } \delta \geq \left\lfloor \frac{r^{-1}r}{s} \right\rfloor - 1 \text{ or } \alpha_2 \geq s - (r^{-1} + 1) \\
0 & \text{otherwise}
\end{cases}
\]

where \( rr^{-1} \equiv 1(\text{mod } s), 0 < r^{-1} < s \).

**Proof.** follows from Proposition 7.

**Proposition 9.** With the above notations \( \#\text{Set}(m) \geq \alpha + 1 \), with equality holding if \( I_m^M \) contains a point of the type \( (x, y) \) where \( r(y - \alpha_2) \equiv 1(\text{mod } s) \).

**Proof.** \( x^{g'} h_{\text{gen}} \), where \( (\alpha_1', \alpha_2') = ((\alpha - n)r + \delta, \alpha_2 + ns), n = 0, \ldots, \alpha \), all belong to \( \text{Set}(m) \). These are the only elements of \( \text{Set}(m) \) with support not excluding points of the mentioned type.
We call the condition in Proposition 9 (C2).

COROLLARY 10. Suppose \( m < m' \). If \( m \) satisfies (C2) then

\[
\#\text{Set}(m') - \#\text{Set}(m) \geq -1.
\]

PROOF. Immediate.

The corresponding result for \( I_m \) follows from Proposition 7, and we state it as

COROLLARY 11. Suppose \( m < m' \). If \( m' \) satisfies (C1) then

\[
\#I_m - \#I_{m'} \geq -1.
\]

PROOF. Immediate.

Consider the following conditions, the first implying (C1), the second implying (C2):

(C1') There exists \( x^g h \in \text{Set}(m) \) with \( \alpha_1 \geq r \) or \( \alpha_2 \geq s \).

(C2') \( I^M_m \) contains at least \( s \) nonempty different lines.

The advantage of this reformulation of (C1) and (C2) is that if \( m < m' \) and \( m \) satisfies (C1') then \( m' \) also satisfies (C1') and if \( m < m' \) and \( m' \) satisfies (C2') then \( m \) also satisfies (C2').

Set \( w_{\text{min}} = \min \{ m \mid \#\text{Set}(m) > \#I_m \} \),
\[
w_{\text{max}} = \max \{ m \mid \#\text{Set}(m) < \#I_m \}.
\]

We can then reformulate Proposition 3:

PROPOSITION 3. \( w_{\text{min}} = w_{\text{max}} - 1 \).

Now we prove

PROPOSITION 12. Set \( w'_{\text{min}} = \min \{ m \mid \#\text{Set}(m) = \#I_m + 1 \} \),
\[
w'_{\text{max}} = \max \{ m \mid \#\text{Set}(m) = \#I_m - 1 \}.
\]

Suppose that \( w'_{\text{min}} \) satisfies (C2') and that \( w'_{\text{max}} \) satisfies (C1'). If \( w'_{\text{min}} < w'_{\text{max}} \) then
\[
w_{\text{min}} = w'_{\text{min}} \text{ and } w_{\text{max}} = w'_{\text{max}}.
\]

PROOF. \( w_{\text{min}} \leq w'_{\text{min}} < w'_{\text{max}} \) means, using Corollaries 10 and 11 that
\[
(\#\text{Set}(w'_{\text{max}}) - \text{Set}(w_{\text{min}})) + (\#I_{w_{\text{min}}} - \#I_{w'_{\text{max}}}) =
\]
\[
(\#\text{Set}(w'_{\text{max}}) - \#I_{w_{\text{max}}}) + (\#I_{w_{\text{min}}} - \#\text{Set}(w_{\text{min}})) \geq -2.
\]

The definition of \( w'_{\text{max}} \) then implies \( (\#I_{w_{\text{min}}} - \#\text{Set}(w_{\text{min}})) \geq -1 \), i.e. \( w_{\text{min}} = w'_{\text{min}} \), and \( w_{\text{max}} = w'_{\text{max}} \) is proved the same way.

i) We first find \( w'_{\text{min}} \).

Let \( x^g h \) correspond to \( w'_{\text{min}} \), where \( g = (\alpha r + \delta, \alpha_2) \) with \( 0 \leq \delta < r \),
0 \leq \alpha_2 < s. Using Propositions 7 and 9 we have \( \alpha + 1 = \gcd(p, q) - \alpha - (0 \text{ or } 1 \text{ or } 2) + 1 \). Since \( 2 | \gcd(p, q) \) we have two possibilities

1) \( \alpha = \gcd(p, q)/2, \delta = \alpha_2 = 0 \) or

2) \( \alpha = \gcd(p, q)/2 - 1, \delta = \left\lfloor \frac{r^{-1} - r}{s} \right\rfloor - 1, \alpha_2 = s - (r^{-1} + 1) \).

But

\[
\frac{\gcd(p, q)/2}{p} > \frac{\gcd(p, q)/2 - 1}{r} + \left\lfloor \frac{r^{-1} - r}{s} \right\rfloor - 1\right)/p + (s - (r^{-1} + 1))/q
\]

shows that \( w'_\min \) corresponds to the second possibility.

ii) Let \( x', y' \) correspond to \( w'_\max, \alpha' \) as in i). We have \( \alpha' + 1 = \gcd(p, q) - \alpha' - (0 \text{ or } 1 \text{ or } 2) - 1 \), giving the possibilities

1) \( \alpha' = \gcd(p, q)/2 - 1, \delta' = \left\lfloor \frac{r^{-1} - r}{s} \right\rfloor - 2, \alpha'_2 = s - (r^{-1} + 1) - 1 \)

2) \( \alpha' = \gcd(p, q)/2 - 2, \delta' = r - 1, \alpha'_2 = s - 1 \).

Comparing as in i), it turns out that 1) is impossible.

iii) We have

\[
((\alpha' - \alpha)r + (\delta' - \delta))/p + (\alpha'_2 - \alpha_2)/q = \gcd(p, q)/pq,
\]

therefore \( w'_\max = w'_\min + 1 \).

iv) That \( w'_\min \) satisfies (C2') is equivalent to

\[
\frac{p - 2}{p} + \frac{q - 2 - s}{q} > (\alpha x + \delta)/p + (\alpha_2)/q
\]

where we suppose that \( q - 2 - s \geq 2 \), which are both satisfied if \( s > 1 \) and \( \gcd(p, q) > 2 \). These two cases must be considered separately.

For \( w'_\max \) to satisfy (C1') it is sufficient to require that

\[
\frac{\gcd(p, q)/2 - 1}{r - 1} \geq r - 1 \text{ or } s - 1 \geq s - 1
\]

which is obvious.

i)-iv) together with Proposition 12 proves Proposition 3.

From Corollary 6 we know that \( \max_{\xi \in H_x^1(f)} \dim o(\xi) = 2 \# I_w^M - 1 \) and using the fact that \( w = w'_\max \) we get

\[
\max_{\xi \in H_x^1(f)} \dim o(\xi) = 2 \# \{(\alpha_1, \alpha_2) | \alpha_1/p + \alpha_2/q > 1 + (p/2 - r - 1)/p + (s - 1)/q, \\
0 \leq \alpha_1 \leq p - 2, 0 \leq \alpha_2 \leq q - 2\} - 1
\]

\[
= \left(\frac{p}{2} - 1\right)\left(\frac{q}{2} - 1\right) - \gcd\left(\frac{p}{2}, \frac{q}{2}\right).
\]
thus proving Theorem 1 under the assumptions $s > 1$ and $\gcd(p, q) > 2$. The $-1$ in Theorem 1 occurs for $s = 1$. The remaining cases are easy to check.

REFERENCES


