RANDERS RICCI SOLITON HOMOGENEOUS NILMANIFOLDS

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Abstract

Let F be a left-invariant Randers metric on a simply connected nilpotent Lie group N, induced by a left-invariant Riemannian metric $\hat{a}$ and a vector field X which is $I_{\hat{a}}(M)$-invariant. We show that if the Ricci flow equation has a unique solution then, (N, F) is a Ricci soliton if and only if (N, F) is a semialgebraic Ricci soliton.

1. Introduction

In 1982, R. S. Hamilton has shown that any compact Riemannian 3-manifold with strictly positive Ricci curvature, admits a Riemannian metric of constant positive curvature (see [12]). He has started with a Riemannian manifold $(M, \hat{a}_0)$ and evolved its metric by the following geometric evolution equation which is named un-normalized Ricci flow

$$\frac{\partial}{\partial t}\hat{a}_{ij} = -2\operatorname{ric}_{ij}, \tag{1.1}$$

where ric_{ij} denotes the Ricci curvature tensor of the Riemannian metric $\hat{a}$. For a compact Riemannian manifold, one can consider the following equation

$$\frac{\partial}{\partial t}\hat{a}_{ij} = \frac{2}{n}\operatorname{sc}(\hat{a})\hat{a}_{ij} - 2\operatorname{ric}_{ij}, \tag{1.2}$$

where $\dim M = n$ and $\operatorname{sc}(\hat{a})$ is the average of scalar curvature [12]. The last equation is called the normalized Ricci flow which is different from un-normalized Ricci flow (1.1) by a change of scale in space and a change of parametrization in time. In this case, the volume of the solution is constant with respect to time (for more details see [12]). A Riemannian metric is a fixed point of the equation (1.1) if and only if it is Ricci flat. For a compact manifold, a Riemannian metric is a fixed point of the equation (1.2) if and only if it is Einstein (see [7]). There is a larger class of solutions, called self-similar

solutions or Ricci solitons, which is considered as a generalization of fixed points. Let $(M, \hat{\boldsymbol{a}}(t))$ be a solution of the equation (1.1) on a time interval (t_1, t_2) containing zero with $\hat{\boldsymbol{a}}_0 = \hat{\boldsymbol{a}}(0)$. Then $\hat{\boldsymbol{a}}(t)$ is a self-similar solution if there exist scalars $\tau(t)$ and diffeomorphisms ϕ_t such that $\hat{\boldsymbol{a}}(t) = \tau(t)\phi_t^*\hat{\boldsymbol{a}}_0$. On the other hand, a Riemannian manifold $(M, \hat{\boldsymbol{a}}_0)$ is named a Ricci soliton if

$$\mathscr{L}_X \hat{\boldsymbol{a}}_0 = 2(\lambda\hat{\boldsymbol{a}}_0 - \mathrm{ric}(\hat{\boldsymbol{a}}_0)),$$

where $\mathscr{L}_X\hat{\boldsymbol{a}}$ denotes the Lie derivative of $\hat{\boldsymbol{a}}$ with respect to a vector field X. A Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. It can be shown that there exists a bijection between the family of self-similar solutions and the family of Ricci solitons (see Lemma 2.4 of [7]). This fact allows us to consider them as equivalent.

D. Bao, in [2], has used Akbar-Zadeh's Ricci tensor [1], to generalize the Ricci flow to Finsler geometry which is a natural extension of Riemannian geometry. In [5], the concept of Ricci soliton is developed to Finsler geometry by B. Bidabad and M. Yarahmadi.

In 1941 an important family of Finsler metrics, which are called Randers metrics now, introduced by G. Randers for its application in general relativity (see [17]). Also, they are the only Finsler metrics which are the solutions of the navigation problem on a Riemannian manifold. In fact the perturbation of a Riemannian metric $\hat{\boldsymbol{a}}$ by a vector field W, with $\|W\|_{\hat{a}} < 1$, generates a Randers metric and conversely every Randers metric can be accomplished through the perturbation of a Riemannian metric $\hat{\boldsymbol{a}}$ by a vector field W satisfying $\|W\|_{\hat{a}} < 1$ (see [4]). The importance of Randers spaces, among Finsler manifolds, leads us to study Ricci solitons on them. In this work, we study the conditions under which a left-invariant Randers metric on a simply connected nilpotent Lie group N, is a Ricci soliton.

In [16], J. Lauret has shown that a Riemannian nilmanifold $(N, \hat{\boldsymbol{a}})$ is a Ricci soliton if and only if its $(1, 1)$ Ricci tensor satisfies the equation

$$\mathrm{Ric}_{\hat{a}} = c\mathrm{Id} + \frac{1}{2}(\mathrm{D} + \mathrm{D}^t),$$

for some $c \in \mathbb{R}$ and some $\mathrm{D} \in \mathrm{Der}(\mathfrak{n})$, where N is a nilpotent Lie group with Lie algebra $\mathfrak{n}$, $\hat{\boldsymbol{a}}$ is a left-invariant Riemannian metric on N and D^t is the transpose of D with respect to $\hat{\boldsymbol{a}}$. In the general case, a homogeneous Riemannian manifold is called semi-algebraic if it satisfies the homogeneous space version of the above equation (for more details see [14]). In this case, if D is symmetric then the homogeneous Riemannian manifold is called algebraic Ricci soliton. Recently, M. Jablonski has proved that every homogeneous Riemannian Ricci soliton must be semi-algebraic [15]. He has completed his result in his next

paper [14], by showing that every semi-algebraic Ricci soliton is algebraic. This means that homogeneous Riemannian Ricci solitons are algebraic.

We say that a vector field X on a Riemannian manifold $(M, \hat{a})$ is $I_{\hat{a}}(M)$-invariant if for any $\phi \in I_{\hat{a}}(M)$ and $x \in M$ we have $\phi_*(X_x) = X_{\phi(x)}$. In this article, we generalize the definition of semi-algebraic Ricci soliton to Finsler spaces. We show that for a left-invariant Randers metric F induced by a left-invariant Riemannian metric $\hat{a}$ and an $I_{\hat{a}}(M)$-invariant vector field X, on a simply connected nilpotent Lie group N, the Finslerian nilmanifold (N, F) is a Ricci soliton if and only if (N, F) is a semialgebraic Ricci soliton. This is a generalization of Lauret's result in [16] to Randers spaces.

2. Preliminaries

In this section we review some fundamental definitions and concepts of Finsler geometry. For more details we refer the readers to [3] and [8].

DEFINITION 2.1. A Finsler manifold (M, F) is a smooth manifold M equipped with a map (which is called a Finsler metric) $F: TM \rightarrow [0, \infty)$ such that

  (i) F be a smooth function on $TM \setminus \{0\}$,

  (ii) $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$, and

  (iii) the hessian matrix $(g_{ij}) = \left(\dfrac{1}{2} \dfrac{\partial^2 F^2}{\partial y^i \partial y^j} \right)$ is positive definite for all $(x, y) \in TM \setminus \{0\}$.

An important special type of Finsler metrics is Randers metric. This metric is of the form $F(x, y) = \sqrt{\hat{a}(y, y)} + \beta(x, y)$, where $\hat{a}(y, y) = \hat{a}_{ij} y^i y^j$ and $\beta(x, y) = b_i y^i$ are a Riemannian metric and a $1-$form on M, respectively, such that $\|\beta_x\|_\alpha = \sqrt{\hat{a}^{ij}(x) b_i(x) b_j(x)} < 1$ ([6]). Using the induced inner product by the Riemannian metric $\hat{a}$ on any cotangent space $T_x^* M$, the 1-form β corresponds to a vector field X on M such that

$$\hat{a}(y, X(x)) = \beta(x, y).$$

Therefore we can consider the Randers metrics as follows

$$F(x, y) = \sqrt{\hat{a}(y, y)} + \hat{a}(X(x), y). \tag{2.1}$$

For any Finsler manifold (M, F) we define

$$C_{ijk} := \frac{1}{4} \left(\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} \right).$$

Suppose that (M, F) is an arbitrary Finsler manifold and $(x^1, x^2, \ldots, x^n)$ is a local coordinate system on an open subset U of M. The fundamental tensor and the Cartan tensor, are defined as follows, respectively

$$g_{(x,y)}(u, v) = g_{ij}(x, y)u^i v^j,$$
$$C_{(x,y)}(u, v, w) = C_{ijk}(x, y)u^i v^j w^k,$$

for any $(x, y) \in TM \setminus \{0\}$, where $u = u^i \frac{\partial}{\partial x^i}\big|_x$, $v = v^i \frac{\partial}{\partial x^i}\big|_x$ and $w = w^i \frac{\partial}{\partial x^i}\big|_x$. We define $C^i_{jk} := g^{is} C_{sjk}$, where (g^{ij}) denotes the inverse of the matrix (g_{ij}). Now we can have the Christoffel symbols of the second kind, on $TU \setminus \{0\}$, as follows

$$\gamma^i_{jk} := \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right).$$

We also define the nonlinear connection N^i_j by the following equation

$$N^i_j := \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_{rs} y^r y^s,$$

where $y = y^i \frac{\partial}{\partial x^i}$. Now we have the following theorem.

THEOREM 2.2 (see [3]). *Suppose that (M, F) is a Finsler manifold. The pull-back bundle $\pi^* TM$ admits a unique torsion free linear connection ∇, called the Chern connection, which is almost g-compatible. The coefficients of the connection are of the form*

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j} \right),$$

where $\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}$.

The concept of sectional curvature in Riemannian geometry is generalized to Finsler geometry where it is called *flag curvature*. The flag curvature is defined by

$$K(P, (x, y)) = \frac{g_{(x,y)}\big(R_{(x,y)}(u), u\big)}{g_{(x,y)}(y, y)g_{(x,y)}(u, u) - g^2_{(x,y)}(y, u)},$$

where $P = \mathrm{span}\{u, y\}$ and R is the Riemann curvature tensor defined by $R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$ (for more details see [3]).

Now for any $(x, y) \in TM \setminus \{0\}$ we define a linear transformation from $T_x M$ to itself by $R_{(x,y)} := R(\cdot, y)y$.

Then the Ricci curvature is defined as the trace of $R_{(x,y)}$. So we have

$$\mathscr{R}ic(x,y) = F^2(x,y) \sum_{i=1}^{n-1} K(P_i,(x,y)),$$

where $P_i := \mathrm{span}\{e_i,y\}$ and $\left\{e_1,\ldots,e_{n-1},\frac{y}{F(x,y)}\right\}$ is a $g_{(x,y)}$-orthonormal basis of $T_x M$.

Akbar-zadeh's Ricci tensor of a Finsler metric F is defined as follows (see [1])

$$(\mathrm{ric}_F)_{ij} := \frac{1}{2}\left(\frac{\partial^2 \mathscr{R}ic(x,y)}{\partial y^i \partial y^j}\right).$$

The following equation is considered by D. Bao, as the natural extension of un-normalised Ricci flow in Finsler geometry (see [2]),

$$\frac{\partial g_t}{\partial t} = -2\,\mathrm{ric}_{F_t}, \quad F(t=0) = F_0. \tag{2.2}$$

The concept of Ricci soliton is generalized to Finsler geometry by B. Bidabad and M. Yarahmadi as follows (see [5]).

Suppose that V is a vector field on the manifold M, such that $V = V^i \frac{\partial}{\partial x^i}$, for a local coordinate system. Then the complete lift of V on TM, which is denoted by V^c, is a vector field on TM locally defined by $V^c = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}$.

Definition 2.3. A Finsler manifold (M,F) is said to be a Finslerian Ricci soliton if it satisfies the following equation,

$$\mathrm{ric}_F = cg + L_{V^c}g,$$

for some vector field $V \in \mathfrak{X}(M)$, where V^c denotes the complete lift of V on TM and g is the Hessian related to F (see [5]).

In Theorem 3.4 of [5], the authors showed that if (M,F_0) is a Finslerian Ricci soliton on a compact manifold M, then there exists a solution (M,F_t) in the form

$$F_t^2 = \tau(t)\tilde{\varphi}_t^* F_0^2, \tag{2.3}$$

to the Ricci flow equation (2.2). Conversely, if F_t, of the form (2.3), is a solution to the Ricci flow equation (2.2), then there exists a vector field V on M such that (M,F_0) is a Finslerian Ricci soliton in the sense of definition 2.3.

Here we mention that, similar to the Riemannian case, a Finsler metric F on a Lie group G is called *left invariant* if

$$F(x,y) = F(e,(l_{x^{-1}})_* y), \quad \forall x \in G, y \in T_x G,$$

where e denotes the unit element of G.

3. Main Results

Here we generalize the concept of homogeneous Riemannian nilmanifold to Finsler spaces.

DEFINITION 3.1. Let (M, F) be a connected Finsler manifold with the Lie group of isometries $G = I_F(M)$. M is said to be a homogeneous Finsler nilmanifold if G contains a nilpotent Lie subgroup which acts transitively on M.

REMARK 3.2. Recall that for a Finsler manifold (M, F), the group of isometries of M is a Lie transformation group with respect to the compact-open topology (see [8] and [9]).

The following proposition is a natural generalization of Theorem 2 of [19].

PROPOSITION 3.3. *Let (M, F) be a Finsler manifold such that $I_F(M) \subset I_{\hat{a}}(M)$, for a Riemannian metric $\hat{a}$ on M, where $I_{\hat{a}}(M)$ denotes the Lie group of isometries of $(M, \hat{a})$. Suppose that (M, F) is a homogeneous Finsler nilmanifold and H is a nilpotent Lie subgroup of $I_F(M)$ acting transitively on M with N the connected component of the identity in H. Then,*

(1) *N acts simply transitively on M and $N \trianglelefteq G := I_{\hat{a}}(M)$,*

(2) *$\mathfrak{n} = \mathfrak{nil}(\mathfrak{g})$, where $\mathfrak{n}$ and $\mathfrak{g}$ denote the Lie algebra of N and G, respectively and $\mathfrak{nil}(\mathfrak{g})$ denotes the nilradical of $\mathfrak{g}$,*

(3) *For any point $p_0 \in M$, G is the semi-direct product of N and the isotropy subgroup $K_{p_0} = \{k \in G : kp_0 = p_0\}$,*

(4) *N is the unique nilpotent subgroup of G acting simply transitively on G.*

REMARK 3.4. Let $F(x, y) = \sqrt{\hat{a}(y, y)} + \hat{a}(X_x, y)$ be a Randers metric on a Lie group G, arising from a left-invariant Riemannian metric $\hat{a}$ and a vector field X. In [18, Proposition 3.6] , we have shown that, F is left invariant if and only if X is left invariant.

PROPOSITION 3.5. *Let $F(x, y) = \sqrt{\hat{a}(y, y)} + \hat{a}(X_x, y)$ be a Randers metric on an arbitrary connected manifold M. Suppose that $I_F(M)$ and $I_{\hat{a}}(M)$ denote the Lie groups of isometries of (M, F) and $(M, \hat{a})$, respectively. Then, $I_F(M) = I_{\hat{a}}(M)$ if and only if X is an $I_{\hat{a}}(M)$-invariant vector field.*

PROOF. Proposition 1.3 of [10] shows that, for an arbitrary Randers metric F, $I_F(M) \subseteq I_{\hat{a}}(M)$. Suppose that X is an $I_{\hat{a}}(M)$-invariant vector field of

$(M, \hat{a})$ and $\phi \in I_{\hat{a}}(M)$, so we have

$$F(\phi(x), \phi_*(y)) = \sqrt{\hat{a}(\phi_*(y), \phi_*(y))} + \hat{a}(X_{\phi(x)}, \phi_*(y))$$

$$= \sqrt{\hat{a}(\phi_*(y), \phi_*(y))} + \hat{a}(\phi_*(X_x), \phi_*(y))$$

$$= \sqrt{\hat{a}(y, y)} + \hat{a}(X_x, y) = F(x, y),$$

which shows that $\phi \in I_F(M)$.

Conversely, suppose that $I_F(M) = I_{\hat{a}}(M)$. If $\phi \in I_F(M)$ then

$$F(x, y) = F(\phi(x), \phi_*(y)) = \sqrt{\hat{a}(\phi_*(y), \phi_*(y))} + \hat{a}(X_{\phi(x)}, \phi_*(y)).$$

On the other hand, $\phi \in I_{\hat{a}}(M)$, so we have

$$F(x, y) = \sqrt{\hat{a}(\phi_*(y), \phi_*(y))} + \hat{a}(\phi_*(X_x), \phi_*(y)).$$

Hence for any $y \in T_x M$ we have

$$\hat{a}(\phi_*(X_x) - X_{\phi(x)}, \phi_*(y)) = 0.$$

So the proof is completed.

EXAMPLE 3.6. Suppose that $\mathfrak{n}_{5,6}$ is the five-dimensional real Lie algebra defined by the following non-vanishing commutators (see [11] and [13])

$$[E_1, E_2] = aE_3 + bE_4 + cE_5, \quad [E_1, E_3] = dE_4 + fE_5,$$

$$[E_1, E_4] = gE_5, \quad [E_2, E_3] = hE_5,$$

where $a, b, c, d, f, g, h \in \mathbb{R}$ with $a, d, g, h \neq 0$ and $b^2 + f^2 \neq 0$. Easily, we see that $\mathfrak{n}_{5,6}$ is a four-step nilpotent Lie algebra. Assume that $N_{5,6}$ is the connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}_{5,6}$ and $\hat{a}$ is the left-invariant Riemannian metric on $N_{5,6}$ such that $\{E_1, \ldots, E_5\}$ is an orthonormal set. In [11], it is shown that $I_{\hat{a}}(N_{5,6}) = N_{5,6}$.

Let X be an arbitrary left-invariant vector field on $N_{5,6}$ with $\|X\|_{\hat{a}} < 1$, and F be the Randers metric defined by $\hat{a}$, X and equation (2.1). The vector field X is left invariant, on the other hand $I_{\hat{a}}(N_{5,6}) = N_{5,6}$, so the vector field X is $I_{\hat{a}}(N_{5,6})$-invariant. Using the Proposition 3.5, we see that $I_F(N_{5,6}) = I_{\hat{a}}(N_{5,6})$.

Recently, we classified all Berwald and Douglas (α, β)-metrics defined by a left-invariant Riemannian metric and a left-invariant vector field on simply connected five-dimensional nilpotent Lie groups (see [13]). We see that if $X \in \mathrm{span}\{E_3, E_4, E_5\}$ then F is nether of Berwald type nor of Douglas type, because, in this case, the vector field X is not orthogonal to the derived Lie algebra (for more details see [13]).

EXAMPLE 3.7. With the notations of the previous example, consider the three-step nilpotent Lie algebra $\mathfrak{n}_{5,5}$ with the non-vanishing commutators (see [11] and [13])

$$[E_1, E_2] = aE_4 + bE_5, \quad [E_1, E_3] = cE_5,$$

$$[E_1, E_4] = dE_5, \quad [E_2, E_3] = eE_5,$$

where $a, b, c, d, e \in \mathbb{R}$ with $a, d, e \neq 0$, $b > 0$ and $c > 0$. If $X \in \text{span}\{E_4, E_5\}$ then F is a Randers metric which is not of Douglas type and also we have $I_F(N_{5,5}) = I_{\hat{a}}(N_{5,5}) = N_{5,5}$.

COROLLARY 3.8. *Suppose that* $F(x, y) = \sqrt{\hat{a}(y, y)} + \hat{a}(X_x, y)$ *is a Randers metric on an arbitrary connected manifold* M *such that* X *is an* $I_{\hat{a}}(M)$-*invariant vector field. Then* (M, F) *is a homogeneous Finsler nilmanifold if and only if* $(M, \hat{a})$ *is a homogeneous Riemannian nilmanifold.*

Proposition 3.5 together with Theorem 2 of [19] lead to the following corollary, which is very basic in this paper.

COROLLARY 3.9. *Suppose that* (M, F) *is a homogeneous Finsler nilmanifold such that* F *is a Randers metric defined by* $F(x, y) = \sqrt{\hat{a}(y, y)} + \hat{a}(X_x, y)$, *where* X *is* $I_{\hat{a}}(M)$-*invariant. Suppose that* H *is a nilpotent Lie subgroup of* $I_F(M)$ *acting transitively on* M *with* N *the connected component of the identity in* H. *Then*

(1) *N acts simply transitively on M and $N \trianglelefteq G := I_F(M)$,*

(2) $\mathfrak{n} = \mathfrak{nil}(\mathfrak{g})$,

(3) *for any point $p_0 \in M$, G is the semi-direct product of N and the isotropy subgroup K_{p_0},*

(4) *N is the unique nilpotent subgroup of G acting simply transitively on G.*

Now we use the above results to study Randers Ricci soliton.

DEFINITION 3.10. A Finslerian Ricci soliton (G, F) is said to be *semial-gebraic* if its $(1, 1)$ Ricci tensor satisfies the equation,

$$\text{Ric}_F = c\text{Id} + \frac{1}{2}(D + D^t), \tag{3.1}$$

for some $c \in \mathbb{R}$ and some $D \in \text{Der}(\mathfrak{g})$, where G is a Lie group with Lie algebra $\mathfrak{g}$, F is a left-invariant Finsler metric on G and for any $y \in T_eG$, D^t denotes the transpose of D with respect to $g_{(e,y)}$.

THEOREM 3.11. *Suppose that* $F(x, y) = \sqrt{\hat{a}_x(y, y)} + \hat{a}_x(X_x, y)$ *is a left-invariant Randers metric on a simply connected nilpotent Lie group N, where*

$\hat{a}$ *is a left-invariant Riemannian metric on* N *and* X *is an* $I_{\hat{a}}(M)$-*invariant vector field. If the Ricci flow equation has a unique solution then,* (N, F) *is a Ricci soliton if and only if* (N, F) *is a semialgebraic Ricci soliton.*

PROOF. Let F be a Ricci soliton. Then, there exists a one-parameter group of diffeomorphisms φ_t on N such that $g_t = \tau(t)\tilde{\varphi}_t^* g$ is the solution to the Ricci flow

$$\frac{\partial g_t}{\partial t} = -2\operatorname{ric}_{F_t},$$

with $g_0 = g$, where g_t denotes the fundamental tensor of F_t and (see [2] and [5]). The uniqueness of the solution shows that $\tilde{\varphi}_t^* g$ is also left invariant for any t. Similar to the Riemannian case, fix a point $p \in N$ (see [15]). The smooth curve $\varphi_t(p) \subset N$ induces a smooth curve $h(t) \subset G = I_F(N)$ such that

$$l_{h(t)}.p = \varphi_t(p) \subset N \simeq G/G_p,$$

where G_p denotes the isotropy at p. Let $\psi_t = l_{h(t)^{-1}} \circ \phi_t$. Since, for all t, $g_t = \tau(t)\tilde{\varphi}_t^* g$ is left invariant, so $g_t = \tau(t)\tilde{\psi}_t^* g$ is a solution to the Ricci flow. Moreover, ψ_t fixes the point p. Suppose that $\sigma \in I_F(N) = G$ is an arbitrary isometry, so we have,

$$\frac{\partial}{\partial t}\tilde{\sigma}^* g_t = \tilde{\sigma}^* \frac{\partial}{\partial t} g_t = -2\tilde{\sigma}^* \operatorname{ric}_{F_t} = -2\operatorname{ric}_{\tilde{\sigma}^* F_t}.$$

Therefore $\tilde{\sigma}^* g_t$ is a solution of the Ricci flow,

$$\frac{\partial}{\partial t} g_t = -2\operatorname{ric}_{F_t},$$

with initial condition $\tilde{\sigma}^* g_0 = g$. Again, the uniqueness of solutions shows that $\tilde{\sigma}^* g_t = g_t$, for all t. In other words, $I_{F_t}(N) = I_F(N) = G$.

Let $\sigma \in I_F(N) = G$, so we have,

$$\psi_t \circ \sigma \circ \psi_t^{-1} \in I_{F_t}(N) = I_F(N) = G.$$

Hence, for any $\sigma \in G$ we have a smooth family

$$\Phi_t(\sigma) = \psi_t \circ \sigma \circ \psi_t^{-1} \in \operatorname{Aut}(G).$$

We can easily see that $\Phi_t(G_p) = G_p$. Now Corollary 3.9 shows that $\psi_t = \phi_t$, where $\phi_t \in \operatorname{Diff}(N)$ is the diffeomorphism defined by Φ_t on N.

This means that there exists a one-parameter group ψ_t of automorphisms of N such that $\tilde{\varphi}_t^* g = \tilde{\psi}_t^* g$, for all t. Now, let $\psi_t = e^{-t\mathrm{D}}$, where $\mathrm{D} \in \operatorname{Der}(\mathfrak{n})$.

Then for any $(x, y) \in TN$ and $v, w \in \pi^* TN_{(x,y)}$, we have

$$\frac{\partial}{\partial t}\Big|_{t=0} \tilde{\psi}_t^* g_{(x,y)}(v, w) = g_{(x,y)}(-(\mathrm{D} + \mathrm{D}^t)v, w).$$

On the other hand, we have

$$\frac{\partial}{\partial t} g_t \Big|_{t=0} = \tau'(0)g - g((\mathrm{D} + \mathrm{D}^t)\cdot, \cdot).$$

Therefore

$$\mathrm{Ric}_F = c\,\mathrm{Id} + \frac{1}{2}(\mathrm{D} + \mathrm{D}^t),$$

where $c = -\tau'(0)/2$.

REMARK 3.12. We can easily see every (non-necessarily nilpotent) Finsler Lie group (G, F), satisfying equation (3.1), is a Finslerian Ricci soliton. It is sufficient to use the one-parameter group of automorphisms determined by D.

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