

SUBALGEBRAS GENERATED IN DEGREE TWO WITH MINIMAL HILBERT FUNCTION

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Abstract

What can be said about the subalgebras of the polynomial ring, with minimal or maximal Hilbert function? This question was discussed in a recent paper by M. Boij and A. Conca. In this paper we study the subalgebras generated in degree two with minimal Hilbert function. The problem to determine the generators of these algebras transfers into a combinatorial problem on counting maximal north-east lattice paths inside a shifted Ferrers diagram. We conjecture that the subalgebras generated in degree two with minimal Hilbert function are generated by an initial Lex or RevLex segment.

1. Introduction

In a recent paper by Boij and Conca [1] the Hilbert function of a subalgebra of the polynomial ring is studied. They ask what can be said about the upper and lower bounds for the Hilbert function, in terms of the number of variables, the number of generators of the subalgebra, and the degree of the generators. This question is inspired by the Fröberg conjecture [5] on the minimal Hilbert series of the quotient of a the polynomial ring with a homogeneous ideal. For a review of the Fröberg conjecture and related problems, see [6]. In this note we will focus of subalgebras generated in degree two, with minimal Hilbert function. We conjecture that these algebras are always given by a Lex or RevLex segment, see Conjecture 3.4. This conjecture is proved for three large classes of algebras in Theorem 3.8. For the first class, the proof is by a computer computation, and for the other two by inductive arguments, using the first class as the base.

Let \mathbb{k} be a field, and let $R = \mathbb{k}[x_1, \dots, x_n]$ be the standard graded polynomial ring in n variables. Let R_d denote the \mathbb{k} -space of homogeneous polynomials of degree d in R . For a linearly independent subset $W \subseteq R_d$, let $\mathbb{k}[W] \subseteq R$ be the subring of R generated by the elements in W . Define the Hilbert function of such an algebra $\mathbb{k}[W]$ as $\text{HF}(\mathbb{k}[W], i) = \dim_{\mathbb{k}}(\text{span } W^i)$. Given positive integers u and i , how should we choose W so that $|W| = u$ and $\text{HF}(\mathbb{k}[W], i)$ takes the smallest possible value? Proposition 3.3 in [1] states that we should choose W as a strongly stable set of monomials.

DEFINITION 1.1. A set W of monomials in R_d is called *strongly stable* if $m \in W$ and $x_i|m$ implies $(x_j/x_i)m \in W$ for all $j < i$.

We use the notation $\text{st}(m_1, \dots, m_s)$ for the smallest strongly stable set containing the monomials m_1, \dots, m_s , and we say that m_1, \dots, m_s are *strongly stable generators* of this set.

Let $L(n, d, u, i)$ denote the minimal value of $\text{HF}(\mathbb{k}[W], i)$ among all strongly stable subsets $W \subseteq R_d$ of size u . The following three questions [1, Questions 3.6] are asked, for fixed parameters d, n , and u .

- (1) Is there a W such that $\text{HF}(\mathbb{k}[W], i) = L(n, d, u, i)$ for all i ?
- (2) Given i , can one characterize combinatorially the strongly stable set(s) W such that $\text{HF}(\mathbb{k}[W], i) = L(n, d, u, i)$?
- (3) Suppose we have W such that $\text{HF}(\mathbb{k}[W], 2) = L(n, d, u, 2)$. Does it follow that $\text{HF}(\mathbb{k}[W], i) = L(n, d, u, i)$ for all i ?

We will see in Example 2.3 that the answer to the questions 1 and 3 is “No”. Since there is not one generating set W that minimizes $\text{HF}(\mathbb{k}[W], i)$ for all i , it is not obvious what the meaning of “minimal Hilbert function” should be. For given parameters n, u , and d , there is always a finite number of strongly stable sets to consider. For each set we know that the Hilbert function is given by the Hilbert polynomial, for i large enough. It follows that there will be an algebra with asymptotically minimal Hilbert function. From now on, we say that $\mathbb{k}[W]$ has *minimal Hilbert function* if it is minimal in the asymptotic sense. That is, $\mathbb{k}[W]$ has minimal Hilbert function if there is a number N such that $\text{HF}(\mathbb{k}[W], i) = L(n, d, u, i)$ for all $i > N$. Assuming this definition of minimal Hilbert function, it makes sense to specialize question 2 as follows.

How can the strongly stable set(s) W such that $\mathbb{k}[W]$ has minimal Hilbert function be characterized?

The aim of this paper is to study this question in the smallest non-trivial case with respect to the parameter d . Hence we fix $d = 2$ from now on, and focus on strongly stable sets of monomials of degree two. An advantage with this restriction is the connection to combinatorics, as we will see in §2. The case $d \geq 3$ is discussed in §4.

To minimize the Hilbert function we firstly want to minimize the degree of the Hilbert polynomial. If there are exactly n variables that occurs in the monomials in W , the degree of the Hilbert polynomial is $n - 1$. Hence, for a given u , we want to choose a strongly stable set of u monomials in as few variables as possible. Secondly, we want to minimize the leading coefficient of the Hilbert polynomial. Recall that, if the Hilbert polynomial is of degree

$n - 1$, the leading coefficient multiplied by $(n - 1)!$ is the *multiplicity* of the algebra, which we will denote $e(\mathbb{k}[W])$.

2. The multiplicity of subalgebras generated by a strongly stable set of degree two

Strongly stable sets of quadratic monomials are also considered as bases of *specialized Ferrers ideals*, which are studied in e.g. [3], [4]. These sets can be illustrated by a diagram, as in Figure 1. The box in row i and column j corresponds to the monomial $x_i x_j$. Since $x_i x_j = x_j x_i$ we only need to consider boxes on and above the diagonal in the diagram. That the set is strongly stable means precisely that if the box on position (i, j) is included in the diagram, so is everything above and to the left of (i, j) .

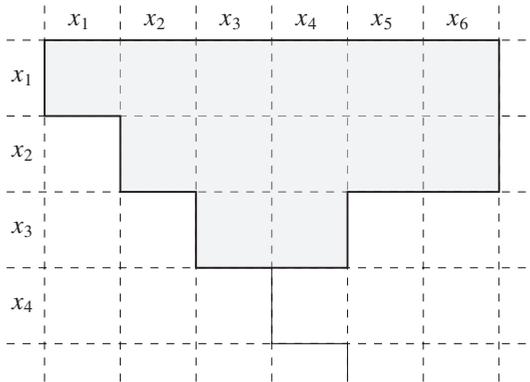


FIGURE 1. $st(x_2x_6, x_3x_4) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3^2, x_3x_4\}$

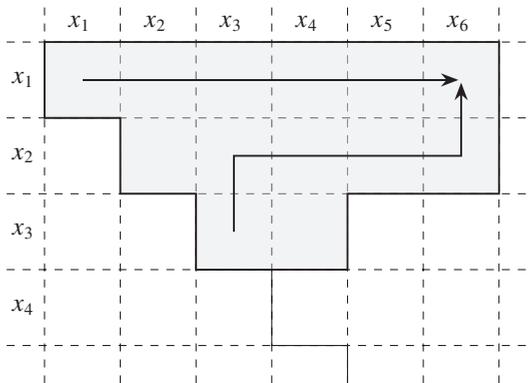


FIGURE 2. Maximal NE-paths in a diagram.

Define an NE-path to be a lattice path in the diagram that can only go up or right (north or east). We say that an NE-path is *maximal* if it is of maximal length, which implies that it starts in x_i^2 (on the diagonal) for some i , and goes to x_1x_n (the upper right corner). An example of two such paths can be found in the Figure 2. It is proved in [4, Theorem 4.2] that if W is a strongly stable set of degree two monomials, then $\mathbb{k}[W]$ is isomorphic to a certain determinantal ring, which has a defining ideal with a quadratic Gröbner basis. It then follows from [2, Corollary 1.9] that the multiplicity is equal to the number of maximal NE-paths in the diagram. We collect this result as a theorem.

THEOREM 2.1. *Let W be a strongly stable set of monomials of degree two. Then $e(\mathbb{k}[W])$ is equal to the number of maximal NE-paths in the diagram representing W .*

For a diagram L of a strongly stable set, we will use the notation $e(L)$ for the number of maximal NE-paths in L . We illustrate the key points from [2] and [4] that provides the proof of Theorem 2.1 in Example 2.2.

EXAMPLE 2.2.

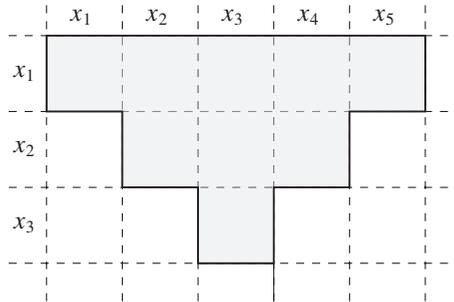


FIGURE 3. $\text{st}(x_1x_5, x_2x_4, x_3^2)$

Let $W = \text{st}(x_1x_5, x_2x_4, x_3^2)$, the set in Figure 3. Let

$$T = \mathbb{k}[y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{22}, y_{23}, y_{24}, y_{33}]$$

and consider the surjective homomorphism $\phi: T \rightarrow \mathbb{k}[W]$ defined by $y_{ij} \mapsto x_i x_j$. The kernel is given by

$$\begin{aligned} J = & (y_{11}y_{22} - y_{12}^2, y_{11}y_{23} - y_{12}y_{13}, y_{11}y_{24} - y_{12}y_{14}, y_{11}y_{33} - y_{13}^2, \\ & y_{12}y_{23} - y_{13}y_{22}, y_{12}y_{24} - y_{14}y_{22}, y_{12}y_{33} - y_{13}y_{23}, y_{13}y_{24} - y_{14}y_{23}, \\ & y_{22}y_{33} - y_{23}^2), \end{aligned}$$

so $\mathbb{k}[W] \cong T/J$, and the Hilbert function of $\mathbb{k}[W]$, as we defined it, is the same as the Hilbert function of T/J given the standard grading. The generating set

given for J is a Gröbner basis, under the Lex order with

$$y_{11} > y_{12} > y_{13} > y_{14} > y_{15} > y_{22} > y_{23} > y_{24} > y_{33}.$$

Hence the Hilbert function of T/J is the same as the Hilbert function of

$$\frac{T}{\text{in}(J)} = \frac{T}{(y_{11}y_{22}, y_{11}y_{23}, y_{11}y_{24}, y_{11}y_{33}, y_{12}y_{23}, y_{12}y_{24}, y_{12}y_{33}, y_{13}y_{24}, y_{22}y_{33})}.$$

This is the Stanley-Reisner ring with the facets

$$\begin{aligned} & y_{11}y_{12}y_{13}y_{14}y_{15}, & y_{22}y_{12}y_{13}y_{14}y_{15}, & y_{22}y_{23}y_{13}y_{14}y_{15}, \\ & y_{22}y_{23}y_{24}y_{14}y_{15}, & y_{33}y_{23}y_{13}y_{14}y_{15}, & y_{33}y_{23}y_{24}y_{14}y_{15}. \end{aligned}$$

Notice that they all have the same dimension, and they correspond exactly to the maximal NE-paths of the diagram in Figure 3. It is a known fact about Stanley-Reisner rings that the multiplicity is equal to the number of facets of maximal dimension, which here are exactly those listed above.

For strongly stable sets in higher degrees, the ideal J need not have a quadratic Gröbner basis. For this reason, Theorem 2.1 does not generalize to higher degrees.

Now, let us return to the questions 1 and 3 in the introduction.

EXAMPLE 2.3. Let $n = 12$, $d = 2$ and $u = 71$. There are five strongly stable sets of size 71, of monomials of degree two, in twelve variables, namely W_1, \dots, W_5 illustrated in Figures 4–8.

To see that these are all the strongly stable sets, we may count the number of ways to remove seven boxes from the diagram of size 78 with 12 completely filled columns. If we remove seven boxes from the last column, we get W_1 . If we remove boxes from the last two columns, there are three options which gives W_2 , W_3 and W_4 . There is only one way to remove seven boxes in the last three columns, which is W_5 . Removing boxes from four or more columns will result in removing more than seven boxes.

A computation in Macaulay2 [7] gives

$$\begin{aligned} e(\mathbb{k}[W_1]) &= 1984, & e(\mathbb{k}[W_2]) &= 2010, & e(\mathbb{k}[W_3]) &= 2018, \\ e(\mathbb{k}[W_4]) &= 2008, & e(\mathbb{k}[W_5]) &= 1980, \end{aligned}$$

which means that $\mathbb{k}[W_5]$ has the minimal Hilbert function, at least in the asymptotic sense. However, a computation of the Hilbert functions shows that $\text{HF}(\mathbb{k}[W_5], i)$ is not minimal for $i = 2$. The first i for which $\text{HF}(\mathbb{k}[W_5], i)$ is minimal is $i = 7$, as we can see in Table 1.

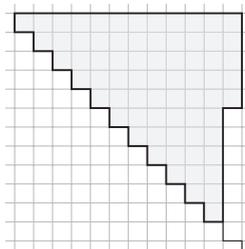
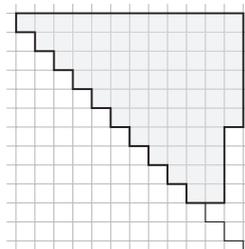
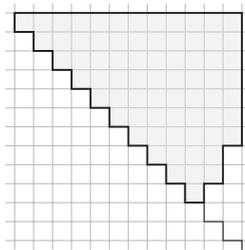
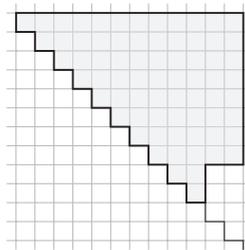
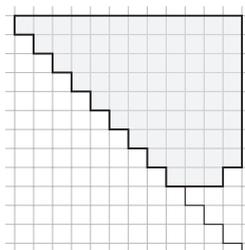
FIGURE 4. $W_1 = \text{st}(x_{11}^2, x_5x_{12})$ FIGURE 5. $W_2 = \text{st}(x_{10}x_{11}, x_6x_{12})$ FIGURE 6. $W_3 = \text{st}(x_{10}^2, x_9x_{11}, x_7x_{12})$ FIGURE 7. $W_4 = \text{st}(x_{10}^2, x_8x_{12})$ FIGURE 8. $W_5 = \text{st}(x_9x_{11}, x_8x_{12})$

TABLE 1

i	2	3	4	5	6	7
$\text{HF}(\mathbf{k}[W_1], i)$	1246	11389	70051	328771	1266005	4188859
$\text{HF}(\mathbf{k}[W_2], i)$	1256	11524	71012	333593	1285193	4253378
$\text{HF}(\mathbf{k}[W_3], i)$	1259	11565	71306	335075	1291108	4273307
$\text{HF}(\mathbf{k}[W_4], i)$	1255	11511	70922	333151	1283464	4247645
$\text{HF}(\mathbf{k}[W_5], i)$	1248	11406	70124	328965	1266265	4188404

This proves that the answer to the questions 1 and 3 is negative.

3. Subalgebras defined by Lex and RevLex segments

Any initial segment of monomials of degree d , according to a monomial ordering in $\mathbb{k}[x_1, \dots, x_n]$, is a strongly stable set. We will now focus on two monomial orderings, namely Lex and (graded) RevLex. In degree two, they may be defined as follows.

DEFINITION 3.1. Let $1 \leq i \leq j \leq n$, and $1 \leq k \leq \ell \leq n$. Then

$$x_i x_j >_{\text{Lex}} x_k x_\ell \quad \text{if } i < k, \text{ or if } i = k \text{ and } j < \ell,$$

and

$$x_i x_j >_{\text{RevLex}} x_k x_\ell \quad \text{if } j < \ell, \text{ or if } j = \ell \text{ and } i < k.$$

In terms of diagrams, we may say that $>_{\text{Lex}}$ orders the monomials firstly by row, and secondly by column, and that $>_{\text{RevLex}}$ orders the monomials firstly by column, and secondly by row.

Recall that, to minimize the degree of the Hilbert polynomial, we want to minimize the number of variables, with respect to the given u . In terms of the diagram, we want the diagram to be such that we can not draw another diagram with the same number of boxes, but fewer columns. Since there are $\binom{n+1}{2}$ monomials of degree two in n variables, we choose n so that $\binom{n}{2} < u \leq \binom{n+1}{2}$. Since $\binom{n+1}{2} - \binom{n}{2} = n$ we may write $u = \binom{n}{2} + r$ for some $0 < r \leq n$, or $u = \binom{n+1}{2} - s$ for some $0 \leq s < n$.

DEFINITION 3.2. For a positive integer u , let n be the unique number such that $\binom{n}{2} < u \leq \binom{n+1}{2}$. Then we let $\text{Lex}(u)$ be the set of the u greatest monomials of degree two, according to $>_{\text{Lex}}$. Similarly, we let $\text{RevLex}(u)$ be the set of the u greatest monomials of degree two, according to $>_{\text{RevLex}}$.

The diagram in Figure 4 is $\text{RevLex}(71)$ and the diagram in Figure 8 is $\text{Lex}(71)$.

REMARK 3.3. For $u = \binom{n+1}{2} - s$ and $s = 0, 1$, or 2 there is only one strongly stable set of size u in n variables, and this set is both a Lex and a RevLex segment. See Figure 9 for an example.

Notice that the subalgebra with the minimal Hilbert function in Example 2.3 was generated by the set $\text{Lex}(71)$, and the ‘‘competition’’ was between $\text{Lex}(71)$ and $\text{RevLex}(71)$.

CONJECTURE 3.4. *One of the algebras $\mathbb{k}[\text{Lex}(u)]$ or $\mathbb{k}[\text{RevLex}(u)]$ has the minimal Hilbert function, for a subalgebra of $\mathbb{k}[x_1, \dots, x_n]$ generated by u forms of degree two.*

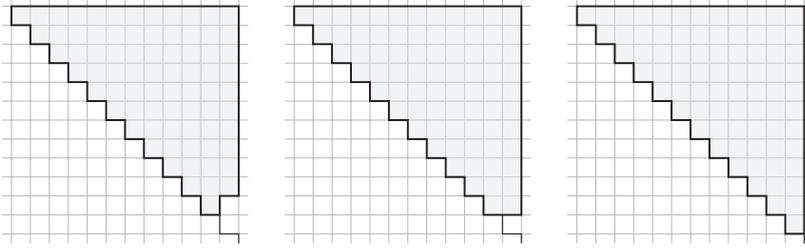


FIGURE 9. The only strongly stable sets of sizes 76, 77, and 78 in 12 variables.

Conjecture 3.4 can also be phrased as a purely combinatorial statement, see Appendix B.

REMARK 3.5. Conjecture 3.4 is true for $n \leq 80$, which means $u \leq 3240$. This is proved by a computation of the multiplicities in Mathematica [8]. The results of the computation, as well as a description of how the computation was made, can be found in Appendix A.

For $u = 7$ and 24 the sets $\text{Lex}(u)$ and $\text{RevLex}(u)$ give the same Hilbert polynomial. For $u = 40$ the sets $\text{Lex}(u)$ and $\text{RevLex}(u)$ give the same multiplicity. Computing their Hilbert polynomials in Macaulay2 gives

$$8! \text{HF}(\mathbb{k}[\text{Lex}(40)], i) = 240i^8 + 4248i^7 + 31640i^6 + 129192i^5 \\ + 315560i^4 + 471072i^3 + 418640i^2 + 201888i + 40320$$

and

$$8! \text{HF}(\mathbb{k}[\text{RevLex}(40)], i) = 240i^8 + 4256i^7 + 31752i^6 + 129752i^5 \\ + 316680i^4 + 471464i^3 + 417408i^2 + 200928i + 40320$$

and we can see that the Lex ordering gives the minimal Hilbert function, as $\text{HF}(\mathbb{k}[\text{Lex}(40)], i)$ has the smaller coefficient for i^7 . For $n \leq 80$ the values $u = 7, 24$, and 40 are the only values of u for which $\text{Lex}(u)$ and $\text{RevLex}(u)$ give the same multiplicity, apart from $u = \binom{n+1}{2} - s$ with $s = 0, 1, 2$, which we saw in Remark 3.3.

Conjecture 3.4 only states that the algebras with minimal Hilbert function are given by Lex or RevLex segments, it does not tell us which of the two orderings it is for a given u . One direct way to find out is of course to compute the multiplicities explicitly.

LEMMA 3.6. For $\binom{n}{2} < u \leq \binom{n+1}{2}$, we have

$$e(\mathbb{k}[\text{RevLex}(u)]) = \begin{cases} 2^{n-1} - 2^{n-r-1}, & \text{for } r < n, \\ 2^{n-1} & \text{for } r = n, \end{cases}$$

where $0 < r \leq n$ such that $u = \binom{n}{2} + r$.

PROOF. The total number of maximal NE-paths from the diagonal to the upper right corner of the diagram is 2^{n-1} , since the paths have length $n - 1$. If $r = n$ all these paths are inside the diagram. In the case $r < n$ we must subtract the number of paths that goes outside the RevLex-diagram. Those are exactly the paths going through the $x_{r+1}x_n$ -box, which is $n - r - 1$ steps from the diagonal. This gives us the formula $2^{n-1} - 2^{n-r-1}$.

LEMMA 3.7. For $\binom{n}{2} < u \leq \binom{n+1}{2}$, let k be the largest integer such that $\text{st}(x_{n-k}x_n) \supseteq \text{Lex}(u)$. If $\text{st}(x_{n-k}x_n) = \text{Lex}(u)$, then

$$e(\mathbb{k}[\text{Lex}(u)]) = \sum_{i=k}^{n-1} \binom{n-1}{i},$$

and otherwise

$$e(\mathbb{k}[\text{Lex}(u)]) = \sum_{i=k}^{n-1} \binom{n-1}{i} - \binom{n-k+j-2}{j-1}$$

where $j = |\text{st}(x_{n-k}x_n)| - |\text{Lex}(u)|$.

PROOF. Let us first compute the number of maximal NE-paths in the diagram of $\text{st}(x_{n-k}x_n)$. The number of maximal NE-paths starting in a row of length i is $\binom{n-1}{i-1}$, as such a path is of length $n - 1$ and should have precisely $i - 1$ steps right. In $\text{st}(x_{n-k}x_n)$ the first row has length n and the last length $k + 1$, so we get $\sum_{i=k}^{n-1} \binom{n-1}{i}$ paths. Now we must subtract those paths that are not inside the diagram of $\text{Lex}(u)$. Those are precisely the paths that go through the $x_{n-k}x_{n-j+1}$ -box, i.e. the first box in the last row that is not contained in $\text{Lex}(u)$. There is only one way to go from the diagonal to $x_{n-k}x_{n-j+1}$. From there $n - k + j - 2$ steps remains, and $j - 1$ of those should be steps right. Hence we should subtract $\binom{n-k+j-2}{j-1}$, which proves the formula.

Even with the formulas for the multiplicities given in Lemma 3.6 and Lemma 3.7, it is not obvious which one gives the smaller value for a given u . Looking at the data in Appendix A, the pattern is still not completely clear. However, two observations can be made.

- (1) For a given n , there are at most three shifts between Lex and RevLex. This has been confirmed by computation for $n \leq 1000$.
- (2) We have RevLex for small r , and Lex for large r in the interval $1 \leq r \leq n$. For $n \geq 80$ we will see in Theorem 3.10 that $\text{Lex}\left(\binom{n}{2} + r\right)$ gives the minimal Hilbert function for $n - 25 \leq r \leq n$, and in Theorem 3.14 that $\text{RevLex}\left(\binom{n}{2} + r\right)$ gives the minimal Hilbert function for $1 \leq r \leq 50$.

We summarize the cases where Conjecture 3.4 is proved in a theorem.

THEOREM 3.8. *One of the algebras $\mathbb{k}[\text{Lex}(u)]$ or $\mathbb{k}[\text{RevLex}(u)]$ has the minimal Hilbert function, for a subalgebra of $\mathbb{k}[x_1, \dots, x_n]$ generated by u forms of degree two, in the following cases.*

- $n \leq 80$, in which case the algebras are listed in Appendix A,
- $u = \binom{n}{2} + r$ with $n \geq 80$ and $1 \leq r \leq 50$, in which case it is given by $\text{RevLex}(u)$,
- $u = \binom{n}{2} + r$ with $n \geq 80$ and $n - 25 \leq r \leq n$, in which case it is given by $\text{Lex}(u)$.

3.1. Lex segments

In this section we will focus on algebras generated by sets $\text{Lex}(u)$, typically for large u in the interval $\binom{n}{2} < u \leq \binom{n+1}{2}$. In this setting it is convenient to use the representation $u = \binom{n+1}{2} - s$.

PROPOSITION 3.9. *Let S and n be fixed integers such that $0 \leq S < n$. Suppose that for all $0 \leq s \leq S$, the algebra $\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s)]$ has minimal multiplicity, among all subalgebras of $\mathbb{k}[x_1, \dots, x_n]$ generated by $\binom{n+1}{2} - s$ forms of degree two. Then $\mathbb{k}[\text{Lex}(\binom{n+2}{2} - s)] \subset \mathbb{k}[x_1, \dots, x_{n+1}]$ has minimal multiplicity for all $0 \leq s \leq S$.*

PROOF. Let L be the diagram of a strongly stable set in $\mathbb{k}[x_1, \dots, x_{n+1}]$ of size $\binom{n+2}{2} - s$, for some $s \leq S$. Let L' be the diagram obtained from L by removing the top row, and let L'' be the diagram obtained from L by removing the last column. Both L' and L'' are diagrams of strongly stable sets in $\mathbb{k}[x_1, \dots, x_n]$. L' is of size $\binom{n+1}{2} - s$ and L'' is of size $\binom{n+1}{2} - s'$ with $s' \leq s$. All maximal NE-path in L ending with a step up can be considered maximal NE-paths in L' by removing the last step. In the same way all maximal NE-paths in L ending with a step right can be considered maximal NE-paths in L'' . It follows that $e(L) = e(L') + e(L'')$. Applying the same argument to $\text{Lex}(\binom{n+2}{2} - s)$ we get

$$e(\mathbb{k}[\text{Lex}(\binom{n+2}{2} - s)]) = e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s)]) + e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s'')])$$

with $s'' \leq s$. Notice that $s'' \geq s'$ as the last column in the diagram of $\text{Lex}(\binom{n+2}{2} - s)$ has at least as many boxes as the last column in L .

By assumption we know that $e(L') \geq e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s)])$ and $e(L') \geq e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s')])$. We now have

$$\begin{aligned} e(L) &= e(L') + e(L'') \geq e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s)]) + e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s')]) \\ &\geq e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s)]) + e(\mathbb{k}[\text{Lex}(\binom{n+1}{2} - s'')]) \\ &= e(\mathbb{k}[\text{Lex}(\binom{n+2}{2} - s)]) \end{aligned}$$

and thus we have proved that $e(\mathbb{k}[\text{Lex}(\binom{n+2}{2} - s)]) \leq e(L)$ for any L of size $\binom{n+2}{2} - s$, with $s \leq S$.

THEOREM 3.10. *Let $u = \binom{n+1}{2} - s$, where $n \geq 80$ and $s \leq 25$. Then $\mathbb{k}[\text{Lex}(u)]$ has the minimal Hilbert function, among all the subalgebras of $\mathbb{k}[x_1, \dots, x_n]$ generated by u forms of degree two.*

PROOF. For $n = 80$, see Appendix A. It follows inductively from Proposition 3.9 that it also holds for all $n > 80$.

3.2. RevLex segments

We will now study algebras generated by sets $\text{RevLex}(u)$, where $u = \binom{n}{2} + r$ and r is small.

LEMMA 3.11. *Let $u_n = \binom{n}{2} + r$ for some fixed r with $1 \leq r \leq n$. Suppose $\mathbb{k}[\text{RevLex}(u_n)]$ has minimal multiplicity among all the subalgebras of $\mathbb{k}[x_1, \dots, x_n]$ generated by u_n forms of degree two. If $\text{Lex}(u_{n+1}) = \text{st}(g)$ for a monomial g , then either $\mathbb{k}[\text{Lex}(u_{n+1})]$ or $\mathbb{k}[\text{RevLex}(u_{n+1})]$ has minimal multiplicity among subalgebras of $\mathbb{k}[x_1, \dots, x_{n+1}]$ generated by u_{n+1} forms. If $\text{Lex}(u_{n+1}) \neq \text{st}(g)$, then $\mathbb{k}[\text{RevLex}(u_{n+1})]$ has minimal multiplicity.*

PROOF. Let L be the diagram of some strongly stable set of size u_{n+1} , which has more than one strongly stable generator. Let L' be the diagram obtained by removing the boxes on the last row of L , except the first box, i.e. the one on the diagonal. Clearly $e(L) \geq e(L')$. Let L'' be the diagram obtained by removing all boxes on the diagonal of L' . This is also a diagram of a strongly stable set, after a shift in the row and column indices. We have $e(L') = 2e(L'')$, since every maximal NE-path in L' comes from adding an up and right step to the beginning to a path of L'' . Notice that the boxes we have removed from L in the two steps were all in different columns. We have not removed any box from the last column, since L had more than one strongly stable generator. This means that $|L| - |L''| \leq n$, and hence $|L''| \geq u_{n+1} - n = u_n$. Let L''' be a diagram

obtained from L'' by, if necessary, removing some arbitrary boxes so that $|L'''| = u_n$. It is clearly possible to do this in such a way so that L''' is still a valid diagram for a strongly stable set. By assumption $e(L''') \geq e(\mathbb{k}[\text{RevLex}(u_n)])$. We now have

$$\begin{aligned} e(L) \geq e(L') &= 2e(L'') \geq 2e(L''') \\ &\geq 2e(\mathbb{k}[\text{RevLex}(u_n)]) = e(\mathbb{k}[\text{RevLex}(u_{n+1})]), \end{aligned}$$

where the last equality follows from Lemma 3.6.

We have now proved that the strongly stable set of size u_{n+1} which gives minimal multiplicity is either $\text{RevLex}(u_{n+1})$ or $\text{st}(g)$ for some monomial g . As $\text{st}(g)$ is a Lex segment, the proof is complete.

As we can see in Lemma 3.11, the situation is a bit more complicated than for the Lex-algebras in Section 3.1. Lemma 3.11 can not be used directly as an induction step, we need to analyze the situation when $\text{Lex}(u_n) = \text{st}(g)$ further. With $u_n = \binom{n}{2} + r$, and r fixed, for which n does this situation occur? The monomial g has to be divisible by x_n , so we have $g = x_{n-k}x_n$ for some number k . The monomials of degree two *not* in $\text{st}(x_{n-k}x_n)$ are the $\binom{k+1}{2}$ monomials in the variables x_{n-k+1}, \dots, x_n . It follows that we can write $u_n = \binom{n+1}{2} - \binom{k+1}{2}$. Then $\binom{n}{2} + r = \binom{n+1}{2} - \binom{k+1}{2}$, and it follows that $n = \binom{k+1}{2} + r$. To summarize, $\text{Lex}(u_n) = \text{st}(x_{n-k}x_n)$ precisely when $n = \binom{k+1}{2} + r$.

Our next goal is to prove that if $\text{Lex}(u_n) = \text{st}(x_{n-k}x_n)$ for some k , and $e(\text{Lex}(u_n)) > e(\text{RevLex}(u_n))$, then $e(\text{Lex}(u_{n'})) > e(\text{RevLex}(u_{n'}))$ for all $n' \geq n$. To do this, we first need the following technical lemma.

LEMMA 3.12. *Let k and s be integers, $s \geq 0$ and $k \geq 3$, and let $m = \binom{k+1}{2} + s$. If k is large enough compared to s , so that $(k-1)(2^k - \frac{3}{2}k) - 1 > s$, then*

$$\sum_{i=k}^m \binom{m}{i} > 2^k \sum_{i=k-1}^{m-k} \binom{m-k}{i}.$$

PROOF. Define

$$F(t) = 2^t \sum_{i=k}^{m-t} \binom{m-t}{i} + (2^t - 1) \binom{m-t}{k-1} + (2^t - t - 1) \binom{m-t}{k-2}$$

for $t \leq m - k = \binom{k}{2} + s$. Note that $F(k)$ is defined, since $k \leq \binom{k}{2} + s$ holds for all $k \geq 3$. The idea of the proof is to show that the two inequalities

$$\sum_{i=k}^m \binom{m}{i} \geq F(k) > 2^k \sum_{i=k-1}^{m-k} \binom{m-k}{i} \quad (1)$$

hold. As $F(0) = \sum_{i=k}^m \binom{m}{i}$, the first inequality is $F(0) \geq F(k)$. We will prove this by showing that $F(t) \geq F(t+1)$. Recall that

$$\binom{m-t}{i} = \binom{m-(t+1)}{i} + \binom{m-(t+1)}{i-1}$$

for $0 < i < m-t$. We get

$$\begin{aligned} & \sum_{i=k}^{m-t} \binom{m-t}{i} \\ &= \sum_{i=k}^{m-(t+1)} \binom{m-t}{i} + 1 \\ &= \sum_{i=k}^{m-(t+1)} \left(\binom{m-(t+1)}{i} + \binom{m-(t+1)}{i-1} \right) + \binom{m-(t+1)}{m-(t+1)} \\ &= \sum_{i=k}^{m-(t+1)} \binom{m-(t+1)}{i} + \sum_{i=k-1}^{m-(t+1)} \binom{m-(t+1)}{i} \\ &= 2 \sum_{i=k}^{m-(t+1)} \binom{m-(t+1)}{i} + \binom{m-(t+1)}{k-1}. \end{aligned}$$

From this we obtain

$$\begin{aligned} F(t) &= 2^t \sum_{i=k}^{m-t} \binom{m-t}{i} + (2^t - 1) \binom{m-t}{k-1} + (2^t - t - 1) \binom{m-t}{k-2} \\ &= 2^{t+1} \sum_{i=k}^{m-(t+1)} \binom{m-(t+1)}{i} + 2^t \binom{m-(t+1)}{k-1} \\ &\quad + (2^t - 1) \left(\binom{m-(t+1)}{k-1} + \binom{m-(t+1)}{k-2} \right) \\ &\quad + (2^t - t - 1) \left(\binom{m-(t+1)}{k-2} + \binom{m-(t+1)}{k-3} \right) \\ &\geq 2^{t+1} \sum_{i=k}^{m-(t+1)} \binom{m-(t+1)}{i} + (2^{t+1} - 1) \binom{m-(t+1)}{k-1} \\ &\quad + (2^{t+1} - (t+1) - 1) \binom{m-(t+1)}{k-2} \\ &= F(t+1). \end{aligned}$$

We have now proved the first inequality of (1). If

$$2^k - k - 1 > \frac{m - 2k + 2}{k - 1} \quad (2)$$

it follows that

$$\begin{aligned} F(k) &= 2^k \sum_{i=k}^{m-k} \binom{m-k}{i} + (2^k - 1) \binom{m-k}{k-1} + (2^k - k - 1) \binom{m-k}{k-2} \\ &> 2^k \sum_{i=k}^{m-k} \binom{m-k}{i} + (2^k - 1) \binom{m-k}{k-1} + \frac{m - 2k + 2}{k - 1} \binom{m-k}{k-2} \\ &= 2^k \sum_{i=k}^{m-k} \binom{m-k}{i} + (2^k - 1) \binom{m-k}{k-1} + \binom{m-k}{k-1} \\ &= 2^k \sum_{i=k-1}^{m-k} \binom{m-k}{i}, \end{aligned}$$

which is the second inequality of (1). Hence we need to verify (2). Since $m = \binom{k+1}{2} + s = \frac{k(k+1)}{2} + s$, (2) is equivalent to

$$2^k > \frac{\frac{k(k+1)}{2} + s - 2k + 2}{k - 1} + k + 1,$$

and the right hand side simplifies to $\frac{3}{2}k + \frac{s+1}{k-1}$. Hence

$$(2) \iff 2^k > \frac{3}{2}k + \frac{s+1}{k-1} \iff (k-1) \left(2^k - \frac{3}{2}k \right) - 1 > s,$$

which is true by assumption. We have now proved both inequalities of (1).

LEMMA 3.13. *Let k_0 and r be integers such that $k_0 \geq 3$, $r \geq 1$, and k_0 large enough compared to r so that $(k_0 - 1)(2^{k_0} - \frac{3}{2}k_0) > r$. Let $u_n = \binom{n}{2} + r$. If $\mathbb{k}[\text{RevLex}(u_n)]$ has minimal multiplicity among all subalgebras of $\mathbb{k}[x_1, \dots, x_n]$ generated by u_n forms of degree two, when $n = \binom{k_0}{2} + r$, then the same holds for all $n \geq \binom{k_0}{2} + r$.*

PROOF. If we can prove $e(\mathbb{k}[\text{RevLex}(u_n)]) < e(\mathbb{k}[\text{Lex}(u_n)])$ for all $n \geq \binom{k_0}{2} + r$ such that $\text{Lex}(u_n)$ has only one strongly stable generator, then we are done by Lemma 3.11. That is, we want to prove

$$e(\mathbb{k}[\text{RevLex}(u_n)]) < e(\mathbb{k}[\text{Lex}(u_n)]) \quad \text{for all } n = \binom{k}{2} + r, \quad k \geq k_0. \quad (3)$$

This is true for $k = k_0$, by assumption. The proof proceeds by induction. We assume that (3) is true for some $k \geq k_0$, and we want to prove it for $k + 1$. Let $n = \binom{k+1}{2} + r$, and notice that $\binom{k}{2} + r = n - k$. Applying Lemma 3.6 and Lemma 3.7 for the multiplicities, we are assuming that

$$2^{n-k-1} - 2^{n-k-r-1} < \sum_{i=k-1}^{n-k-1} \binom{n-k-1}{i},$$

and we want to prove

$$2^{n-1} - 2^{n-r-1} < \sum_{i=k}^{n-1} \binom{n-1}{i}.$$

By the inductive hypothesis we get

$$2^{n-1} - 2^{n-r-1} = 2^k (2^{n-k-1} - 2^{n-k-r-1}) < 2^k \sum_{i=k-1}^{n-k-1} \binom{n-k-1}{i}.$$

As $(k-1)(2^k - \frac{3}{2}k) > r$ holds for any $k \geq k_0$ by the assumption on k_0 , we can apply Lemma 3.12 with $m = n - 1$ and $s = r - 1$. This gives

$$2^k \sum_{i=k-1}^{n-k-1} \binom{n-k-1}{i} < \sum_{i=k}^{n-1} \binom{n-1}{i}$$

and we are done.

Finally we can apply Lemma 3.13 to get a class of minimal RevLex-algebras not included in the table in Appendix A.

THEOREM 3.14. *Let $u_n = \binom{n}{2} + r$ for $n \geq 80$ and $1 \leq r \leq 50$. Then $\mathbb{k}[\text{RevLex}(u_n)]$ has the minimal Hilbert function among the subalgebras of $\mathbb{k}[x_1, \dots, x_n]$ generated by u_n forms of degree two.*

PROOF. We will use Lemma 3.13 with $k_0 = 9$. As $(k_0 - 1)(2^{k_0} - \frac{3}{2}k_0 - 1) = 3988 > 50$ Lemma 3.13 can indeed be applied for $1 \leq r \leq 50$, but let us first consider $1 \leq r \leq 44$. In the table in Appendix A we see that $\mathbb{k}[\text{RevLex}(u_n)]$ gives the minimal multiplicity for all $n = \binom{9}{2} + r = 36 + r$, i.e. $37 \leq n \leq 80$. By Lemma 3.13 $\mathbb{k}[\text{RevLex}(u_n)]$ will have the minimal multiplicity for all $n \geq 80$.

Next, let us consider $45 \leq r \leq 50$. It follows from Appendix A and Lemma 3.11, with $n = 80$, that $\text{RevLex}(u_n)$ gives the minimal Hilbert function for $80 \leq n < 36 + r$, as $n = 36 + r$ is the least $n > 80$ for which $\text{Lex}(u_n) = \text{st}(g)$

for some monomial g . For $n = 36 + r$, Lemma 3.11 only tells us that Lex or RevLex gives the minimal Hilbert function. Using Lemma 3.6 and Lemma 3.7 we can compute $e(\mathbb{k}[\text{Lex}(u_n)])$ and $e(\mathbb{k}[\text{RevLex}(u_n)])$ for all $n = 36 + r$ with $45 \leq r \leq 50$, and verify that $e(\mathbb{k}[\text{RevLex}(u_n)])$ has the smaller value. By Lemma 3.13, this holds also for any $n > 36 + r$, and we are done.

Theorem 3.10 and Theorem 3.14 together with the data in Appendix A now proves Theorem 3.8.

4. Concluding remarks

A next step would be to look for a generalization of Conjecture 3.4 to higher degrees. Example 3.5 in [1] shows that the minimal Hilbert function for a subalgebra of $\mathbb{k}[x_1, x_2, x_3]$ generated by 12 forms of degree five is not given by the Lex or RevLex segment. Hence, Conjecture 3.4 does not generalize directly to higher degrees, one needs to use other monomial orderings. In fact, this can be observed already in degree three.

EXAMPLE 4.1. For $n = 4$, $d = 3$, and $u = 13$ there are eight strongly stable sets, namely

$$\begin{aligned} W_1 &= \text{st}(x_1x_3x_4, x_3^3), & W_2 &= \text{st}(x_2^2x_4, x_3^3), \\ W_3 &= \text{st}(x_1x_4^2, x_2^2x_4), & W_4 &= \text{st}(x_1x_3x_4, x_2^2x_4, x_2x_3^2), \\ W_5 &= \text{st}(x_1x_3^2, x_2^2x_4, x_3^3), & W_6 &= \text{st}(x_1x_2x_4, x_1x_3^2, x_2^2x_4, x_3^3), \\ W_7 &= \text{st}(x_1^2x_4, x_1x_3^2, x_2^2x_4, x_3^3), & W_8 &= \text{st}(x_1x_4^2, x_2x_3^2). \end{aligned}$$

These sets are generated using Macaulay2. Here W_2 is the RevLex segment, and W_3 the Lex segment. The multiplicities are $e([W_1]) = 13$, $e([W_2]) = \dots = e(\mathbb{k}[W_7]) = 15$, and $e(\mathbb{k}[W_8]) = 16$, so $\mathbb{k}[W_1]$ has the minimal Hilbert function.

It is not obvious which monomial ordering(s) that has W_1 in Example 4.1 as an initial segment. Another approach would be to look for a combinatorial description of the strongly stable sets that gives minimal multiplicity.

QUESTIONS 4.2.

- Which monomial orderings define subalgebras with minimal Hilbert function?
- Is there a combinatorial classification of the strongly stable sets giving minimal Hilbert function (not necessarily referring to monomial orderings)?

One may also consider the questions 1 and 3 in [1, Questions 3.6], mentioned in the introduction, again for $d \geq 3$. Does examples such as Example 2.3,

where the Hilbert function is minimal in the asymptotic sense but not minimal for small arguments, exist also in higher degrees? The following example, with $d = 3$, shows that a minimal value of the Hilbert function in $i = 2$ does not imply minimal Hilbert function for all i . That is, the answer to question 3 is negative, also in degree three.

EXAMPLE 4.3. For $n = 6$, $d = 3$, and $u = 43$ there are 672 strongly stable sets. The sets were generated using Macaulay2. Among the algebras generated by those sets, the minimal multiplicity is 176, and this is attained only by the set $W_1 = \text{st}(x_3x_5x_6)$. The minimal value of $\text{HF}(A, 2)$, among the 672 algebras, is 343, and is attained by both W_1 and $W_2 = \text{st}(x_2x_5x_6, x_4x_5^2)$. For $i > 2$ we have $\text{HF}(\mathbb{k}[W_1], i) < \text{HF}(\mathbb{k}[W_2], i)$.

We may also remark that neither W_1 nor W_2 is a Lex or RevLex segment, as the Lex segment is $\text{st}(x_3x_4x_6, x_2x_6^2)$, and the RevLex segment is $\text{st}(x_2x_4x_6, x_3^2x_6, x_5^3)$.

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Appendix A. Data for $n \leq 80$

In Table 2 the monomial orderings giving the algebras on $u = \binom{n}{2} + r$ generators with minimal Hilbert function are given, for $n \leq 80$ and $1 \leq r \leq n - 3$. For $n - 2 \leq r \leq n$ there is only one strongly stable set, as we saw in Remark 3.3.

The data for the table is based only on a computation of the multiplicities, except for three cases where more information was needed, see the discussion after Remark 3.5. The multiplicities are computed recursively, in the following way. We let the strongly stable sets be represented by diagrams, as before. To each box on the diagonal we also associate the number of maximal NE-paths starting in that box. The multiplicity is the sum of those numbers. Suppose that we have all diagrams, including the numbers in the diagonal boxes, of strongly stable sets with precisely n columns and size greater than $\binom{n}{2}$. The following steps generate the diagrams with precisely $n + 1$ columns and size greater than $\binom{n+1}{2}$. The procedure is also illustrated in Figure 10.

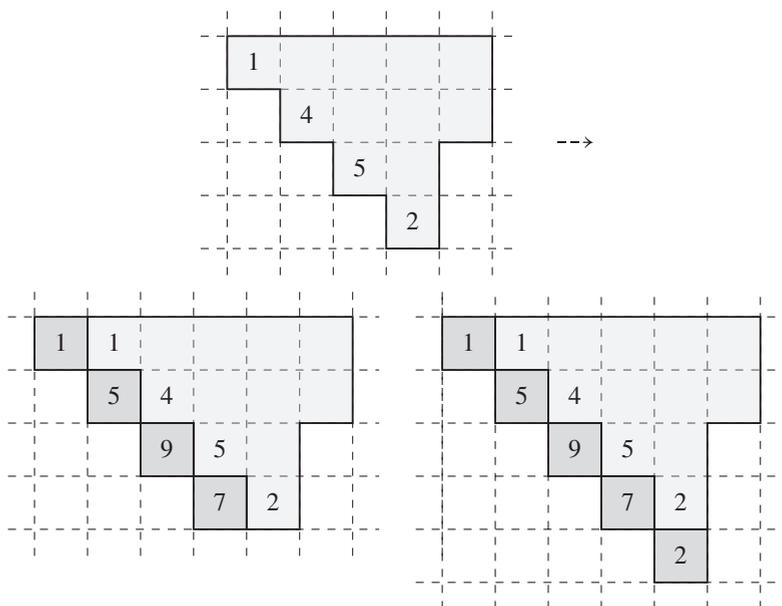


FIGURE 10. A diagram of five columns generates two diagrams of six columns.

- Step 1. To each diagram, add one box to the left in each row. This gives the diagram a new diagonal, to which we shall associate numbers. The box in the first row is given the number 1. To the other boxes, assign the sum of the number above and the number to the right.
- Step 2. For each diagram constructed in Step 1, construct a new diagram by adding a new row with one box. This box is assigned the same number as the box right above.
- Step 3. Take all diagrams produced in Step 1 and 2, and discard those of size less than or equal to $\binom{n+1}{2}$.

The number associated to a box on the new diagonal indeed gives the number of maximal NE-paths, as each path has to start with either a step up or right. Let L be an arbitrary diagram with $n + 1$ columns and size greater than $\binom{n+1}{2}$. Let L' be the diagram with n columns obtained by removing the diagonal from L . Then Step 1 or Step 2 above applied to L' will produce L , but we need to verify that L' has size greater than $\binom{n}{2}$. If the diagonal of L has at most n boxes, L' has size greater than $\binom{n+1}{2} - n = \binom{n}{2}$. If the diagonal of L has $n + 1$ boxes it means that L is the largest possible diagram with $n + 1$ columns, which has size $\binom{n+2}{2}$. Then L' has size $\binom{n+1}{2} - (n + 1) = \binom{n+1}{2}$.

Starting from the single strongly stable set $\{x_1^2\}$ on one variable we can

produce all strongly stable sets on n variables of size greater than $\binom{n}{2}$, for any given n .

To implement the algorithm, each strongly stable set can be represented by a vector containing the number of boxes in each row of the diagram.

TABLE 2. Monomial orderings for $n \leq 80$

u	n	r	ordering	u	n	r	ordering
7	4	1	Lex/RevLex	254–264	23	1–11	RevLex
11	5	1	RevLex	265–273	23	12–20	Lex
12	5	2	Lex	277–287	24	1–11	RevLex
16–17	6	1–2	RevLex	288–297	24	12–21	Lex
18	6	3	Lex	301–312	25	1–12	RevLex
22–23	7	1–2	RevLex	313–322	25	13–22	Lex
24	7	3	Lex/RevLex	326–338	26	1–13	RevLex
25	7	4	Lex	339–348	26	14–23	Lex
29–31	8	1–3	RevLex	352–362	27	1–11	RevLex
32–33	8	4–5	Lex	363	27	12	Lex
37–38	9	1–2	RevLex	364–365	27	13–14	RevLex
39–42	9	3–6	Lex	366–375	27	15–24	Lex
46–48	10	1–3	RevLex	379–390	28	1–12	RevLex
49–52	10	4–7	Lex	391	28	13	Lex
56–59	11	1–4	RevLex	392–393	28	14–15	RevLex
60–63	11	5–8	Lex	394–403	28	16–25	Lex
67–70	12	1–4	RevLex	407–419	29	1–13	RevLex
71–75	12	5–9	Lex	420	29	14	Lex
79–83	13	1–5	RevLex	421–422	29	15–16	RevLex
84–88	13	6–10	Lex	423–432	29	17–26	Lex
92–97	14	1–6	RevLex	436–449	30	1–14	RevLex
98–102	14	7–11	Lex	450	30	15	Lex
106–112	15	1–7	RevLex	451	30	16	RevLex
113–117	15	8–12	Lex	452–462	30	17–27	Lex
121–125	16	1–5	RevLex	466–480	31	1–15	RevLex
126	16	6	Lex	481	31	16	Lex
127	16	7	RevLex	482	31	17	RevLex
128–133	16	8–13	Lex	483–493	31	18–28	Lex
137–142	17	1–6	RevLex	497–511	32	1–15	RevLex
143	17	7	Lex	512–525	32	16–29	Lex
144	17	8	RevLex	529–544	33	1–16	RevLex
145–150	17	9–14	Lex	545–558	33	17–30	Lex
154–160	18	1–7	RevLex	562–578	34	1–17	RevLex
161–168	18	8–15	Lex	579–592	34	18–31	Lex
172–179	19	1–8	RevLex	596–613	35	1–18	RevLex
180–187	19	9–16	Lex	614–627	35	19–32	Lex
191–198	20	1–8	RevLex	631–649	36	1–19	RevLex
199–207	20	9–17	Lex	650–663	36	20–33	Lex
211–219	21	1–9	RevLex	667–685	37	1–19	RevLex
220–228	21	10–18	Lex	686–700	37	20–34	Lex
232–241	22	1–10	RevLex	704–723	38	1–20	RevLex
242–250	22	11–19	Lex	724–738	38	21–35	Lex

table continues

TABLE 2. Monomial orderings for $n \leq 80$ (continued)

u	n	r	ordering	u	n	r	ordering
742–762	39	1–21	RevLex	1432–1462	54	1–31	RevLex
763–777	39	22–36	Lex	1463–1482	54	32–51	Lex
781–802	40	1–22	RevLex	1486–1517	55	1–32	RevLex
803–817	40	23–37	Lex	1518–1537	55	33–52	Lex
821–843	41	1–23	RevLex	1541–1573	56	1–33	RevLex
844–858	41	24–38	Lex	1574–1593	56	34–53	Lex
862–885	42	1–24	RevLex	1597–1629	57	1–33	RevLex
886–900	42	25–39	Lex	1630–1650	57	34–54	Lex
904–928	43	1–25	RevLex	1654–1687	58	1–34	RevLex
929–943	43	26–40	Lex	1688–1708	58	35–55	Lex
947–968	44	1–22	RevLex	1712–1746	59	1–35	RevLex
969	44	23	Lex	1747–1767	59	36–56	Lex
970–971	44	24–25	RevLex	1771–1806	60	1–36	RevLex
972–987	44	26–41	Lex	1807–1827	60	37–57	Lex
991–1013	45	1–23	RevLex	1831–1867	61	1–37	RevLex
1014	45	24	Lex	1868–1888	61	38–58	Lex
1015–1016	45	25–26	RevLex	1892–1929	62	1–38	RevLex
1017–1032	45	27–42	Lex	1930–1950	62	39–59	Lex
1036–1059	46	1–24	RevLex	1954–1992	63	1–39	RevLex
1060	46	25	Lex	1993–2013	63	40–60	Lex
1061–1062	46	26–27	RevLex	2017–2056	64	1–40	RevLex
1063–1078	46	28–43	Lex	2057–2077	64	41–61	Lex
1082–1106	47	1–25	RevLex	2081–2121	65	1–41	RevLex
1107	47	26	Lex	2122–2142	65	42–62	Lex
1108–1109	47	27–28	RevLex	2146–2186	66	1–41	RevLex
1110–1125	47	29–44	Lex	2187–2288	66	42–63	Lex
1129–1154	48	1–26	RevLex	2212–2253	67	1–42	RevLex
1155	48	27	Lex	2254–2275	67	43–64	Lex
1156–1157	48	28–29	RevLex	2279–2321	68	1–43	RevLex
1158–1173	48	30–45	Lex	2322–2343	68	44–65	Lex
1177–1203	49	1–27	RevLex	2347–2390	69	1–44	RevLex
1204	49	28	Lex	2391–2412	69	45–66	Lex
1205–1206	49	29–30	RevLex	2416–2460	70	1–45	RevLex
1207–1222	49	31–46	Lex	2461–2482	70	46–67	Lex
1226–1252	50	1–27	RevLex	2486–2527	71	1–42	RevLex
1253–1254	50	28–29	Lex	2528	71	43	Lex
1255–1256	50	30–31	RevLex	2529–2531	71	44–46	RevLex
1257–1272	50	32–47	Lex	2532–2553	71	47–68	Lex
1276–1303	51	1–28	RevLex	2557–2599	72	1–43	RevLex
1304–1305	51	29–30	Lex	2600	72	44	Lex
1306	51	31	RevLex	2601–2603	72	45–47	RevLex
1307–1323	51	32–48	Lex	2604–2625	72	48–69	Lex
1327–1355	52	1–29	RevLex	2629–2672	73	1–44	RevLex
1356–1357	52	30–31	Lex	2673	73	45	Lex
1358	52	32	RevLex	2674–2676	73	46–48	RevLex
1359–1375	52	33–49	Lex	2677–2698	73	49–70	Lex
1379–1408	53	1–30	RevLex	2702–2746	74	1–45	RevLex
1409–1428	53	31–50	Lex	2747	74	46	Lex

table continues

TABLE 2. Monomial orderings for $n \leq 80$ (continued)

u	n	r	ordering	u	n	r	ordering
2748–2750	74	47–49	RevLex	2978–3000	77	52–74	Lex
2751–2772	74	50–71	Lex	3004–3052	78	1–49	RevLex
2776–2821	75	1–46	RevLex	3053	78	50	Lex
2822	75	47	Lex	3054–3055	78	51–52	RevLex
2823–2824	75	48–49	RevLex	3056–3078	78	53–75	Lex
2825–2847	75	50–72	Lex	3082–3130	79	1–49	RevLex
2851–2897	76	1–47	RevLex	3131–3132	79	50–51	Lex
2898	76	48	Lex	3133–3134	79	52–53	RevLex
2899–2900	76	49–50	RevLex	3135–3157	79	54–76	Lex
2901–2923	76	51–73	Lex	3161–3210	80	1–50	RevLex
2927–2974	77	1–48	RevLex	3211–3212	80	51–52	Lex
2975	77	49	Lex	3213–3214	80	53–54	RevLex
2976–2977	77	50–51	RevLex	3215–3237	80	55–77	Lex

Appendix B. A combinatorial analogue of Conjecture 3.4

Each strongly stable set W of degree two monomials in $\mathbb{k}[x_1, \dots, x_n]$ corresponds to the integer partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of $u = |W|$ defined by

$$\lambda_i = |\{x_i x_j \in W \mid j \geq i\}| = \text{the length of row } i \text{ in } L,$$

where L is the diagram representing W . The partition λ will have *distinct parts* in the sense that $\lambda_i \geq \lambda_{i+1}$ with equality only when $\lambda_i = \lambda_{i+1} = 0$. The diagram L is also called the *shifted Ferrers diagram* of the partition λ . We assume that $\binom{n}{2} < u \leq \binom{n+1}{2}$ as before, meaning that the diagram has precisely n columns, or equivalently that $\lambda_1 = n$. The set $\text{RevLex}(u)$ corresponds to the partition $(n, \dots, \widehat{i}, \dots, 1)$ meaning that we list all integers between n and 1, except i , where i is chosen uniquely so that the parts add up to u . For example, the set $\text{RevLex}(71)$ displayed in Figure 4 corresponds to the partition $(12, \dots, \widehat{7}, \dots, 1) = (12, 11, 10, 9, 8, 6, 5, 4, 3, 2, 1)$. The set $\text{Lex}(u)$ corresponds to the partition $(n, n - 1, \dots, j, k)$ where again j and k are chosen uniquely so that the sum is u , and with the condition that $j > k$. For example the set $\text{Lex}(71)$ in Figure 8 gives $(12, 11, 10, 9, 8, 7, 6, 5, 3)$.

If we fix n , the set W is uniquely determined by $\lambda' = (\lambda_2, \dots, \lambda_n)$, as $\lambda_1 = n$. This is a partition of $v = u - n$, and $\binom{n-1}{2} \leq v \leq \binom{n}{2}$. A maximal NE-path in the diagram L defines a subdiagram by taking the boxes on the path, and those to the left of it, as in Figure 11. This, in turn, gives a subpartition $\mu \subseteq \lambda$ with distinct parts, i.e. $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \leq \lambda_i$. As we will always have $\mu_1 = n$, it is enough to consider $\mu' = (\mu_2, \dots, \mu_n) \subseteq \lambda'$. In this way we have a bijection between the subpartitions μ' with distinct parts, and the maximal NE-paths of L . Conjecture 3.4 can now be stated as follows.

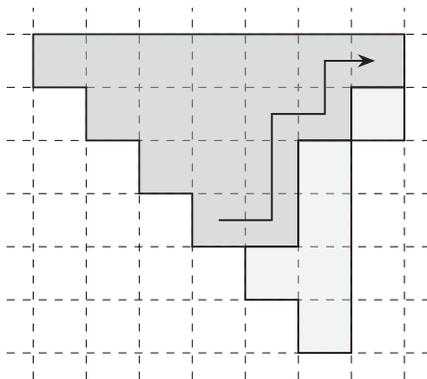


FIGURE 11. A maximal NE-path defines a subdiagram.

CONJECTURE B.1. For fixed positive integers N and v such that $\binom{N}{2} \leq v \leq \binom{N+1}{2}$ let \mathcal{P} be the set of integer partition of v into distinct parts, with largest part at most N . The member of \mathcal{P} that has the minimal number of subpartitions with distinct parts is

$$(N, \dots, \widehat{i}, \dots, 1) \quad \text{or} \quad (N, N-1, \dots, j, k) \quad \text{with } j > k.$$

Recall that the *dominance order* on the set of partitions of a number v is defined as follows. Let $\tau = (\tau_1, \dots, \tau_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be partitions of v with $\tau_1 \geq \tau_2 \geq \dots \geq \tau_n \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. We say that $\tau \leq \lambda$ if

$$\tau_1 + \dots + \tau_k \leq \lambda_1 + \dots + \lambda_k \quad \text{for all } 1 \leq k \leq n.$$

We can also say that $\tau \leq \lambda$ if the diagram of τ can be obtained by that of λ by “moving boxes down to the left” in the (shifted) Ferrers diagram. With this ordering, \mathcal{P} is a bounded poset, with the two partitions in Conjecture B.1 as lower and upper bound.

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