WEAKLY UNCONDITIONALLY CONVERGENT SERIES IN M-IDEALS

GILLES GODEFROY AND PAULETTE SAAB*

Abstract.

Every Banach space $X$ which is an $M$-ideal in its bidual has the property (V) of Pełczynski. If $E$ is separable complex Banach space with the approximation property and $K(E)$ is an $M$-ideal in $L(E)$, then $E$ is isomorphic to a complemented subspace of a space with a shrinking unconditional finite dimensional decomposition.

Introduction.

The concept of an $M$-ideal has been introduced by Alfsen and Effros in 1972 ([1], [2]) and has attracted a lot of attention since then (see e.g. [4], [17], [28]). The present work is a contribution to the study of their structure in Banach spaces and Banach algebras. We first show that if a Banach space $X$ is an $M$-ideal in its bidual $X^{**}$ then $X$ has the property (V) of Pełczynski [30], that is if $T: X \to E$ is a non weakly compact operator from $X$ into a Banach space $E$, then $X$ contains a subspace $Y$ isomorphic to $c_0$ such that the restriction of $T$ to $Y$ is an isomorphism between $Y$ and $T(Y)$. We use different techniques for showing that if $E$ is a separable complex Banach space and $K(E)$ is an $M$-ideal in $L(E)$, then an operator $T \in L(E^{**})$ is a conjugate operator if and only if it is the weak*-sum of a weakly unconditionally convergent series of compact operators of $K(E)$; this applies of course to the identity operator, and this permits to show that if $E$ is separable complex Banach space with the approximation property and $K(E)$ is an $M$-ideal in $L(E)$ then $E$ is isomorphic to a complemented subspace of a space with a shrinking unconditional finite dimensional decomposition. We also show that if a separable complex space $E$ is reflexive and $K(E)$ is an $M$-ideal in $L(E)$, then $K(E)$ has the property (u) of Pełczynski.

We are glad to thank D. Werner and A. Lima for very useful comments on early versions of this paper.

* Supported in Part By an NSF Grant DMS-87500750

Received November 16, 1987; in revised form February 8, 1988
NOTATIONS. The Banach spaces we consider are real or complex. The space of compact (resp. bounded) linear operators on a Banach space $E$ is denoted by $K(E)$ (resp. $L(E)$). The closed unit ball of a Banach space $X$ will be denoted by $X_1$, and the unit sphere is $S_1(X)$. A weakly unconditionally convergent series is a sequence $(x_n)_{n \geq 1}$ in $X$ such that

$$\sum_{n=1}^{\infty} |y^*(x_n)| < \infty \text{ for every } y^* \in X^*.$$ 

It is an easy consequence of the uniform boundedness principle that this condition is equivalent to

$$\sup \|\sum \varepsilon_i x_i\| < \infty,$$

where the supremum is taken over the finite sequences of $\varepsilon_i$ with $|\varepsilon_i| = 1$.

Since the sequence $S_n = \sum_{i=1}^{n} x_i$ is clearly weakly Cauchy, we can let

$$\sum^* x_n = \lim_{n \to \infty} S_n \text{ in } (X^{**}, \text{ weak}).$$

A space $X$ is said to have property (u) [30] if every $z \in X^{**}$ which is in the sequential closure of $X$ in $(X^{**}, \text{ weak}^*)$ may be written

$$z = \sum^* x_n,$$

where $(x_n)_{n \geq 1}$ is a weakly unconditionally convergent series in $X$. A subspace $X$ of a Banach space $E$ is an $M$-ideal in $E$ if there exists a subspace $Z$ of $E^*$ such that $E^* = X^\perp \oplus Z$, where $X^\perp$ denotes the annihilator of $X$ in $E^*$, and $\oplus Z$ means $\|u + v\| = \|u\| + \|v\|$ for every $u \in X^\perp$ and every $v \in Z$. The conjugate of an operator $T$ is denoted by $T^*$, and we note $T^{**} = (T^*)^*$. All subspaces we consider are supposed to be norm closed.

Results.

Our first result was announced in [11].

THEOREM 1. Let $X$ be a Banach space which is an $M$-ideal in its bidual $X^{**}$. Then $X$ has the property (V) of Pelczynski.

PROOF. We consider the set

$$D = \{z \in X^{**} \mid \|z\| = 1 = \text{dist } (z, X)\}.$$ 

Since $X$ is proximinal in $X^{**}$ [18], the linear span of $(X \cup D)$ is $X^{**}$. Hence if $T : X \to Y$ is a non weakly compact operator, there exists $z \in D$ such that $T^{**}(z) \notin Y$. We let $\alpha = \text{dist } (T^{**}(z), Y) > 0$, and pick $\varepsilon \in (0, \alpha)$. 
We stop now to prove the following lemma

**Lemma 2.** For every finite subset \((x_i)_{1 \leq i \leq n}\) of \(X\) with \(\|x_i\| < 1\) for \(1 \leq i \leq n\), there exists \(x \in X\) such that \(\|x\| < 1\), \(\|x - x_i\| < 1\) for \(1 \leq i \leq n\) and \(\|T(x)\| > \varepsilon\).

**Proof of Lemma 2.** Pick \(\eta > 0\) such that \((1 + \eta)\|x_i\| < 1\) for \(1 \leq i \leq n\) and \(\varepsilon < \alpha(1 + \eta)^{-2}\). For \(z \in D\) such that \(\text{dist}(T^{**}(z), Y) = \alpha > 0\), let

\[
P(z) = \{x \in X \mid \|z - x\| = 1\}.
\]

The set \(P(z)\) is a pseudo ball [4] and thus there exists \(x_0 \in P(z)\) such that \((x_0 + (1 + \eta)x_i) \in P(z)\) for \(1 \leq i \leq n\) [4]. Since \(x_0 \in X\), we have

\[
\|T^{**}(z - x_0)\| \geq \text{dist}(T^{**}(z), Y) = \alpha,
\]

hence there exists \(y \in Y^*\) with \(\|y\| = 1\) such that

\[
\langle T^{**}(z - x_0), y \rangle = \langle z - x_0, T^*(y) \rangle > \alpha(1 + \eta)^{-1}.
\]

Let \(V\) be the linear span of \(\{z, x_0, x_1, \ldots, x_n\}\); by the local reflexivity principle [22], there is an operator \(A: V \to X\) such that

(i) \(\|A\| < 1 + \eta\).

(ii) \(A(u) = u\) for every \(u \in V \cap X\).

(iii) \(\langle A(z - x_0), T^*(y) \rangle > \alpha(1 + \eta)^{-1}\).

We let now

\[
x = (1 + \eta)^{-1} A(z - x_0)
\]

This element \(x\) works; indeed

\[
\|x\| \leq (1 + \eta)^{-1} \|A\| \|z - x_0\| < 1
\]

Moreover, for \(1 \leq i \leq n\), we have

\[
\|z - x_0 - (1 + \eta)x_i\| \leq 1
\]

and thus

\[
\|A(z - x_0) - (1 + \eta)x_i\| < 1 + \eta,
\]

which implies that \(\|x - x_i\| < 1\); finally, the condition (iii) implies

\[
\langle x, T^*(y) \rangle = \langle T(x), y \rangle > \alpha(1 + \eta)^{-2} > \varepsilon
\]

and therefore

\[
\|T(x)\| > \varepsilon.
\]

Let us now resume the proof of Theorem 1. Since \(\|T^{**}(z)\| > \varepsilon\), there exists \(u_0 \in X\) with \(\|u_0\| < 1\) and \(\|T(u_0)\| > \varepsilon\). We apply the lemma 2 to the family \(\{u_0, -u_0\}\) to
find $u_1 \in X$ with

(i) \[ \|u_1\| < 1, \|u_0 + u_1\| < 1, \|u_1 - u_0\| < 1 \]

and

(ii) \[ \|T(u_1)\| > \varepsilon. \]

We apply now the lemma 2 to

\[ \{\varepsilon_0 u_0 + \varepsilon_1 u_1 \mid \varepsilon_i = \pm 1\}, \]

and we continue in this way to construct by induction a sequence $(u_i)_{i \geq 1}$ such that

(i) \[ \left\| \sum_{i=1}^{n} \varepsilon_i u_i \right\| < 1 \quad \forall n \quad \text{and} \quad \forall \varepsilon_i = \pm 1 \]

and

(ii) \[ \|T(u_i)\| > \varepsilon \quad \forall i; \]

the result follows easily.

Let us observe that the above proof is making a crucial use of the techniques of [4] and [17].

**Remarks 3.**

1) If $X$ is an $M$-ideal in its bidual, then $X^*$ is weakly sequentially complete [12] and since, by Theorem 1 $X$ has (V), this implies that every operator from $X$ to $X^*$ is weakly compact. In particular if $X$ a Banach algebra, then $X$ is Arens-regular [13].

2) If $X$ has (V), then $X^*$ has (V*) [30] and thus by Theorem 2, if $Y$ is such that $Y^{**} = Y \oplus_1 Y'$ with $Y'$ weak* closed, then $Y$ has the property (V*). It is an open question to know whether the assumption put on $Y'$ to be weak* closed is actually necessary.

3) The space $X = (\sum \oplus l_1^1)_{\ell_0}$ is an $M$-ideal in its bidual; however $X^{**}$ contains a complemented copy of $l_1$ [19] and thus $X^{**}$ does not have the property (V).

4) It is an open question to know whether a separable Banach space that is an $M$-ideal in its bidual has the property (u); This question will be answered below in the affirmative in an important special case (Corollary 8), see: Added in proof.

We will now prove a structural theorem for the complex spaces $E$ such that $K(E)$ is an $M$-ideal in $L(E)$. Let us state our main result.

**Theorem 4.** Let $E$ be separable complex Banach space such that $K(E)$ is an $M$-ideal in $L(E)$. Then $K(E)^{**}$ is canonically isometric to $L(E^{**})$ and for $T \in L(E^{**})$ the following are equivalent:

1) There exists $T_0 \in L(E^*)$ such that $T_0^* = T$.
2) There is a sequence $(K_n)_{n \geq 1}$ in $K(E)$ such that:
(i) \( \| \sum \varepsilon_i K_i \| \leq M \) for every finite sequence of \( |\varepsilon_i| = 1 \).

(ii) \( \langle T(z), y \rangle = \sum_{n=1}^{\infty} \langle z, K_n^* y \rangle \) \( \forall y \in E^* \) and \( \forall z \in E^{**} \).

**Proof.** If \( K(E) \) is an \( M \)-ideal in \( L(E) \), then \( E \) and \( E^* \) have the compact approximation property (C.A.P) [17] and \( E^* \) has the Radon Nikodym Property (RNP). The Feder-Saphar technique [9] permits to show that \( K(E)^{**} \) is canonically isometric to \( L(E^{**}) \) [14]; where canonical means that the diagram

\[
\begin{array}{ccc}
L(E) & \xrightarrow{i^{**}} & L(E^{**}) \\
\downarrow j & & \uparrow l \\
K(E) & \xrightarrow{i} & K(E)^{**}
\end{array}
\]

is commutative, where \( i \) and \( j \) are the canonical injections, \( l \) is the isometry and \( i^{**}(T) = T^{**} \).

Let us now proceed to the proof of the equivalence. To show that 1) implies 2) we need to prove the following crucial lemma which relies heavily on ([28], lemma 2.4.).

**Lemma 5.** Let \( A \) be a complex Banach algebra with unit \( e \). Let \( X \) be a separable subspace of \( A \) which is an \( M \)-ideal in \( A \); if we write \( A^* = X^\perp \oplus Y \), then there is a weakly unconditionally convergent series \((x_n)_{n \geq 1}\) in \( X \) such that

\[
e(y) = \sum_{n=1}^{\infty} x_n(y) \quad \forall y \in Y.
\]

**Proof.** Let

\[
S = \{ y \in A^* | \| y \| = 1 = y(e) \}
\]

be the state space of \( A \). Since \( X \) is an \( M \)-ideal in \( A \), the sets

\[F = X^\perp \cap S \text{ and } F' = Y \cap S \]

form a pair of split faces of \( S \) such that \( S = \text{conv}(F \cup F') \), and moreover \( X^\perp \) (resp., \( Y \)) is algebraically spanned by \( F \) (resp., \( F' \)), [28]. Let \( \Pi: A^* \to Y \)

be the projection having as kernel \( X^\perp \), and let \( z = \Pi^*(e) \in A^{**} \). It is clear that \( z_{|F} = 0 \) and \( z_{|F'} = 1 \). Since \( S = \text{conv}(F \cup F') \) we have \( 0 \leq z \leq 1 \) on \( S \) and for every \( \lambda \in [0,1] \) the set

\[S_{\lambda} = S \cap z^{-1}((-\infty, \lambda]) \]

may be written

\[S_{\lambda} = \{ \mu t + (1 - \mu) t' | t \in F, t' \in S, 1 - \lambda \leq \mu \leq 1 \}.\]
Since $F$ is $w^{*}$-compact, the set $S_{\delta}$ is $w^{*}$-closed. The projection $\Pi$ is continuous from $(A^{*}, w^{*})$ to $(Y, \sigma(Y, X))$ and therefore the set $S_{0} = \Pi(S)$ is $\sigma(Y, X)$-compact. Moreover since $z = e \circ \Pi = \Pi^{*}(e)$ we have $0 \leq z \leq 1$ on $S_{0}$ and

$$S_{0} \cap z^{-1}((-\infty, \lambda]) = \Pi(S_{\delta}),$$

and this shows that $z$ is lower semi-continuous on $(S_{0}, \sigma(Y, X))$; therefore there exists an increasing sequence $(f_{n})_{n \geq 1}$ of continuous functions on $(S_{0}, \sigma(Y, X))$ which converges pointwise to $z$; in particular we have

(i) $\sum_{n=1}^{\infty} |f_{n}(y)| < \infty \quad \forall y \in S_{0},$

(ii) $z(y) = \sum_{n=1}^{\infty} f_{n}(y) \quad \forall y \in S_{0}.$

But we also have $z \in X^{\perp\perp}$ and a fortiori $z$ belongs to the pointwise closure on $S_{0}$ of $X_{1}$. Hence by a classical lemma (see [24], p. 32) there is a sequence $(x_{n})_{n \geq 1}$ in $X_{1}$ such that

(iii) $\sum_{n=1}^{\infty} |\lambda_{n}(y)| < \infty \quad \forall y \in S_{0},$

(iv) $z(y) = \sum_{n=1}^{\infty} x_{n}(y) \quad \forall y \in S_{0}.$

The numerical radius defines an equivalent norm on $A$ [3], thus $A^{*} = \text{span}(S)$ and $Y = \text{span}(S_{0})$; hence the conditions (iii) and (iv) hold also for $y \in Y$; this finishes the proof of the lemma since $z(y) = e(y) \quad \forall y \in Y$.

Let us now proceed to the proof of Theorem 4. We apply Lemma 5 to $A = L(E)$ and $X = K(E)$. For every $y \in E^{*}$ and every $z \in E^{**}$ with $\|y\| = \|z\| = 1$, let us consider the linear form $z \otimes y$ in $L(E)^{*}$ where

$$\langle z \otimes y, T \rangle = \langle z, T^{*}(y) \rangle;$$

clearly

$$\|z \otimes y\| = 1 \text{ in } L(E)^{*},$$

but also in

$$K(E)^{*} = L(E)^{*}/K(E)^{\perp}.$$ 

Hence if

$$L(E)^{*} = K(E)^{\perp} \oplus_{1} Y$$

we have

$$z \otimes y \in Y \text{ if } \|y\| \|z\| = 1$$
and thus
\[ z \otimes y \in Y \text{ for every } y \in E^* \text{ and } z \in E^{**}. \]

Hence by Lemma 5, there is a sequence \((S_n)_{n \geq 1}\) in \(K(E)\) such that
\[ \| \sum \varepsilon_i S_i \| \leq M \]
for every finite sequence \(|\varepsilon_i| = 1\), and such that
\[ (*) \quad \langle z, y \rangle = \sum_{n=1}^{\infty} \langle z, S_n^*(y) \rangle \ \forall y \in E^* \text{ and } \forall z \in E^{**}; \]

consider now \(T \in L(E^{**})\) such that there is \(T_0 \in L(E^*)\) with \(T = T_0^*\) and apply (*) to \(z = T(z')\) to get
\[ \langle T(z'), y \rangle = \sum_{n=1}^{\infty} \langle T(z'), S_n^*(y) \rangle \ \forall y \in E^* \text{ and } \forall z' \in E^{**} \]
and thus
\[ \langle T(z'), y \rangle = \sum_{n=1}^{\infty} \langle z', T_0 S_n^*(y) \rangle, \]
but since \(S_n^*\) is compact, it is \(\text{weak}^*\) to norm continuous on bounded sets and so is \(T_0 S_n^*\), hence \(T_0 S_n^*\) is \(\text{weak}^*\) to \(\text{weak}^*\) continuous and compact and thus there exists \(K_n \in K(E)\) such that \(K_n^* = T_0 S_n^*\). Finally we have
\[ \| \sum \varepsilon_i K_i \| = \| \sum \varepsilon_i K_i^* \| = \| \sum \varepsilon_i T_0 S_n^* \| \leq \| T_0 \| \| \sum \varepsilon_i S_n^* \| \leq \| T_0 \| M, \]
and this concludes the proof of 1) implies 2) in Theorem 4.

Conversely, we will prove a much stronger result than 2) implies 1), namely if there exists a sequence \((V_n)_{n \geq 1}\) in \(L(E^*)\) such that
\[ (**) \quad \langle T(z), y \rangle = \lim_{n \to \infty} \langle z, V_n(y) \rangle \ \forall y \in E^* \text{ and } \forall z \in E^{**}, \]
then there is \(T_0 \in L(E^*)\) such that \(T_0^* = T\). Indeed (**) implies that
\[ T^*(y) = \lim_{n \to \infty} V_n(y) \text{ in } (E^{***}, \text{weak}^*), \]
but \(K(E)\) being \(M\)-ideal in \(L(E)\) implies that \(E\) is an \(M\)-ideal in \(E^{**}\) [21] and thus \(E^*\) is weakly sequentially complete [12]; hence \(T^*(y) \in E^*\) and if we define \(T_0\) to be the restriction of \(T^*\) to \(E^*\) we have \(T_0^* = T\).
Our first application of theorem 4 is a structural result for the complex spaces $E$ for which $K(E)$ is an $M$-ideal in $L(E)$. Let us observe that such a space has always the metric compact approximation property [17]; it is unknown whether it has necessarily the approximation property (A.P). Our next result asserts that if the A.P. holds, then a much stronger property is satisfied.

**Corollary 6.** Let $E$ be a separable complex Banach space such that $K(E)$ is an $M$-ideal in $L(E)$. Then the following statements are equivalent:

1) $E$ has the A.P.

2) $E$ has the metric A.P.

3) $E^*$ has the A.P.

4) $E$ is isomorphic to a complemented subspace of a space $a$ with a shrinking unconditional finite dimensional decomposition.

**Proof.** The implications 4) implies 3) and 2) implies 1) are obvious. To see that 3) implies 2) notice that $E^*$ has the RNP and therefore $E^*$ has the metric A.P. if it has A.P. (see [23]); and it is always true that $E$ has the metric A.P. if $E^*$ does (see [23]).

For 1) implies 4), apply Theorem 4, to find a sequence $(S_n)_{n \geq 1}$ in $K(E)$ such that

(i) $\|\sum e_i S_i\| \leq M$ for every finite sequence of $|e_i| = 1$

(ii) $\langle z, y \rangle = \sum_{n=1}^{\infty} \langle z, S_n^* y \rangle \forall y \in E^*$ and $z \in E^{**}$.

Since $E$ has the A.P. there exists a sequence $(R_n)_{n \geq 1}$ of finite rank operators such that

$$\|S_n - R_n\| < 2^{-n-1}.$$  

Following the lines of ([27], proposition 3), we observe that for every $x \in E$, the series

$$S(x) = \sum R_n(x)$$

is weakly unconditionally convergent and thus defines an operator from $E$ into $E^{**}$; but $S$ actually takes its value in $E$; indeed for every $N$, we have

$$\left\| \sum_{i > N} R_i(x) - \sum_{i > N} S_i(x) \right\| \leq \|x\| \sum_{i > N} \|R_i - S_i\|$$

$$\leq \|x\| 2^{-N-1}$$

and thus

$$\left\| S(x) - \sum_{n=1}^{N} R_n(x) - x + \sum_{n=1}^{N} S_n(x) \right\| \leq \|x\| 2^{-N-1},$$

which shows that

$$\text{dist} (S(x), E) \leq \|x\| 2^{-N-1} \text{ for every } N$$
and thus \( S(x) \in E \). Moreover we clearly have that \( \| \text{Id}_E - S \| \leq 2^{-1} \) and thus \( S = U^{-1} \) is an invertible operator; if we consider now the finite rank operators \( U_n = UR_n \) we have:

\[
x = \sum_{n=1}^{\infty} U_n(x) \quad \forall x \in E
\]

and the convergence is unconditional. In other words \( E \) has the unconditional approximation property (in the terminology of [8]). Thus by [31], the space \( E \) is isomorphic to a complemented subspace of the unconditional sum [31]

\[
X = \sum_u U_n(E).
\]

For completing the proof let us observe that \( E \) is an \( M \)-ideal in \( E^{**} [21] \) and thus \( E \) is an Asplund space; that is \( E \) is an Asplund complemented subspace of a space \( X \) which has an unconditional finite dimensional decomposition. Under these assumptions, it is possible [20] to adapt the proof of Theorem 3.3 of [10] to show that \( E \) is isomorphic to a complemented subspace of a space with a shrinking unconditional finite dimensional decomposition: in the notation of ([10], with \( E = A(Z) \)), one needs to observe that the set

\[
W = \overline{\text{conv}} \left\{ \left( \sum_{n=1}^{k} \varepsilon_n P_n \right) E_1 | \varepsilon_n \in \{-1,1\}^N, k \geq 1 \right\}
\]

where the \( P_n \)'s are the "coordinate projections" associated with the finite dimensional decomposition is weak*-sequentially compact and apply the interpolation technique of [5].

**Remarks 7.**

1) The above condition 4) implies in particular that \( E^* \) is complemented in a space with an unconditional boundedly complete finite dimensional decomposition.

2) If moreover \( E \) is reflexive, we can show like in ([10], Theorem 3.3) that \( E \) is isomorphic to a complemented subspace of a reflexive space with an unconditional finite dimensional decomposition. It suffices indeed to reproduce the above proof and to observe [20] that the corresponding set \( W \) is weakly compact.

In the case where \( E \) is reflexive we obtain without assuming the A.P. the following corollary:

**Corollary 8.** Let \( E \) be a separable reflexive complex Banach space such that \( K(E) \) is and \( M \)-ideal in \( L(E) \). Then \( K(E) \) has the property \((u)\).

**Proof.** By [17], \( K(E)^{**} \) is canonically isometric to \( L(E) \). By Theorem 4, there exists a sequence \( (K_n)_{n \geq 1} \) in \( K(E) \) such that
\[(i) \quad \left\| \sum_{i=1}^{n} e_i K_i \right\| \leq M \quad \forall |e_i| = 1, \quad \forall n \geq 1\]

\[(ii) \quad \langle y, x \rangle = \sum_{n=1}^{\infty} \langle y, K_n(x) \rangle \quad \forall x \in X, \quad \forall y \in X^*.\]

Condition (ii) means that \(\text{Id}_E = \sum K_n\) in \(K(E)^{**} = L(E)\): if now \(T \in L(E)\) is any operator, then we have

\[T = \sum S_n\]

since the multiplication in \(L(E)\) is weak*-separately continuous if \(E\) is reflexive; and it is clear that \(S_n \in K(E)\) and the \(S_n\)'s satisfy condition (i).

**Examples, remarks and questions.**

1) It is easy to deduce from the results of [8] and [14] that if \(X\) has a shrinking unconditional finite dimensional decomposition such that the weak* and the weak topology coincide on the unit sphere \(S_1(X^*)\) of \(X^*\) and \(E\) is a subspace of \(X\) then saying that \(E\) has the approximation property is equivalent to saying that \(E^*\) has the metric approximation property and this in turn is equivalent to asserting that \(E\) has the unconditional approximation property.

2) Let \(A\) be a subset of an abelian discrete group \(\Gamma = \hat{\Gamma}\); let \(\mathcal{C} = \mathcal{C}(G)\) and \(\Lambda'\) be the complement of \((-\Lambda)\) in \(\Gamma\), then the following statements were shown to be equivalent in [15]:

   (i) \(\mathcal{C}/\mathcal{C}_{\Lambda'}\) is an \(M\)-ideal in its bidual.

   (ii) The unit ball \(B_A\) of \(L_A^1(G)\) is closed for the topology \(\tau\) of convergence in measure, and the Fourier coefficients \(\hat{F}(f) = \hat{f}(\alpha)\) are continuous on \((B_A, \tau)\).

   It is not known if these \(M\)-ideals have the property (u) in general. This is true if \(\Gamma = \mathbb{Z}\) and \(\Lambda = \mathbb{N}\) since \(\mathcal{C}(T)/A_0(D)\) is isometric to a subspace of \(K(l_2)\) [15], see: Added in proof.

   Observe that the convolution induces a structure of Banach algebra on \(\mathcal{C}/\mathcal{C}_{\Lambda'}\), since \(\mathcal{C}_{\Lambda'}\) is an ideal of \((\mathcal{C}, *)\), but the bidual space \(L^\infty/L^\infty_{\Lambda'}\) has no unit in general.

3) If \(X\) is a separable complex Banach algebra such that:

   a) \(X\) is an \(M\)-ideal in its bidual \(X^{**}\)

   b) \(X\) is an ideal of the algebra \(X^{**}\)

   c) \(X^{**}\) is a Banach algebra with unit

   then it is easy to deduce from Lemma 5 that \(X\) has the property (u). We do not know whether the statement a) implies the statement b); this is true if \(X\) is commutative [29], see: Added in proof.

4) Any space which has an unconditional finite dimensional decomposition is a subspace of a space with an unconditional basis ([23] Theorem 1.g.5), hence by Corollary 6 and [10], any separable complex Banach space with the approxi-
mation property such that $K(E)$ is an $M$-ideal in $L(E)$ is a subspace of a space $X$ with a shrinking unconditional basis. If moreover $E$ is reflexive, the space $X$ can be taken reflexive as well.

5) On which separable spaces $E$ does there exist an equivalent norm such that $K(E)$ is an $M$-ideal in $L(E)$ when $L(E)$ is equipped with the operator norm? Observe that by Corollary 6 and [26] (resp. [23]) the spaces

$$E = l_p \hat{\otimes} l_p; \quad 1 < p < 2$$

(resp.)

$$F = (\sum \oplus L_1^{1 + 1/n})_2$$

which are reflexive spaces with basis, do not admit such a renorming. Note also that if a complex space $E$ is reflexive, separable and $K(E)$ is an $M$-ideal in $L(E)$, then Corollary 8 permits to show easily that $K(E)^* = E^* \hat{\otimes} E$ has the property $(X)$ [16] or equivalently $K(E)^* < l_1$ in Edgar’s ordering [6].

6) If $E = l_2$, let $N(E)$ be the space of nuclear operators on $E$. It is well known that $N(E) = K(E)^* = E \hat{\otimes} E$; let $H$ be the subspace of "upper triangular operators", that is the closed linear span of \{ $e_i \otimes e_j \mid j \geq i$ \} where $(e_n/n \geq 1$ the usual basis of $E$. It is easily seen that $H$ is weak* closed in $K(E)^*$, hence $N(E)/H \equiv (H^\tau)^*$ and since $H^\tau$ is a subspace of $K(E)$ which has the property (u), $H^\tau$ has (u) as well; therefore $N(E)/H$ has $(X)$ (see [6] and [16]), so it has $(V^*)$ and hence it is weakly sequentially complete; actually the space $N(E)/H$ shares most of the infinite dimensional geometrical properties of its "commutative relative" $L_1^1(T)/H^1(D)$.

7) If $E$ is an $M$-ideal in $E^{**}$ and thus if $K(E)$ is an $M$-ideal in $L(E)$ then $E$ is weakly compactly generated [7]. Hence the assumptions of separability we made can be deleted mutatis mutandis with standard but tedious technicalities.

8) If $A$ is a real Banach algebra, the state space $S$ does not separate $A$ in general; a classical example is $A = l_2^2 \hat{\otimes} l_2^2$. Hence for being able to apply our crucial Lemma 5, we have to limit ourselves to the complex situation. This restriction is probably unnecessary; however, it seems technically uneasy to complexify the Banach algebras we are using while respecting the $M$-ideal structure.

**ADDED IN PROOF.** After this paper was accepted, D. Li and the first-named author showed that any $M$-ideal in its bidual has property $u$ (Ann. Inst. Fourier 39(1989), 361–371). It follows in particular that our results on $K(E)$ are still valid if $E$ is a real Banach space.

REFERENCES