SIMPLE SINGULARITIES OF FUNCTIONS ON SUPERMANIFOLDS

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Introduction.

The study of singular points of a smooth function on a manifold is related as is shown in works by Arnold and his school [A] with objects from different branches of mathematics: groups generated by reflections, Lie algebras, automorphic forms, etc.

Inspired by remarkable achievements of singularity theory we decided to consider functions on supermanifolds from this point of view.

In this paper we classify simple (with respect to the group of diffeomorphisms of the pre-image) germs of functions on a supermanifold. An attempt to find the relations of our classification with other branches of mathematics failed as yet, cf. [A] with [S].

However we have decided to publish the results of the classification in the hope to draw attention to this subject which, we are sure, deserves it.

The paper consists of two parts. The first one introduces the main notions: stable equivalence, local algebra, etc., and certain general results are proved: analogues of the Morse Lemma, theorems on finite definiteness, etc. The second part contains the classification of simple germs itself.

The main differences of the supercase as compared with the conventional one are perhaps the following. The fact that a singularity is isolated is not equivalent any more to its finite multiplicity. Moreover, not all the singular points of an even function in which the corank of the second differential equals $p/q$, where $q > 0$, are isolated. Perhaps that is why we cannot find for our classification relations similar to the even case (since to define the corresponding geometric objects: intersection forms, monodromy groups, etc. isolatedness is crucial). The absence for supermanifolds of adequate analogues of the usual topological notions like the degree of the map is the reason for our algebraic definition of Milnor's number as the dimension of the local superalgebra of the singularity. A certain justifica-

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tion for doing so is that Milnor's number coincides (as in the even, manifold case) with the dimension of the base of a versal deformation.

A few words about our future plans. For the time being we consider only the action of the diffeomorphism group $\text{Diff}_0$ on the space of even germs. Elsewhere we will classify the functions with respect to the supgroup of diffeomorphisms whose underlying is $\text{Diff}_0$. This will enable us to consider not only even functions and 2) give a correct definition of the modality of a singularity (as a minimal pair $m/n$, such that a neighborhood of the orbit of a singular germ may be covered by a finite number of $m/n$-parameter families of orbits). One more direction which might lead to the phenomena reflecting hidden as yet supersymmetry of a singularity is the study of its bifurcational diagram (an algebraic subsupervariety in the base of the versal deformation corresponding to the singular germs).

1. Beginning of the classification of germs of functions on supermanifolds.

1.1. Definitions and notations. Let $\mathcal{M} = (M, \mathcal{O}_M)$ be a supermanifold. The functions on $M$ are arbitrary sections of the structural sheaf $\mathcal{O}_M$. We will study the normal forms to which a function on $M$ can be locally reduced. The majority of the classification work is performed for the case when $M$ is complex and the functions are analytic. Nevertheless, all the results of Chapter 1 which are simple generalizations of the corresponding facts from the purely even theory [A]) hold in either of the three categories: analytic (over $\mathbb{R}$ or $\mathbb{C}$), smooth ($C^\infty$) or formal. All the singularities of finite multiplicity are by Theorem 2.1.2 of a polynomial form in some coordinate system which effaces the differences between these three categories but the problem of description of real forms still remains open. We do not study this problem here.

We will only be interested in the local behaviour of functions and therefore we will assume that $M$ is a subsuperslot in the $(p, q)$-dimensional linear superspace $\mathbb{C}^{p/q}$ (resp. $\mathbb{R}^{p/q}$) with even coordinates $x_1, x_2, \ldots, x_p$ and odd ones $\xi_1, \ldots, \xi_q$, i.e. $M$ is an open subset of $\mathbb{C}^p$ (resp. $\mathbb{R}^p$) with coordinate system $(x_1, \ldots, x_p)$ and $\mathcal{O}_M = \mathcal{O}_M \otimes \wedge^* V$, where $\mathcal{O}_M$ is a ring of analytic (infinitely differentiable) functions on $M$ and $V = \langle \xi_1, \ldots, \xi_q \rangle$ the $q$-dimensional vector space over $\mathbb{C}$ (resp. $\mathbb{R}$).

We will denote by $\mathcal{O}_{\mathcal{M}, m}$ the ring of germs of functions at $m \in M$. It is a local supercommutative superalgebra with the maximal ideal $\mathfrak{M}_m$ consisting of the functions whose restriction to $M$ vanishes at $m$. A choice of a coordinate system in a neighborhood of $m$ determines an isomorphism of $\mathcal{O}_{\mathcal{M}, m}$ and the superalgebra $\mathcal{O}_{p/q}$ of germs of analytic (smooth) functions at the origin of $\mathbb{C}^{p/q}$ (resp. $\mathbb{R}^{p/q}$).

1.2. Equivalence of germs.

1.2.1. On the space $\mathcal{O}_{\mathcal{M}, m}$, the group $G = \text{Diff}_0$ of diffeomorphisms of the supermanifold $M$ preserving $m$ acts; we will identify $G$ with the group $G_{p, q}$ of
automorphisms of $\mathcal{C}_{p/q}$. Any such diffeomorphism (automorphism) may be considered as a change of variables

$$x_i \mapsto y_i = f_{0,i} + \sum f_{2,i}^{j,j_2} \xi_j + \sum f_{4,i}^{j_2,j_3,j_4} \xi_{j_1} \xi_{j_2} \xi_{j_3} \xi_{j_4} + \ldots$$

$$\xi_j \mapsto \eta_j = \sum f_{j,j_2}^{i,j_2} \xi_{j_1} + \sum f_{3,j}^{j_2,j_3} \xi_{j_1} \xi_{j_3} + \ldots,$$

where $i = 1, \ldots, p, j = 1, \ldots, q, f_{k,i}^{j_1,j_2,\ldots,j_k} \in \mathcal{C}_p$, and the matrices

$$\left( \frac{\partial f_{p,i}}{\partial x_k} \right), k = 1, \ldots, p \text{ and } (f_{l,i}^j), l = 1, \ldots, q$$

are invertible.

The germs belonging to one $G$-orbit are called equivalent. (In [A] this equivalence is called $R$-equivalence).

1.2.2. The problem on local normal forms of functions is to describe the orbits of the $G_{p,q}$-action in $\mathcal{C}_{p/q}$. The group and the space of orbits (however understood) are infinite dimensional. Therefore the complete solution of this problem is a rather hopeless task. But on the orbits there is a natural stratification and the first terms of this stratification are subject to complete investigation.

1.2.3. Remark. $G_{p/q}$ is the underlying group of the supergroup $\hat{G}_{p,q}$ of the diffeomorphisms of $M$ which also acts on $\mathcal{C}_{p/q}$. Therefore, from the supermanifold theory point of view it would be more correct to study the quotient space $\mathcal{C}_{p/q}/\hat{G}_{p,q}$; this will be done elsewhere. However, it is clear now why we confine ourselves to homogeneous germs: the $G_{p/q}$-equivalence for non-homogeneous elements of $\mathcal{C}_{p/q}$ is too weak and the corresponding classification is the description of pairs consisting of an even and an odd function.

1.3. Non-singular germs.

1.3.1. A point $m \in M$ is called a singular point of a function $f \in \mathcal{C}_{\mathfrak{m}}$ if all the partial derivatives $\partial f/\partial x_i$ and $\partial f/\partial \xi_j$ vanish at $m$. A germ $f \in \mathcal{C}_{\mathfrak{m},m}$ is called singular if $m$ is singular for a representative of the germ.

In other words, "$f$ is a singular germ" means that $f \equiv \text{const} \mod \mathfrak{m}^2$.

Almost all germs are non-singular and the first classification result refers to this most general case.

1.3.2. Proposition. Let a function $f$ be non-singular at $m \in M$. Then the germ of $\rho$ at $m$ is equivalent to $\text{const} + x_1$ if $f$ is an even function and $\xi_1$ if $f$ is odd.

Proof. The implicit function theorem, see [L].
2. Morse’s lemma and stable equivalence.

2.1. Morse’s lemma.

2.1.1. By Proposition 1.3.2 we may now only consider singular germs of even and odd functions. Any even germ is of the form $f = a + f_2$, where $a \in \mathbb{C}$, and $f_2 \in m^2$. The $G_{p/q}$-action does not affect the constant $a$, therefore to describe all the orbits in the space of germs of the functions it suffices to study the orbits in the space of germs of functions that vanish at the origin together with their differential. In other words, we will consider the $G_{p/q}$-action in the space $m^2_{p/q}$ then the 2-jet of the germ $f$ is a bilinear form on $T_m M$ called the second differential of $f$. The rank of the second differential of a singular germ $f$ (or just the rank of $f$) is the rank at $m$ of the matrix

$$
\begin{pmatrix}
\frac{\partial^2 f}{\partial x_i x_j} & \frac{\partial^2 f}{\partial x_i \partial \xi_k} \\
\frac{\partial^2 f}{\partial \xi_i \partial x_j} & \frac{\partial^2 f}{\partial \xi_i \partial \xi_k}
\end{pmatrix}
$$

The corank, cork $f = p/q - \text{rk } f$ is an important invariant of a singular germ.

Notice that for the non-singular germs the quadratic part of the Taylor series is not invariantly defined.

2.1.2. For germs homogeneous with respect to parity the following analogue of Morse’s lemma holds.

Let $f$ be a homogeneous germ of $m^2$ whose second differential is non-degenerate. Then $f$ is equivalent to its quadratic part.

We will prove this statement in a trifle more general form called the parametric Morse’s lemma (2.1.4).

2.1.3. Let $f_1(x, \xi)$ be a function on a supermanifold $M_1$ and $f_2(y, \eta)$ a function on a supermanifold $M_2$. The function $f_1(x, \xi) + f_2(y, \eta)$ on $M = M_1 \times M_2$ is called the direct sum of $f_1$ and $f_2$.

2.1.4. Theorem. Let $f \in m^2_{p/q}$ be homogeneous and $\text{rk} f = k/l$. Then $f$ is equivalent to the direct sum of a non-degenerate quadratic form $b(x, \xi) \in m^2_{k/l}$ and a germ $g(y, \eta) \in m^3_{p-k/q-1}$ whose 2-jet is zero.

Proof. First consider the case when $f$ is even. Let $\mathfrak{n}$ be the ideal of $O_{p/q}$ generated by the odd variables. Reducing $f$ modulo $\mathfrak{n}$ we get a function $f$ in $p$ even variables. Applying the conventional parametric Morse’s lemma [A] reduce $f$ to the form

$$
f = x_1^2 + \ldots + x_k^2 + \tilde{g}(y_1, \ldots, y_{p-k}), \quad \tilde{g} \in m^3_{p-k}.
$$
Complement the even coordinate functions $x_i$, $y_j$ with odd ones $\xi_1, \ldots, \xi_i, 
eta_1, \ldots, \neta_q-i$ so that the second differential in the resulting coordinate system is of the form $x_1^2 + \ldots + x_k^2 + \xi_1 \xi_2 + \xi_3 \xi_4 + \ldots$.

Then $f$ takes in these coordinates the form

$$f = x_1^2 + \ldots + x_k^2 + x_1 f_1 + x_2 f_2 + \ldots + x_k f_k + \xi_1 \xi_2 + \xi_3 \xi_4 + \ldots + \xi_1 \phi_2 +$$

$$+ \phi_1 \xi_2 + \ldots + g$$

where $f_i \in \mathbb{n}^2$, $\phi_j \in \mathbb{n}$. Performing the change of variables

$$x_i \mapsto x_i - \frac{1}{2} f_i, \quad \xi_j \mapsto \xi_j - \phi_i,$$

we reduce $f$ to the form

$$f = x_1^2 + x_2^2 + \ldots + x_1 \tilde{f}_1 + x_2 \tilde{f}_2 + \ldots + \xi_1 \xi_2 + \xi_3 \xi_4 + \ldots + \xi_1 \tilde{\phi}_2 + \tilde{\phi}_1 \xi_2 +$$

$$+ \xi_3 \tilde{\phi}_4 + \tilde{\phi}_3 \xi_4 + \ldots + \tilde{g},$$

where now $\tilde{f}_i \in \mathbb{n}^4$, $\tilde{\phi}_j \in \mathbb{n}^3$. Repeating this procedure several times we get a representation in which $\tilde{f}_i$, $\tilde{\phi}_j \in \mathbb{n}^{q+1}$. But $\mathbb{n}^{q+1} = 0$ therefore this is the desired representation of $f$.

For an odd germ the proof is still simpler. In fact, expressing $f$ in a coordinate system in which its second differential is of the form $x_1 \xi_1 + x_2 \xi_2 + \ldots + x_p \xi_p$, we get

$$f = x_1 \xi_1 + \ldots + x_p \xi_p + \xi_1 f_1 + \ldots + \xi_p f_p + g(x, \eta),$$

where $f_i \in \mathbb{n}^2$, $g \in \mathbb{n}^3$. The change $x_i \mapsto x_i - f_i$ reduces to a representation where $f_i \in \mathbb{n}^4$. At the following step we get $f_i \in \mathbb{n}^8$, and in several steps we get $f_i \in \mathbb{n}^{q+1} = 0$ as required.

2.2. Stable equivalence.

2.2.1. The parametric Morse’s lemma reduces the classification of singular germs to the description of germs with zero second differential, i.e. to the study of $G_{p,q}$-orbits in $\mathbb{m}^{3}_{p/q}$. Besides, it shows that it is more reasonable to classify the germs not by the number of even or odd variables on which it depends but by the corank of the second differential. More exactly these considerations are formulated in the following definitions.

2.2.2. The germs $f_1 \in \mathcal{O}_{p/q}$ and $f_2 \in \mathcal{O}_{m/n}$ are called stably equivalent if there exist non-degenerate quadratic forms $b_1$ and $b_2$ in $k_1/l_1$ and $k_2/l_2$ variables respectively such that $p + k_1 = m + k_2$ and $q + l_1 = h + l_2$ and the germs $f_1 \oplus b_1$ and $f_2 \oplus b_2$ are equivalent.

This definition immediately implies that the corank is an invariant of stable equivalence. Moreover for homogeneous functions depending on the same number of even and odd variables stable equivalence is equivalent to the usual equivalence. Therefore, the following definition is correct.
2.2.3. A singularity (of parity $\varepsilon \in \mathbb{Z}_2$) is a class of stable equivalence of germs. Before we prove the above statement let us give one more definition and a useful lemma.

2.2.4. The gradient ideal $I_f$ of a germ $f \in \mathcal{O}_{p/q}$ is the ideal generated by the partial derivatives $\partial f/\partial x_i, \partial f/\partial \xi_j$.

2.2.4.1. Remark. The gradient ideal is an invariant of a germ which is the tangent superspace to the orbit of $f$ under the action of the supergroup of all diffeomorphisms preserving $m \in \mathcal{M}$.

2.2.4.2. Example. The gradient ideal of a non-singular germ coincides with the whole of $\mathcal{O}_{p/q}$. Conversely, if $I_f = \mathcal{O}_{p/q}$ then $f$ is a non-singular germ.

2.2.4.3. Example. If the quadratic differential of a singular homogeneous germ $f$ is non-degenerate then $I_f = m$. The converse is also true.

2.2.5. Proposition. Let $f$ and $\varphi$ be homogeneous germs of the same parity such that $\varphi \in m I_f$ and $I_\varphi \subset m I_f$. Then $f$ and $f + \varphi$ are equivalent.

Proof. Is based on the homotopy method whose essence is the following.

Instead of one diffeomorphism $F$ sending $f + \varphi$ into $f$ we will seek a family of diffeomorphisms $F_t$ such that

$$F_t^*(f + t\varphi) = f, \quad F_0 = \text{id}, \quad F_t(0) = 0.$$ 

Differentiating the first identity with respect to $t$ we get

$$(*) \quad V(F_t(x, \xi), t)(f + t\varphi) + \varphi(F(x, \xi)) = 0,$$

where $V(F_t(x, \xi), t) := \frac{d}{dt} F_t(x, \xi)$ is the vector field corresponding to the family of diffeomorphisms $F_t$. The map $(x, \xi, t) \mapsto (F(x, \xi), t)$ is a diffeomorphism and therefore solving $(*)$ reduces to seeking a vector field $V(x, \xi, t)$ on $M \times \mathbb{R}$ such that $V \cdot (f + t\varphi) = -\varphi$. If, besides, $V(0, 0, t) = 0$ then a solution of the initial value problem

$$\frac{d}{dt} F_t(x, \xi) = V(F_t(x, \xi), t), \quad F_0(x, \xi) = (x, \xi)$$

exists for sufficiently small $x$ for $t \in [0, 1]$. The existence is guaranteed by Shander's theorem [Sh] and the extendability to the segment follows from the fact that $F(0, 0, t) = 0$ is a solution for $(x, \xi) = 0$.

Let $\mathcal{O}$ be the superalgebra of germs of functions in $(x\xi)$ and $m$ its maximal ideal. Let $\hat{\mathcal{O}} = \mathcal{O} \otimes \mathcal{O}_{[0, 1]}$ be the superalgebra of families of germs that depend smoothly (analytically) on a parameter $t \in [0, 1]$. We seek germs $a_t \in [m \cdot \hat{\mathcal{O}}]_0$ and
\[ \alpha_j \in [m \cdot \widetilde{O}]_{\widetilde{I}} \] such that
\[ \sum a_i \frac{\partial}{\partial x_i} (f + t\varphi) + \sum \alpha_j \frac{\partial}{\partial \xi_j} (f + t\varphi) = -\varphi \]

For \( t = 0 \) such germs can be found since \( \varphi \in m I_f \). Let us show that \( I_{\varphi} \subset m I_f \) implies \( I_f \subset \widetilde{I}_f + t\varphi \), where \( \widetilde{I}_f + t\varphi \) is the ideal of \( \widetilde{O} \) generated by the \( \frac{\partial}{\partial x_i} (f + t\varphi), \frac{\partial}{\partial \xi_j} (f + t\varphi) \). In this case \( \varphi \in m \widetilde{I}_f + t\varphi \) and the coefficients \( a_i, \alpha_j \) in the decomposition
\[ \varphi = \sum a_i \frac{\partial}{\partial x_i} (f + t\varphi) + \sum \alpha_j \frac{\partial}{\partial \xi_j} (f + t\varphi) \]
give all that is needed.

The inclusion \( I_f \subset \widetilde{I}_f + t\varphi \) is proved by the following trick (a variant of Nakayama’s lemma). Making use of the fact that \( (\frac{\partial}{\partial x_i}) \varphi \) and \( (\frac{\partial}{\partial \xi_j}) \varphi \) belong to \( m I_f \) we find \( b_{ki}, c_{ij}, \in [m \widetilde{O}]_{\widetilde{I}} \) and \( \beta_{kj}, \gamma_{li} \in [m \widetilde{O}]_{\widetilde{I}} \) such that
\[ \frac{\partial}{\partial x_k} f = \frac{\partial}{\partial x_k} (f + t\varphi) - t \frac{\partial}{\partial x_k} \varphi = \frac{\partial}{\partial x_k} (f + t\varphi) - t \sum b_{ki} \frac{\partial}{\partial x_i} f - t \sum \beta_{kj} \frac{\partial}{\partial \xi_j} f, \]
\[ \frac{\partial}{\partial \xi_l} f = \frac{\partial}{\partial \xi_l} (f + t\varphi) - t \frac{\partial}{\partial \xi_l} \varphi = \frac{\partial}{\partial \xi_l} (f + t\varphi) - t \sum \gamma_{li} \frac{\partial}{\partial x_i} f - t \sum c_{ij} \frac{\partial}{\partial \xi_j} f. \]

Transporting to the left-hand side the terms with partial derivatives of \( f \) we get a system of linear equations
\[ (1 + T) \text{grad} f = \text{grad} (f + t\varphi), \]
where \( T = \begin{pmatrix} (b_{ki}) & (\beta_{kj}) \\ (\gamma_{li}) & (c_{ij}) \end{pmatrix} \). All the matrix elements of \( T \) belong to \( m \widetilde{O} \) therefore \( 1 + T \) is invertible and
\[ \text{grad} f = (1 + T)^{-1} \text{grad} (f + t\varphi). \]

Therefore \( I_f \subset \widetilde{I}_f + t\varphi \) and we are done.

2.2.5.1. Corollary. If \( f \) and \( \varphi \) are homogeneous germs of the same parity, \( f \in m^3 \) and \( \varphi \in t^2 \) then \( f \mod \varphi \) is equivalent to \( f \).

Proof. The conditions of the Proposition hold since \( I_f \subset m^2 \) and partial derivation sends \( I_f \) into \( m \) and \( I_f^2 \) into \( m I_f \).

2.2.5.2. Corollary: Morse’s lemma.

In fact, let \( f \) be a non-degenerate quadratic form (either even or odd) then \( I_f = m \) and \( \varphi \in m^3 \) immediately implies \( I_{\varphi} \subset m I_f \).
One more corollary of Proposition 2.2.3 is the theorem on finite determinacy 3.2.1.

2.2.6. **Theorem.** Let homogeneous functions \( f_1, f_2 \in \mathcal{C}_{p/q} \) be stably equivalent. Then they are equivalent.

**Proof.** Apply to the functions \( f_1 \) and \( f_2 \) the parametric Morse's lemma and reduce them to the form \( f_1^0 \oplus b_1 \) and \( f_2^0 \oplus b_2 \), where \( b_1 \) and \( b_2 \) are non-degenerate quadratic forms and \( f_1^0 \) and \( f_2^0 \) are germs with zero 2-jet. The stable equivalence of \( f_1 \) and \( f_2 \) means that there exist non-degenerate quadratic forms \( b'_1 \) and \( b'_2 \) of the same rank since \( f_1 \) and \( f_2 \) depend on the same number of variables) for which the germs \( f_1^0 \oplus b_1 \oplus b'_1 \) and \( f_2^0 \oplus b_2 \oplus b'_2 \) are equivalent. Then the forms \( b_1 \oplus b'_1 \) and \( b_2 \oplus b'_2 \) are isomorphic and therefore it suffices to prove the statement for stably equivalent functions with zero 2-jet.

Thus, let now \( f_1 \) and \( f_2 \) be functions of variables \( (x_1, \ldots, x_q, \xi_1, \ldots, \xi_q) \) belonging to \( m^3 \) and \( b_1 \) and \( b_2 \) non-degenerate quadratic forms in variables \( (y_1, \ldots, y_2, \eta_1, \ldots, \eta_2) \) such that the germ of \( f_1 + b_1 \) is equivalent to the germ of \( f_2 + b_2 \). Performing a linear change of variables \( (y, \eta) \), we may assume that the forms \( b_1 \) and \( b_2 \) are reduced to the canonical forms

\[
\begin{align*}
 b(y, \eta) &= y_1^2 + \ldots + y_k^2 + \eta_1 \eta_2 + \ldots + \eta_{2l-1} \eta_{2l} \\
 b(y, \eta) &= y_1 \eta_1 + \ldots + y_k \eta_k
\end{align*}
\]

for even functions and

\[
\begin{align*}
 b(y, \eta) &= y_1 \eta_1 + \ldots + y_k \eta_k
\end{align*}
\]

for odd ones.

Consider a diffeomorphism \( F(x, \xi, y, \eta)_+ \in G \), sending \( f_1 \oplus b \) into \( f_2 \oplus b \). The linear part of \( F \) preserves \( b \) therefore the Jacobi matrix \( J \) of \( F \) is of the form

\[
\begin{pmatrix}
 (x, \xi) \\
 (x, \xi) \\
 (y, \eta)
\end{pmatrix}
\begin{pmatrix}
 A & B \\
 O & C
\end{pmatrix}
\]

Therefore, the invertibility of \( J \) implies that of the matrix \( A \) which is the Jacobi matrix for the map \( F' : C^{p/q} \to C^{p/q} \) which is the composition of an embedding \( C^{p/q} \to C^{p+r/q+s} \) and the projection \( C^{p+r/q+s} \to C^{p/q} \). By the implicit function theorem \( F' \) is a diffeomorphism and therefore replacing \( f_1(x, \xi) \) by an equivalent germ \( f'_1 = f_1(F(x, \xi)) \) we get a pair of equivalent germs \( f'_1 + b_1 \) and \( f_2 + b_2 \) and the change of variables sending \( f_2 + b_2 \) into \( f'_1 + b_1 \) is of the form

\[
\begin{align*}
 x_i &\mapsto x_i + \sum y_k a_k(x, \xi) + \sum \eta_l \Psi_i(x, \xi) + \ldots \\
 \xi_j &\mapsto \xi_j + \sum y_k \beta_{kj}(x, \xi) + \sum \eta_l d_{kl}(x, \xi) + \ldots
\end{align*}
\]
where dots stands for the terms that belong to the square of $I' = \langle y, \eta \rangle$. (To make the linear part of this change of variables the identity in $y, \eta$ this diffeomorphism should be multiplied by a linear transformation preserving $b$).

Now let us show that $h = f'_1 - f_2$ belongs to the square of the gradient ideal $I_{f_2} \subset O_{p,q}$. By Corollary 1.5.4.1 this would mean that $f_2$ and $f'_1$ are equivalent. In fact, for even functions we get modulo $I'^2$:

$$f'_1 + y_1^2 + \ldots + b_1 = f_2 + y_2^2 + \ldots + h =$$

$$= f_2(x + ya + \eta d) + \sum (y_k + c_k(x, \xi) + \ldots)^2 + \sum (\eta_l + \gamma_l(x, \xi) + \ldots) \cdot$$

$$(\eta_{l+1} + \gamma_l(x, \xi) + \ldots) =$$

$$= f_2 + y_i a_i \frac{\partial f_2}{\partial y_i} + \sum \eta_l d_l \frac{\partial f_2}{\partial \eta_l} + \ldots + \sum y_k^2 + 2 y_k c_k + c^2 k +$$

$$+ \sum (\eta_l \eta_{l+1} + \eta_l \gamma_{l+1} + \eta_{l+1} \gamma_l + \ldots + \gamma_l \gamma_{l+1} + \ldots).$$

Equating the coefficients of $y_k, \eta_l$ and the terms we get

$$c_k = - \frac{1}{2} a_k \frac{\partial f_2}{\partial y_k}, \quad \gamma_{2l+1} = d_{2l} \frac{\partial f_2}{\partial \eta_{2l+1}},$$

$$\gamma_{2l} = - x_{2l} \frac{\partial f_2}{\partial \eta_{2l}}, \quad h = \sum c_k^2 + \sum \gamma_l \gamma_k,$$

yielding $h \in I^2 f_2$ and by Proposition 2.2.2, the functions $f'_1$ and $f_2$ are equivalent. But $f_1$ and $f'_1$ are also equivalent therefore so are $f_1$ and $f_2$. For even functions the end of the proof is similar.

3. Local algebra and deformations.

3.1. Germs of finite multiplicity. One of the main algebraic invariants of a singularity is its local algebra.

3.1.1. The local superalgebra of a singular germ $f$ is $O_f := O/I_f$. If $O_f$ is of finite dimensions then the singular point is said to be of finite multiplicity and $\mu = \mu_0/\mu_1 = \dim O_f$ is called its multiplicity (or Milnor's number).

3.1.2. Remark. Clearly, if of two germs $f_1$ and $f_2$ at least one is of finite multiplicity then $O_{f_1} \oplus O_{f_2} = O_{f_1} \otimes O_{f_2}$. If $f_2$ is a non-singular quadratic form then $O_{f_1} = \mathbb{C}$ (cf. 2.2.4.3) therefore the local algebra does not depend on the choice of representatives of the class of stable equivalence and is an invariant of a singularity.
3.1.3. REMARK. In a purely even situation (on manifolds) the multiplicity of a singular point equals the number of non-degenerate singular points into which it splits under a small perturbation. In the supercase the geometric meaning of the multiplicity is unclear as yet. The finite multiplicity of a germ on a manifold is equivalent to the isolatedness of a singular point. In general, on a supermanifold non-isolated singular points can be of finite multiplicity, e.g. the germ of \( x^3 + x^2 \xi_1 \xi_2 \) is singular on the hypersurface \( x = 0 \) though the dimension of its local ring is \( 3/2 \) and with a basis 1, \( x, \xi_1, \xi_2 \) (even) and \( \xi_1, \xi_2 \) (odd). A necessary and sufficient condition for finite multiplicity of an even singularity is isolatedness of its restriction onto the underlying manifolds.

3.1.4. PROPOSITION. Let \( f \in \mathcal{O}_{p/q, \bar{0}} \). Denote by \( \bar{f} \in \mathcal{O}_p \) the restriction of \( f \) onto the underlying manifold singled out by the equations \( \xi_1 = \ldots = \xi_q = 0 \). Then the origin is a singular point of \( \bar{f} \) of finite multiplicity if and only if \( \bar{f} \) has an isolated singularity at the origin.

PROOF. Under the reduction \( \mathcal{O}_{p/q} \rightarrow \mathcal{O}_{p/q}/\langle \xi_1, \ldots, \xi_q \rangle = \mathcal{O}_p \) the gradient ideal \( I_\bar{f} \) and therefore an epimorphism \( \mathcal{O}_f \rightarrow \mathcal{O}_{\bar{f}} \) is defined and finite dimensionally of \( \mathcal{O}_f \) implies that of \( \mathcal{O}_{\bar{f}} \) and this implies due to \( [A] \) that the singularity of \( \bar{f} \) is isolated.

Conversely, if the origin is an isolated singularity for \( \bar{f} \) then \( \mu = \dim \mathcal{O}_{\bar{f}} < \infty \). As we will see the finite dimensionality of the local algebra implies the nilpotency of its maximal ideal. Let the images of \( x_i \) in \( \mathcal{O}_{\bar{f}} \) satisfy \( \bar{x}_i^N = 0 \), then \( x_i^N \in \ker(\mathcal{O}_f \rightarrow \mathcal{O}_{\bar{f}}) \). But this kernel is generated by the images of odd variables \( \xi_j \) for \( \mathcal{O}_{p/q} \) and therefore \( \ker^{a+1} = 0 \) implying \( x_i^{N(q+1)} = 0 \) in \( \mathcal{O}_f \). Thus any elements from \( \mathcal{O}_f \) can be presented as a linear combination of monomials in which the powers of variables \( x_i \) do not exceed \( N(q + 1) \). Therefore \( \dim \mathcal{O}_f < \infty \).

It remains to verify the following fact.

3.1.5. LEMMA. Let \( A \) be a finite dimensional local algebra of dimension \( \mu \) and \( m \) its maximal ideal. Then \( m^\mu = 0 \).

PROOF. Let \( a_1, \ldots, a_\mu \in m \) be arbitrary. The elements 1, \( a_1, a_1a_2, \ldots, a_1 \ldots a_\mu \) are linearly dependent, say, \( \lambda_0 + \lambda_1a_1 + \ldots + \lambda_\mu a_1 \ldots a_\mu = 0 \). Select the minimal \( i \) such that \( \lambda_i \neq 0 \) then \( a_1 \ldots a_i(\lambda_i + \lambda_{i+1}a_{i+1} + \ldots + \lambda_\mu a_1 \ldots a_\mu) = 0 \). The element in parenthesis does not belong to \( m \) and therefore is invertible implying \( a_1 \ldots a_i = 0 \) and \( a_1 \ldots a_\mu = 0 \).

3.2. Sufficient jets. The germs of finite multiplicity possess the following property of finite determinacy.

3.2.1. THEOREM. A germ with the singular point of finite multiplicity is equivalent to a polynomial in \( x, \xi \) (notably, to a segment of its Taylor series).
PROOF. Let the multiplicity of the singular point of \( f \) equal \( \mu = \mu_0/\mu_1 \). Then by Lemma 3.1.5 \( I_f \supseteq m^{\mu_0 + \mu_1} \), and denoting by \( \tilde{f} \) the jet of \( f \) of order \( \mu_0 + \mu_1 + 1 \) we see that \( \varphi = \tilde{f} - f \) belongs to \( m^{\mu_0 + \mu_1 + 2} \subseteq m^2 I_f \). Further, the partial derivatives of \( \varphi \) belong to \( m^{\mu_0 + \mu_1 + 1} \subseteq m I_f \). Therefore, we may apply Proposition 2.2.5 and deduce that \( f \) is equivalent to \( \tilde{f} \).

3.2.2. We say that the \( k \)-jet of \( f_0 \in \mathcal{C}/m^{k+1} \) is sufficient if any function \( f \) with \( f^k(f) = f_0 \) is equivalent to \( f_0 \).

In 2.2 we have proved that if the multiplicity of a germ is \( \mu_0/\mu_1 \) then its jet of order \( \mu_0 + \mu_1 + 1 \) is sufficient.

3.2.3. We say that a function \( f_1 \) is adjacent to a function \( f_2 \) if the orbit of \( f_1 \) belongs to the closure of the orbit of \( f_2 \).

In other words \( f_1 \) is adjacent to \( f_2 \) if in any neighbourhood \( f_1 \) a function equivalent to \( f_2 \) is contained.

The adjacency is a natural order relation on the equivalence classes of singularities. The local algebra enables us to trace this relation algebraically.

3.2.4. PROPOSITION. The multiplicity is semicontinuous from below on the space of germs of finite multiplicity (i.e. under a small jiggling of a germ its multiplicity \( \mu = \mu_0/\mu_1 \) may only diminish).

PROOF. Let \( f \) be a germ with a singular point of multiplicity \( \mu = \mu_0/\mu_1 \). Then its \( k \)-jet is sufficient for \( k \geq \mu_0 + \mu_1 + 1 \) and under a small jiggling of \( f \) the image of the gradient ideal \( I_{f_1} \), where \( f_1 \) is the perturbed germ, in the finite dimensional space of \( k \)-jets \( \mathcal{C}/m^{k+1} \) is the vector subsuperspace spanned by the products of partial derivatives of \( f_1 \). Now the proof follows from the obvious lemma:

3.2.5. LEMMA. Let \( v_1(t), \ldots, v_p(t), w_1(t), \ldots, w_q(t) \) be a family of \( p \) even and \( q \) odd vectors of a linear superspace \( U \) continuously depending on a parameter \( t \). Then the function

\[
d(t) = (d_0(t) \mid d_1(t)) = \dim \langle v_1(t), \ldots, v_p(t), w_1(t), \ldots, w_q(t) \rangle
\]

is semicontinuous from above (\( d(t) \) cannot diminish under a small jiggling).

3.3. Deformations.

3.3.1. A deformation of a germ \( f \in \mathcal{C}^{\mu_0/\mu_1} \) with base \( \mathbb{C}^{\tau_0} \) is a germ morphism of spaces with fixed points \( F: (\mathbb{C}^{\mu_0/\mu_1} \times \mathbb{C}^{\tau_0}, 0) \to (\mathbb{C}, 0) \) whose restriction onto \( \mathbb{C}^{\mu_0/\mu_1} \) coincides with \( f \) (where \( f \) is considered as a map \( (\mathbb{C}^{\mu_0/\mu_1}, 0) \to (\mathbb{C}, 0) \)). Two deformations \( F \) and \( \tilde{F} \) with base \( \mathbb{C}^{\tau_0} \) are called equivalent if \( F \) reduces to \( \tilde{F} \) by an isomorphism of germs \( G: \mathbb{C}^{\mu_0/\mu_1} \times \mathbb{C}^{\tau_0} \to \mathbb{C}^{\mu_0/\mu_1} \times \mathbb{C}^{\tau_0} \) of the form

\[
(3.3.1) \quad G(x, \xi; t, \tau) = (g(x, \xi, t, \tau); t, \tau),
\]

where \( g: \mathbb{C}^{\mu_0/\mu_1 + s} \to \mathbb{C}^{\mu_0/\mu_1} \) is a map such that \( g(x, \xi, 0, 0) = (x, \xi) \).
A deformation $F$ of $f$ is called \textit{versal} if any deformation $\tilde{F}$ of this germ with base $C^{r/s}$ is equivalent to a deformation obtained from $F$ by a reparametrization, i.e. if there exists a map of bases $h: (C^{r/s}, 0) \to (C^{r/s}, 0)$ such that $\tilde{F}'(x, \xi, t, \tau) = F(x, \xi; h(t, \tau))$.

(In the deformation theory of complex structures, see e.g. references in [W], such deformations are usually called \textit{full} and what is called a versal deformation should in addition possess the unique differential of the inducing morphism. In singularity theory the deformations with this additional property are called \textit{miniversal}).

As in the manifold case the finite multiplicity of $f$ is equivalent to the existence of a versal deformation.

3.3.2. \textbf{Versality theorem.} \textit{Let the multiplicity of a germ $f: C^{p/q} \to C$ be finite and $e_{\mu_0}, \ldots, e_{\mu_1}$ be odd elements from $\mathcal{O}_{p/q}$ generating $\mathcal{O}_f$. Then the function}

$$F(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q, a_1, \ldots, a_{\mu_0}, \alpha_1, \ldots, \alpha_{\mu_1}) = f + \sum a_i e_i + \sum \alpha_j e_j \in \mathcal{O}_p + \mu_0/q + \mu_1$$

\textit{is a versal deformation of $f \in \mathcal{O}_{p/q}$. The dimension of the base of any other versal deformation of $f$ is no less than $\mu_0/\mu_1$. Conversely, if $f$ has a versal deformation then the multiplicity of $f$ is finite.}

\textbf{Proof.} A deformation of an analytic function $f$ is a particular case of the notion of a germ of a relative complex superspace $f: X \to B$. The latter is the triple consisting of a germ of a complex superspace $S$ (the base of deformation), a germ of a relative superspace $F: X \to B \times S$ flat over $S$ and a morphism $i: X \to X$ over $B$ identifying $X$ with $F^{-1}(B \times \{S_0\})$ (for definitions see [W]). The general deformation theory of complex structures on superspace (see [W]) implies that the set of equivalence classes of deformations over the germ of $D = (\ast, \mathcal{O}_D)$, where $\mathcal{O}_D = C\{x, \xi\}/(x^2, x\xi)$ (such deformation is called \textit{infinitesimal}) is endowed with a natural vector superspace structure (intuitively it may be considered as the tangent superspace at $f$ to any transversal to the orbit of $f$ in the space of all germs over $C$). If the dimension $a/b$ of this superspace in finite then the germ $f: X \to B$ has a versal deformation with an $(a, b)$-dimensional base $S$ which is in general singular). The base of a versal deformation $S$ is smooth if all the obstructions to the extension of infinitesimal deformations to deformations with base $C^{1/0}$ (or $C^{0/1}$) vanish. In our case when $X = C^{p/q}$ and $B = C$ any deformation of a germ of $C^{p/q}$ is trivial ([W]) and we get the maps: $F: C^{p/q} \times S \to C$, i.e. a deformation of the function $f$. Let us calculate the space of infinitesimal deformations of $f$. Let $F(x, \xi; t, \tau)$ be a deformation depending on 1 even and 1 odd parameter. Its infinitesimality means that we are only interested in the linear terms of the decomposition in the power series in parameters:

$$F(x, \xi; t, \tau) = f(x, \xi) + ta(x, \xi) + \tau\alpha(x, \xi) + \ldots.$$
Let the change of variables (3.3.1) be given by the map
\[ g: \mathbb{C}^{p+1/q+1} \to \mathbb{C}^{p/q}: g = 1_{p/q} + t b + \tau \beta + \ldots, \]
where \( b = (b_1, \ldots, b_{p+q}), \beta = (\beta_1, \ldots, \beta_{p+q}); b_1, \ldots, b_p, \beta_{p+1}, \ldots, \beta_{p+q} \in \mathcal{O}_{p/q, \bar{0}}, \) \( b_{p+1}, \ldots, b_{p+q}, \beta_1, \ldots, \beta_p \in \mathcal{O}_{p/q, \bar{1}}, \) and dots denote the term of higher degrees in \( t \) and \( \tau. \) Then
\[ F(g(x, \xi; t, \tau); t, \tau) = f(g(x, \xi; t, \tau)) + t a(g(x, \xi; t, \tau) + \tau \alpha(g(x, \xi; t, \tau)) + \ldots = \]
\[ = f(x, \xi) + t \left( \sum_{i=1}^{p} b_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{q} b_{p+i} \frac{\partial f}{\partial \xi_i} \right) + \tau \left( \sum_{i=1}^{p} \beta_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{q} \beta_{p+i} \frac{\partial f}{\partial \xi_i} \right) + \ldots + t a(x, \xi) + \tau \alpha(x, \xi) + \ldots \]
Therefore the infinitesimal deformations given by the pairs \((a', \alpha')\) and \((a, \alpha),\)
where \( a, a' \in \mathcal{O}_{p/q, \bar{0}}, \alpha, \alpha' \in \mathcal{O}_{p/q, \bar{1}} \) are equivalent if \( a' - a \in \mathcal{I}_f, \bar{0} \) and \( \alpha' - \alpha \in \mathcal{I}_f, \bar{1}. \)
Identifying the set of pairs \((a, \alpha)\) with \( \mathcal{O}_{p/q} \) we finally see that the space of infinitesimal deformations is isomorphic to \( \mathcal{O}_{p/q}/\mathcal{I}_f = \mathcal{C}_f. \) Therefore if \( \mu = \dim \mathcal{C}_f < \infty \) the germ of the relative complex superspace \( f: \mathbb{C}^{p/q} \to \mathbb{C} \) has a \( \mu \)-dimensional base. For such a germ all the obstruction vanish (cf. [W]) therefore the base is the germ of the supermanifold \( \mathbb{C}^\mu. \)

3.3.3.4. Remark. The above arguments hold if the analytic functions are replaced by smooth ones since by Theorem 2.1.2 a germ of finite multiplicity is equivalent to a polynomial.

4. Simple singularities

4.1. Modality. As is shown by V. Arnold the most natural results are obtained when singularities are classified with respect to the number of parameter (moduli) on which nearby orbits depend. Let us give an exact definition.

4.1.1. Let a Lie group \( G \) act on a manifold \( M. \) The modality of a point \( f \in M \) is the minimal \( m \) such that a neighbourhood of \( f \) may be covered by a finite number of \( m \)-parameter families of \( G \)-orbits.

The points of modality 0, i.e. such that in their neighbourhoods there is only a finite number of orbits, are called simple.

The manifold of germs of even functions on \( (\mathbb{C}^{p/q}, 0) \) with a singular point with zero critical value at the origin and the group \( G_{p/q} \) which acts on this manifold are both infinite dimensional. Nevertheless, all the singularities of finite multiplicity are of finite modality. Indeed, by 3.2.1 any such germ has a sufficient \( k \)-jet therefore considering the action of \( k \)-jets of diffeomorphisms on the space \([\mathcal{O}/m^{k+1}]_0\) for \( k \geq \mu_0 + \mu_1 + 1 \) we reduce all the considerations to a finite dimensional situation.
(The germs close to these considered also have sufficient $k$-jets since the multiplicity does not grow under small jiggling thanks to 3.2.4. Therefore, the modality of a germ as calculated above does not depend on $k$).

Before we start describing simple germs let us make several remarks.

4.1.2. The above definition of modality is a mere surrogate of a correct definition (that will be given elsewhere) according to which and the general principles of supergeometry we should consider not only the manifold of even germs and jets but the supermanifolds of all germs on which the supergroup of automorphisms acts. Modality considered this way is not one number but a pair of numbers since the families of orbits may depend both on even and odd parameters. In the subsequent paper we will list the $(1,0)$-modal singularities and hope to describe also $(0,1)$-modal ones.

4.1.3. As is often the case when superizing different mathematical notions that of modality splits into two. The point is that the statement of Gabrielov's theorem to the effect that the modality of a (purely even) singularity equals the dimension of the stratum $\mu = \text{const}$ in the base of the miniversal (even if we disregard Remark 3.1.3) deformation is false in the supercase. Thus, the dimension of the stratum $\mu = \text{const}$ is also the legitimate pretender for the role of the superanalogue of modality. We will call this dimension the inner modality and denote it by $\mu_0$. It is not difficult to show that $\mu_0 \leq \mu$ so that in the list of singularities with inner modality 0 all the simple singularities will be found but not only they. In particular, this list contains the germs $\xi_1, \xi_2, \ldots, \xi_q$, etc. Perhaps the classification of germs in terms of their inner modality leads to more final (algebraic) results.

4.2. Lists of singularities. In what follows (Tables 1–3) we list the normal forms of all the simple and bordering singularities (i.e. families of singularities that depend on at least one modulus, but all the other singularities in a neighbourhood fall into finitely many equivalence classes) and also the normal forms of germs of cork 1/4. Proofs of Theorems 4.2.1–4.2.5 follow from the Classifier of Singularities 4.3 and Lemmas 4.4.

Remark. Our notations for the types of singularities unlike Arnold's ones have no deep meaning. In particular, the notations of Tables 2, 3 are compatible with those of Table 1, as far as corank is concerned. This explains the appearance of the types VIII in Table 2 between the types VI and VII of any places: the singularities of cork 1/8–1/9 do not occur in Table 1.

4.2.1. Theorem. Any singularity is either equivalent to one of the singularities from Table 1 or is adjacent to one of the singularities from Table 2.

4.2.2. Theorem. The modality of any singularity from Table 2 is no less than 1, i.e. they are bordering ones.
4.2.3. Theorem. The singularities from Table 1 are not adjacent to any of the singularities from Table 2.

4.2.3.1. Corollary. All the singularities from Table 1 are simple and any simple singularity is equivalent to one of them.

4.2.4. Theorem. Any singularity of cork 1/4 is equivalent to one of the singularities from Table 3.

4.2.5. Theorem. All the singularities from Tables 1 and 2 are pair-wise nonequivalent.

Table 1.

Simple singularities

<table>
<thead>
<tr>
<th>Type</th>
<th>Normal form</th>
<th>Cork</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_n</td>
<td>x^{n+1}</td>
<td>1/0–1/1</td>
<td>n \geq 2</td>
</tr>
<tr>
<td>D_n</td>
<td>x^{n-1} + xy^2</td>
<td>2/0–2/1</td>
<td>n \geq 4</td>
</tr>
<tr>
<td>E_6</td>
<td>x^3 + y^4</td>
<td>2/0–2/1</td>
<td></td>
</tr>
<tr>
<td>E_7</td>
<td>x^3 + xy^3</td>
<td>2/0–2/1</td>
<td></td>
</tr>
<tr>
<td>E_8</td>
<td>x^3 + y^4</td>
<td>2/0–2/1</td>
<td></td>
</tr>
<tr>
<td>I_1</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4</td>
<td>0/4–0/7</td>
<td></td>
</tr>
<tr>
<td>I_2</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6</td>
<td>0/6–0/7</td>
<td></td>
</tr>
<tr>
<td>I_3</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6 + \xi_3 \xi_4 \xi_5 \xi_6</td>
<td>0/6–0/7</td>
<td></td>
</tr>
<tr>
<td>I_4</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6</td>
<td>0/6–0/7</td>
<td></td>
</tr>
<tr>
<td>I_5</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_5 \xi_6 \xi_7</td>
<td>0/7</td>
<td></td>
</tr>
<tr>
<td>I_6</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_3 \xi_5 \xi_7</td>
<td>0/7</td>
<td></td>
</tr>
<tr>
<td>I_7</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_3 \xi_5 \xi_7</td>
<td>0/7</td>
<td></td>
</tr>
<tr>
<td>I_8</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_3 \xi_5 \xi_7 + \xi_2 \xi_4 \xi_5 \xi_7</td>
<td>0/7</td>
<td></td>
</tr>
<tr>
<td>I_9</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_3 \xi_5 \xi_7 + \xi_2 \xi_4 \xi_6 \xi_7</td>
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<td></td>
</tr>
<tr>
<td>I_{10}</td>
<td>\xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_3 \xi_5 \xi_7 + \xi_2 \xi_3 \xi_6 \xi_7 + \xi_3 \xi_5 \xi_6 \xi_7</td>
<td>0/7</td>
<td></td>
</tr>
<tr>
<td>I_{11}</td>
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<td>0/1–0/7</td>
<td></td>
</tr>
<tr>
<td>Table 1 (continued)</td>
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<td></td>
</tr>
<tr>
<td>---------------------</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>$x^n + x^m \xi_1 \xi_2$</td>
<td>$1/2 - 1/3$</td>
<td>$1 \leq m \leq n - 1$</td>
</tr>
<tr>
<td>III$_1(n, m)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4$</td>
<td>$1/4$</td>
<td>$1 \leq m \leq n - 1$</td>
</tr>
<tr>
<td>III$_2(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2$</td>
<td>$1/4$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>III$_3$</td>
<td>$x^3 + \xi_1 \xi_2 \xi_3 \xi_4$</td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td>III$_4$</td>
<td>$x^3 + x^m \xi_1 \xi_2 \xi_3 \xi_4$</td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td>III$_5$</td>
<td>$x^3$</td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td>IV$_1(n, m)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_1 \xi_3 \xi_4 \xi_5$</td>
<td>$1/5$</td>
<td>$2 \leq m \leq n - 1$</td>
</tr>
<tr>
<td>IV$_2(n, m)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4$</td>
<td>$1/5$</td>
<td>$1 \leq m \leq n - 1$</td>
</tr>
<tr>
<td>IV$_3(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2$</td>
<td>$1/5$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>IV$_4(n, m)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_1 \xi_3 \xi_4 \xi_5$</td>
<td>$1/5$</td>
<td>$2 \leq m \leq n - 1$</td>
</tr>
<tr>
<td>IV$_5$</td>
<td>$x^3 + x^m \xi_1 \xi_2 \xi_3 \xi_4$</td>
<td>$1/5$</td>
<td></td>
</tr>
<tr>
<td>IV$_6$</td>
<td>$x^3$</td>
<td>$1/5$</td>
<td></td>
</tr>
<tr>
<td>V(n)</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6$</td>
<td>$1/6$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_1(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6$</td>
<td>$1/7$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_2(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 \xi_1 \xi_3 \xi_5$</td>
<td>$1/7$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_3(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 \xi_1 (\xi_3 \xi_5 + \xi_4 \xi_6)$</td>
<td>$1/7$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_4(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 (\xi_1 \xi_3 \xi_5 + \xi_2 \xi_4 \xi_6)$</td>
<td>$1/7$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_5(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 (\xi_1 \xi_3 \xi_5 + \xi_2 \xi_4 \xi_6 + \xi_2 \xi_5 \xi_6)$</td>
<td>$1/7$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_6(n)$</td>
<td>$x^n + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 \xi_5 \xi_6 \xi_1 + \xi_7 \xi_5 \xi_2 \xi_3 + \xi_7 \xi_6 \xi_2 \xi_4$</td>
<td>$1/7$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VI$_7$</td>
<td>$x^3 + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_7 \xi_6 \xi_5 \xi_1 + \xi_7 \xi_6 \xi_2 \xi_3 + \xi_7 \xi_5 \xi_3 \xi_2$</td>
<td>$1/7$</td>
<td></td>
</tr>
<tr>
<td>VI$_8$</td>
<td>$x^3 + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_7 \xi_6 \xi_5 \xi_1 + \xi_7 \xi_6 \xi_2 \xi_3 + \xi_7 \xi_5 \xi_2 \xi_4$</td>
<td>$1/7$</td>
<td></td>
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<tr>
<td>VI$_9$</td>
<td>$x^3 + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_7 \xi_6 \xi_5 \xi_1 + \xi_7 \xi_6 \xi_2 \xi_3 + \xi_7 \xi_5 \xi_3 \xi_2$</td>
<td>$1/7$</td>
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<tr>
<td>VI$_{10}$</td>
<td>$x^3 + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_7 \xi_6 (\xi_5 \xi_1 + \xi_3 \xi_4)$</td>
<td>$1/7$</td>
<td></td>
</tr>
<tr>
<td>VI$_{11}$</td>
<td>$x^3 + x^m \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \xi_7 \xi_6 \xi_5 \xi_1$</td>
<td>$1/7$</td>
<td></td>
</tr>
<tr>
<td>VII$_1(n)$</td>
<td>$x^2 y^3 + x^m \xi_1 \xi_2 + y \xi_2 \xi_3$</td>
<td>$2/3$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>VII$_2$</td>
<td>$x^3 + y^4 + x^m \xi_1 \xi_2 + y \xi_2 \xi_3$</td>
<td>$2/3$</td>
<td></td>
</tr>
<tr>
<td>VII$_3$</td>
<td>$x^3 + y^3 + x^m \xi_1 \xi_2 + y \xi_2 \xi_3$</td>
<td>$2/3$</td>
<td></td>
</tr>
<tr>
<td>VII$_4$</td>
<td>$x^3 + y^5 + x^m \xi_1 \xi_2 + y \xi_2 \xi_3$</td>
<td>$2/3$</td>
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</tbody>
</table>
Table 2.
Bordering unimodal singularities

<table>
<thead>
<tr>
<th>Type</th>
<th>Singularity</th>
<th>Cork</th>
<th>Modality</th>
</tr>
</thead>
<tbody>
<tr>
<td>I'</td>
<td>generic $\Psi \in \Lambda^4 \langle \xi_1, \ldots, \xi_8 \rangle$</td>
<td>0/8</td>
<td>$\geq 6$</td>
</tr>
<tr>
<td>I''</td>
<td>generic $\Psi \in \Lambda^4 \langle \xi_1, \ldots, \xi_9 \rangle$</td>
<td>0/9</td>
<td>$\geq 45$</td>
</tr>
<tr>
<td>III'</td>
<td>$x^4 + x^2 \xi_1 \xi_2 + x^2 \xi_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4$, $a \neq \frac{1}{2}$</td>
<td>1/4</td>
<td>1</td>
</tr>
<tr>
<td>IV'</td>
<td>$x^4 + x^2 \xi_1 \xi_2 + x^2 \xi_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4$, $a \neq \frac{1}{2}$</td>
<td>1/4</td>
<td>1</td>
</tr>
<tr>
<td>V'</td>
<td>$x^3 + x_1 \xi_2 + x_3 \xi_4 + x^2 \xi_5 \xi_6 + a \xi_5 \xi_6 \xi_7 (\frac{1}{a} \xi_1 \xi_2 + a \xi_3 \xi_4)$</td>
<td>1/6</td>
<td>1</td>
</tr>
<tr>
<td>VI'_</td>
<td>$x^3 + x_1 \xi_2 + x_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_4 \xi_6 \xi_7 + \xi_5 \xi_6 \xi_7 \xi_8 \xi_9 \xi_1 \xi_2 + a \xi_3 \xi_4$</td>
<td>1/7</td>
<td>1</td>
</tr>
<tr>
<td>VI''</td>
<td>$x^3 + x_1 \xi_2 + x_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_4 \xi_6 \xi_7 + \xi_5 \xi_6 \xi_7 \xi_8 \xi_9 \xi_1 \xi_2 + a \xi_3 \xi_4$</td>
<td>1/7</td>
<td>$\geq 1$</td>
</tr>
<tr>
<td>VIII'</td>
<td>$x^3 + x_1 \xi_2 + x_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_4 \xi_6 \xi_7 + \xi_5 \xi_6 \xi_7 \xi_8 \xi_9 \xi_1 \xi_2 + a \xi_3 \xi_4 + \Psi$; $\Psi \in \Lambda^4 \langle \xi_1, \ldots, \xi_8 \rangle$</td>
<td>1/8</td>
<td>$\geq 6$</td>
</tr>
<tr>
<td>VIII''</td>
<td>$x^3 + x_1 \xi_2 + x_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_4 \xi_6 \xi_7 + \xi_5 \xi_6 \xi_7 \xi_8 \xi_9 \xi_1 \xi_2 + a \xi_3 \xi_4 + \Psi$; $\Psi \in \Lambda^4 \langle \xi_1, \ldots, \xi_9 \rangle$</td>
<td>1/9</td>
<td>$\geq 45$</td>
</tr>
<tr>
<td>VII'</td>
<td>$x^3 + y^6 + ax_2 y^2$</td>
<td>2/0-2/1</td>
<td>1</td>
</tr>
<tr>
<td>VII''</td>
<td>$x^4 + y^4 + ax_2 y^2$</td>
<td>2/0-2/1</td>
<td>1</td>
</tr>
<tr>
<td>VII''</td>
<td>$x^3 + x_2 y^2 + x_1 \xi_2 + ay_1 \xi_2$</td>
<td>2/2-2/3</td>
<td>1</td>
</tr>
<tr>
<td>VII''</td>
<td>$x^3 + y^3 + z^3 + axyz$</td>
<td>3/0-3/1</td>
<td>1</td>
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</tbody>
</table>

Table 3.
Singularities of cork 1/4

<table>
<thead>
<tr>
<th>Type</th>
<th>Normal form</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_1(n, m_1, m_2, l)$</td>
<td>$x^n + n^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4 \cdot (1 + a_1 x + \ldots + a_k x^k), \neq 0$</td>
<td>$\begin{cases} \min (n - m_1 - 2, m_2 - 2 - l) &amp; k = \begin{cases} \text{for } m - m_1 - m_2 \geq l \text{ or } a \neq m_2/n &amp; m_2 - l - 1 \text{ otherwise} \ n - 1 &gt; m_2 &gt; m_1 &gt; l + 1 &amp; m_2 - l - 1 \text{ otherwise} \end{cases} \end{cases}$</td>
</tr>
<tr>
<td>$\Pi_2(n, m_1, m_2)$</td>
<td>$x^n + n^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4$</td>
<td>$n - 1 &gt; m_2 &gt; m_1$</td>
</tr>
<tr>
<td>$\Pi_3(n, m, l)$</td>
<td>$x^n + n^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4$</td>
<td>$n - m \geq l$</td>
</tr>
<tr>
<td>$\Pi_4(n, m)$</td>
<td>$x^n + n^{m_1} \xi_1 \xi_2$</td>
<td>$n - 1 &gt; m$</td>
</tr>
<tr>
<td>$\Pi_5(n, m)$</td>
<td>$x^n + n^{m_1} \xi_1 \xi_2 \xi_3 \xi_4$</td>
<td>$n - 1 \geq m$</td>
</tr>
</tbody>
</table>

4.3. Classifier of singularities. Like the classifiers of plants, minerals, etc., the classifier of singularities is a device that helps one to assign a singularity its place in the accepted classification system.

Its other aim is to help organizing the proof of the right classification.
Notations:

- $m$ the maximal ideal of the ring of germs $\mathcal{O}_{p/q}$;
- $n$ the ideal of $\mathcal{O}_{p/q}$ generated by the odd elements;
- $(M)$ the ideal of $\mathcal{O}_{p/q}$ generated by a set $M$; in particular
- $(x)$ the ideal generated by the even generators of $\mathcal{O}_{p/q}$ (it is not $G$-invariant);
- $\langle M \rangle$ the linear span of a set $M$ (over $\mathbb{C}$);
- $\text{ord } f = \min \{ k \mid f \in m^k \}$ the degree of $f$ in the grading
  adjoined to the filtration $\mathcal{O} \supseteq m \supseteq m^2 \supseteq \ldots$;
- $\text{ord}_x f$ the degree of $f$ in the grading adjoined to the filtration
  $\mathcal{O} \supseteq (x) \supseteq (x)^2 \supseteq \ldots$ (this degree is not $G$-invariant);
- $f = f^{(0)} + f^{(2)} + f^{(4)} + \ldots, f^{(2k)} \in A^{2k}[\xi_1, \ldots, \xi_q]$, the decomposition of an
  even function in the power series in odd variables;
- $r_k$ the minimal number of summands in the representations of
  $r \in A^k[\xi_1, \ldots, \xi_q]$ as the sum of decomposable polyvectors
  $\rightarrow$ is adjacent to
  $\Rightarrow$ implies, yields
  $\Rightarrow$ see
  $\iff$ only one of the following possibilities may occur

In parenthesis we indicate the numbers of the lemmas in which the statements
of Classifier are proved; the number at the end of a subsection is the number of
a subsection that refers to this one (unless the one that refers is the immediately
preceding one).

Let cork $f = p/q$

1. If $\mu(f) < \infty$ then $\iff$
   1) $q = 0$ or $1 \Rightarrow f = f^{(0)}$ and $f$ does not depend on odd variables and
      belongs to one of the types $A_n, D_n, E_n, VII_1, VII_2, VII_3, VII_4 ([A])$;
   2) $p \geq 3 \Rightarrow f \rightarrow VII_4([A])$;
   3) $p = 2, q = 2$ or $q \geq 4 \Rightarrow VII_3$ (5.19–5.20);
   4) $p = 2, q = 3 \Rightarrow 24$
   5) $p = 1, q \geq 8 \Rightarrow f \rightarrow VIII_1$ or $VIII_2$ depending on the parity of $q$
   6) $p = 1, 2 \leq q \leq 7 \Rightarrow 2$;
   7) $p = 0, q \geq 8 \Rightarrow f \rightarrow I_1$ or $I_2$ depending on the parity of $q$
   8) $p = 0, 2 \leq q \leq 7 \Rightarrow 26$.

2. If $p = 1, 2 \leq q \leq 7$ then $\iff$
   1) $q = 2$ or $q = 3 \Rightarrow f \sim (n, m)$ (5.1.1);
   2) $q = 4 \Rightarrow 3$;
   3) $q \geq 5 \Rightarrow 7$.  

3. If \( p = 1, q = 4 \) then \( \Rightarrow \ (5.1) \)
   1) \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + f^{(4)}, m_1, m_2 \leq n - 2 \Rightarrow 4; \)
   2) \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + f^{(4)}, m_1 \leq n - 2 \Rightarrow 5; \)
   3) \( f \sim x^n + f^{(4)} \Rightarrow 6. \)

4. If \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + f^{(4)}, m_1 \leq m_2 \leq n - 2 \), then \( \Rightarrow \)
   1) \( \text{ord}_x f^{(4)} \geq m_1 - 1 \Rightarrow f \sim \Pi_{b}(n, m_1, m_2) \ (5.2a); \)
   2) \( \text{ord}_x f^{(4)} = m_1 - 2 \Rightarrow f \sim \Pi_{a}(n, m_1, m_2, l) \ (5.2a). \)

5. If \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + f^{(4)} \) then \( \Rightarrow \)
   1) \( \text{ord}_x f^{(4)} \geq m_1 - 1 \Rightarrow f \sim \Pi_{c}(n, m_1) \ (5.2b); \)
   2) \( \text{ord}_x f^{(4)} = l \leq m_1 - 2 \Rightarrow f \sim \Pi_{c}(n, m_1, l) \ (5.2b). \)

6. If \( f \sim x^n + f^{(4)} \) then \( \Rightarrow \)
   1) \( \text{ord}_x f^{(4)} \geq n - 1 \Rightarrow f \sim \Pi_{c}(n, n - 1) \ (5.2c); \)
   2) \( \text{ord}_x f^{(4)} = l \leq n - 2 \Rightarrow f \sim \Pi_{c}(n, l) \ (5.2c). \)

7. If \( p = 1, 5 \leq q \leq 7 \) then \( \Rightarrow \)
   1) \( q = 5 \Rightarrow 8; \)
   2) \( q = 6 \Rightarrow 17; \)
   3) \( q = 7 \Rightarrow 18. \)

8. If \( p = 1, q = 5 \) then \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + f^{(4)}, m_1 \leq m_2 \ (5.1) \)
   and \( \Rightarrow \)
   1) \( m_1 \geq 3 \Rightarrow 9; \)
   2) \( m_1 = 2 \Rightarrow 10; \)
   3) \( m_1 = 1 \Rightarrow 13. \)

9. If \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + f^{(4)}, 3 \leq m_1 \leq m_2 \) then \( \Rightarrow \)
   1) \( n = 3 \Rightarrow 16; \)
   2) \( n \geq 4 \Rightarrow f \rightarrow IV' \ (5.4.1) \)

10. If \( f \sim x^n + x^2 \xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}, m \geq 2 \) then \( \Rightarrow \)
    1) \( n = 3 \Rightarrow 16; \)
    2) \( n \geq 4, m = 2 \Rightarrow 11; \)
    3) \( n \geq 4, m \geq 3 \Rightarrow 12. \)

11. If \( f \sim x^n + x^2 (\xi_1 \xi_2 + \xi_3 \xi_4) + f^{(4)}, n \geq 4 \) then \( \Rightarrow \)
    1) \( f^{(4)} \in (\xi_1 \xi_2) + (x) \Rightarrow f \rightarrow IV' \ (5.4.3); \)
    2) \( f^{(4)} \in (\xi_1 \xi_2 \xi_3 \xi_4) + (x) \Rightarrow f \sim IV_4(n, 2) \ (5.5). \)

12. If \( f \sim x^n + x^2 \xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}, n \geq 4, m \geq 3 \) then \( \Rightarrow \)
    1) \( f^{(4)} \in (\xi_1 \xi_2) + (x) \Rightarrow f \rightarrow IV' \ (5.4.2); \)
    2) \( f^{(4)} \notin (\xi_1 \xi_2) + (x) \) and \( m \leq n - 1 \Rightarrow f \sim IV_4(n, m) \ (5.5); \)
    3) \( f^{(4)} \notin (\xi_1 \xi_2) + (x) \) and \( m \geq n \Rightarrow f \sim IV_4(n, n - 1) \ (5.5). \)
13(8). If $f \sim x^n + x\xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}$ then

1) $m = 1 \Rightarrow f \sim IV_2 (n, 1)$ (5.6a);
2) $2 \leq m \leq n - 2 \Rightarrow 14$;
3) $m \geq n - 1 \Rightarrow 15$.

14. If $f \sim x^n + x\xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}$, $2 \leq m \leq n - 2$ then

1) $f^{(4)} \in (\xi_1 \xi_2) + (x) \Rightarrow f \sim IV_2 (n, m)$ (5.6a);
2) $f^{(4)} \notin (\xi_1 \xi_2) + (x) \Rightarrow f \sim IV_1 (n, m)$ (5.6b).

15(13). If $f \sim x^n + x\xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}$ then

$f \sim x^n + x\xi_1 \xi_2 + f^{(4)}$ (5.1) and

1) $f^{(4)} \in (x) \Rightarrow f \sim IV_3 (n)$ (5.7a);
2) $f^{(4)} \notin (x)$ and $f^{(n)} \in (x) + (\xi_1 \xi_2) \Rightarrow f \sim IV_2 (n, n - 1)$ (5.7b);
3) $f^{(4)} \notin (x) + (\xi_1 \xi_2) \Rightarrow f \sim IV_1 (n, n - 1)$ (5.7c).

16(9 and 10). If $f \sim x^3 + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + f^{(4)}$, $m_1, m_2 \geq 2$ then

$f \sim x^3 + f^{(4)}$ (5.1) and

1) $f^{(4)} \in (x^2) \Rightarrow f \sim IV_6$ (5.8a);
2) $f^{(4)} \in (x)$ and $f^{(4)} \notin (x^2) \Rightarrow f \sim IV_5$ (5.8b);
3) $f^{(4)} \notin (x) \Rightarrow f \sim IV_4$ (3.2) (5.8c).

17(7). If $p = 1$, $q = 6$ then

$f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + x^{m_3} \xi_5 \xi_6 + f^{(4)} + f^{(6)}$, $m_1 \leq m_2 \leq m_3$ (5.1)

and

1) $m_1 = m_2 = m_3 = 1 \Rightarrow f \sim V(n)$ (5.9);
2) $m_3 \geq 2 \Rightarrow f \rightarrow V'$ (5.10).

18(7). If $p = 1$, $q = 7$ then

$f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + x^{m_3} \xi_5 \xi_6 + f^{(4)} + f^{(6)}$, $m_1 \leq m_2 \leq m_3$, (5.1) and

1) $m_1 = m_2 = m_3 = 1 \Rightarrow 19$;
2) $m_1 = m_2 = 1$, $m_3 \geq 2 \Rightarrow 20$;
3) $m_2 \geq 2 \Rightarrow f \rightarrow VI_2$ (5.15).

19. If $f \sim x^n + x(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) + f^{(4)} + f^{(6)}$ then

$f \sim x^n + x(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) + \xi_7 r$, where $r \in A^3 \langle \xi_1, \ldots, \xi_6 \rangle$ (5.13).

and

1) $r = 0 \Rightarrow f \sim VI_1 (n)$ (5.12 and 5.14);
2) $r k r = 1 \Rightarrow f \sim VI_2 (n)$ (5.12 and 5.14);
3) $r k r = 2$ and $r$ is divisible by a vector $v \in \langle \xi_1, \ldots, \xi_6 \rangle \Rightarrow$

$\Rightarrow f \sim VI_3 (n)$ (5.12 and 5.14);
4) $r k r = 2$ and $r$ is not divisible by a vector $v \Rightarrow f \sim VI_4 (n)$ (5.12 and 5.14);
5) $r k = 3 \Rightarrow f \sim VI_5 (n)$ (5.12 and 5.14).
20(18). If \( f \sim x^n + x\xi_1 \xi_2 + x^{3}\xi_3 \xi_4 + x^{m}\xi_5 \xi_6 + f^{(4)} + f^{(6)}, m \geq 2 \) then \( \Rightarrow \)

1) \( n = 3 \Rightarrow f \sim x^3 + x(\xi_1 \xi_2 + \xi_3 \xi_4) + f^{(4)} + f^{(6)} \) (5.1) \( \Rightarrow \) 22;
2) \( n \geq 4, m = 2 \Rightarrow 21; \)
3) \( n \geq 4, m \geq 3 \Rightarrow f \rightarrow IV' \) (5.16.1).

21. If \( f \sim x^n + x(\xi_1 \xi_2 + \xi_3 \xi_4) + x^2 \xi_5 \xi_6 + f^{(4)} + f^{(6)} \) and \( f^{(4)} \equiv \gamma r \mod((x) + (A^4 \langle \xi_1, \ldots, \xi_6 \rangle) + (\xi_1 \xi_2 + \xi_3 \xi_4)), \) where \( r \in A^3 \langle \xi_1, \ldots, \xi_6 \rangle \) then \( \Rightarrow \)

1) \( \text{rk} r = 3 \Rightarrow f \sim V_6(n) \) (5.16a);
2) \( \text{rk} r \leq 2 \Rightarrow f \rightarrow IV' \) (5.16b).

22(20). If \( f \sim x^3 + x(\xi_1 \xi_2 + \xi_3 \xi_4) + f^{(4)} + f^{(6)} \) then \( \Rightarrow \)

1) \( f^{(4)} \in (A^2(\xi_1, \ldots, \xi_4)) + (x) \Rightarrow f \rightarrow VI'_1 \) (5.17a)
2) \( f^{(4)} \notin (A^2(\xi_1, \ldots, \xi_4)) + (x) \Rightarrow f \sim x^3 + x(\xi_1 \xi_2 + \xi_3 \xi_4) + a\xi_3 \xi_4 \xi_6 \xi_7 + b\xi_2 \xi_4 \xi_5 \xi_7 + c\xi_2 \xi_3 \xi_5 \xi_6 \) (5.17b) \( \Rightarrow \) 23.

23. If \( f \sim x^3 + x(\xi_1 \xi_2 + \xi_3 \xi_4) + a\xi_3 \xi_4 \xi_6 \xi_7 + b\xi_2 \xi_4 \xi_5 \xi_7 + c\xi_2 \xi_3 \xi_5 \xi_6 \) then \( \Rightarrow \) (5.18):

1) \( a, b, c \neq 0 \Rightarrow f \sim VI_4 \) (3);
2) \( a, c \neq 0 \) or \( a, b \neq 0, c = 0 \Rightarrow f \sim VI_7; \)
3) \( a = 0, b, c \neq 0 \Rightarrow f \sim VI_8; \)
4) \( a \neq 0, b = 0, c = 0 \Rightarrow f \sim VI_9; \)
5) \( a = b = 0, c \neq 0 \) or \( a = c = 0, b \neq 0 \Rightarrow f \sim VI_10; \)
6) \( a = b = c = 0 \Rightarrow f \sim VI_{11} . \)

24(1). If \( p = 2, q = 3 \) then \( f = f_0 + r \mod m^4, \) where \( r \in \langle x, y \rangle \otimes A^2 \langle \xi_1, \xi_2, \xi_3 \rangle \) then \( \Rightarrow \)

1) \( r \) is decomposable \( \Rightarrow f \rightarrow VII'_3 \) (5.21a);
2) \( r \) is not decomposable and \( f_0 \) is a simple even singularity \( \Rightarrow 25; \)
3) \( r \) is not decomposable and \( f_0 \) is not simple \( \Rightarrow f \rightarrow VII'_1 \) or \( VII'_2 \) (5.21b).

25. If \( f = f_0 + r \mod m^4, \) where \( r \in \langle x, y \rangle \otimes A^2 \langle \xi_1, \xi_2, \xi_3 \rangle, \) \( \text{cork} k f = 2 \) then \( \Rightarrow \) (5.21c).

1) \( f^0 \) is of type \( D_n \Rightarrow f \sim VII_1(n - 1); \)
2) \( f^0 \) is of type \( E_6 \Rightarrow f \sim VII_2; \)
3) \( f^0 \) is of type \( E_7 \Rightarrow f \sim VII_3; \)
4) \( f^0 \) is of type \( E_8 \Rightarrow f \sim VII_4. \)

26(1). If \( p = 0, 2 \leq q \leq 7 \) then \( f = f^{(4)} + f^{(6)} \) and \( \Rightarrow \) (5.22)

1) \( f = 0 = I_{11}; \)
2) \( f^{(4)} = 0; f^{(6)} = 0; q = 6 \) or \( 7 \) and \( f \sim I_4 \)
3) \( f^{(4)} \neq 0 \Rightarrow f \sim f^{(4)} \) which belongs to one of the types \( I_1 = I_3, I_5 - I_{10}. \)

The classification in this case is the description of the orbits of \( GL(V) \) in \( A^4(V) \) in \( A^4(V) \) where \( \text{dim} V \leq 7. \) The most difficult part of this classifi-
5. Proof of lemmas of the classifier

5.1. Lemma. Let \( c = (1/r), s = [r/2], h = \text{ord}_x f^{(0)}, m_1 = \text{ord}_x f^{(2)}, m_2 = \text{ord}_x (f^{(2)})^2 - m_1, \ldots, m_s = \text{ord}_x (f^{(2)})^s - \ldots - m_{s-1}. \) Then \( m_1 \leq m_2 \leq \ldots \leq m_s. \) Let \( i \) be such that \( m_i < n - 1, m_{i+1} \geq n - 1. \) Then

\[
f \sim x^n + x^{m_1} \xi_1 \xi_2 + \ldots + x^{m_i} \xi_{2i-1} \xi_{2i} + \varphi
\]

for some \( \varphi \in n^4. \)

Proof. Since \( \text{ord}_x a_i \geq \text{ord}_x a^{i-1} + \text{ord}_x a, \) then \( m_1 \leq m_2 \leq \ldots \leq m_s. \) As follows from \([A], f^{(0)} \sim x^n. \) Therefore \( f \) reduces to the form:

\[
f = x^n + f^{(2)} + f^{(4)} + f^{(4)} + \ldots.
\]

Let \( f^{(2)} = \omega \cdot x^{m_1} + \tau, \) where \( \omega \in \Lambda^2(\xi_1, \ldots, \xi_7) \) and \( \tau \in m^{m_1+1}. \) Reduce \( \omega \) to the form

\[
\xi_1 \xi_2 + \xi_3 \xi_4 + \ldots + \xi_{2p-1} \xi_{2p}.
\]

Then \( m_1 = m_2 = \ldots = m_p. \) By Morse's lemma

\[
f^{(2)} \sim x^{m_1}(\xi_1 \xi_2 + \xi_3 \xi_4 + \ldots + \xi_{2p-1} \xi_{2p} + \tilde{t}_1(x, \xi_{2p+1}, \ldots, \xi_7))
\]

and \( \tilde{t}_1 \in m^{m+p+1-m_1}. \) Similarly we may show that

\[
f^{(2)} \sim x^{m_1}(\xi_1 \xi_2 + \ldots + \xi_{2p-1} \xi_{2p}) + x^{m_2}(\xi_{2p+1} \xi_{2p+2} + \ldots \\
\ldots + \xi_{2q+1})_{2q+2} + \tilde{t}_2(x, \xi_{2q+3}, \ldots, \xi_7),
\]

where \( \tilde{t}_2 \in m^{m_{q+1}-m_q}, \) and \( m_q = \ldots = m_{p+1}. \) Repeating this procedure several times we finally reduce \( f^{(2)} \) to the form

\[
f^{(2)} = x^{m_1} \xi_1 \xi_2 + \ldots + x^{m_i} \xi_{2s-1} \xi_{2s}.
\]

We have only used the changes of coordinates preserving \( x. \) Therefore

\[
f \sim x^n + x^{m_1} \xi_1 \xi_2 + \ldots + x^{m_s} \xi_{2s-1} \xi_{2s} + \varphi, \quad \text{where} \quad \varphi \in n^4.
\]

Now applying the change of coordinates

\[
x \mapsto x - \frac{1}{n}(x^{m_1+1-n+1} \xi_{2i+1} + x^{m_2-n+1} \xi_{2i+2} + \ldots + x^{m_s-n+1} \xi_{2s-1} \xi_{2s}),
\]

we reduce \( f \) to the form indicated in the lemma.

5.1.1. Corollary. Let \( c = 1/2 \) (or \( 1/3 \)). Then \( f \sim II(n, m), \) where \( n = \text{ord}_x f^{(0)}, m = \min(\text{ord}_x f^{(2)}, n - 1) \)
5.2. Lemma. Let \( f = 1/4 \).

a) If \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + \varphi \),
where \( m_1, m_2 < n - 1 \), and
\[
\varphi = ax^l \xi_1 \xi_2 \xi_3 \xi_4 (1 + x \Psi(x)),
\]
then \( f \sim III_{a} \) for \( l < m_1 - 1 \) and \( f \sim III_{b} \) for \( l \geq m_1 - 1 \).

b) If \( f \sim x^n + x^{m_1} \xi_1 \xi_2 + \varphi \) and \( \deg \varphi = 1 \), then \( f \sim III_{c} \) for \( l \leq m_1 \) and \( f \sim III_{d} \) for \( l > m_1 \).

c) If \( f \sim x^n + \varphi \) and \( \deg \varphi = l \), then \( f \sim III_{i} (n, l) \) for \( l < n - 1 \), and \( f \sim III_{i} (n, n - 1) \) for \( l \geq n - 1 \).

Proof. Let
\[
f = x^n + x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 + ax^l \xi_1 \xi_2 \xi_3 \xi_4 (1 + x \Psi(x)) (a \neq 0).
\]
Notice that \( f \sim f + f(x) \xi_1 \xi_2 \xi_3 \xi_4 \), if \( \text{ord } f \geq m_1 \), since \( f \) turns into \( f + f(x) \xi_1 \xi_2 \xi_3 \xi_4 \) under the change \( \xi_1 \rightarrow \xi_1 \xi_3 \xi_4 f(x)/x^{m_1} + \xi_1 \). Consecutively, if \( l \geq m_1 \), then \( f \sim III_{b} \).

Let \( l = m_1 - 1 \). Then the composition of the changes \( x \mapsto x - a \xi_3 \xi_4 /m_1 \) and \( \xi_3 \mapsto \xi_3 (1 - nx^{n-1} a)^{-1} \) reduces \( f \) to the same form but with \( a = 0 \), implying \( f \sim III_{b} \).

Let \( l < m_1 - 1 \). Consider the change of coordinates
\[
x \mapsto x + \theta(x) \xi_3 \xi_4 \text{ and } \xi_3 \mapsto \xi_3 (1 - n \theta(x)nx^{n})^{-1},
\]
where \( f_1 = f + \varphi(x) \xi_1 \xi_2 \xi_3 \xi_4 \). Then \( f_1 \) turns into \( f \) for
\[
\theta(x) = \varphi(x)/(m_1 x^{n-1} - ax^l (1 + x \Psi(x)).
\]
Notice that for \( l + k = m_1 - 1 \) or \( a = m_1 /n \) the power of the numerator does not exceed \( \min(m_1 - 1, l + k) \). Therefore
\[
f \sim f + \varphi(x) \xi_1 \xi_2 \xi_3 \xi_4 \text{ if } \deg \varphi \geq \min(m_1 - 1, l + k).
\]
Let \( l + k = m_1 - 1 \) and \( a = m_1 /n \). Then \( f \sim f + \varphi(x) \xi_1 \xi_2 \xi_3 \xi_4 \text{, if } \deg \varphi \geq m_1 \).

This makes it clear that \( f \) reduces to the form \( III_{a} \).

b) Let \( f = x^n + x^{m_1} \xi_1 \xi_2 + \Psi(x) \xi_1 \xi_2 \xi_3 \xi_4 \), where \( \text{ord } \Psi(x) = l \). The change of variables \( \xi_3 \mapsto \Psi(x)/x^l \) reduces \( f \) to the form \( f = x^n + x^{m_1} \xi_1 \xi_2 + x^l \xi_1 \xi_2 \xi_3 \xi_4 \).

If \( l > m_1 \) then \( f \sim x^n + x^{m_1} \xi_1 \xi_2 \) and therefore \( f \sim III_{i} (n, m_1) \). If \( l \leq m_1 \), then \( f \sim III_{d} \).

c) Let \( f = x^n + \Psi(x) \xi_1 \xi_2 \xi_3 \xi_4 \).
This case is similar to the preceding one.
5.2.1. Corollary. Let \( c \)ork \( f = (1/4) \) and \( f \in m^4 \). Then \( f \) is adjacent to III'.

Proof. Let \( g \) be a singularity of the form III' and \( h = f + \lambda g \). For all \( \lambda/y \) except perhaps three values the germ \( h_3 \) falls into the case a) of Lemma 5.2 with \( n = 4, m_1 = m_2 = 2 \). Then by Lemma 5.2 we get \( h_{3} \sim III' \) for almost all \( \lambda \) sufficiently close to 0.

5.2.2. Corollary. Let \( f = x^n + x\xi_1 \xi_2 + x^m \xi_3 \xi_4 + \varphi \), where \( m < n - 1, \varphi \in n^4 \). Then \( f \sim III_1(n, m) \).

5.2.3. Corollary. Let \( f = x^n + x\xi_1 \xi_2 + \varphi \), where \( \varphi \in n^4 \) for \( \text{ord}_{x} \varphi = 0 \), then \( f \sim III_1(m, n - 1) \) and if \( \text{ord}_{x} \varphi > 0 \) then \( f \sim III_2(n) \).

5.2.4. Corollary. Let \( f = x^3 + \varphi \), where \( \varphi \in n^4 \). Then \( f \sim III_5 \), if \( \deg \varphi \geq 2 \), \( f \sim III_4 \), if \( \text{ord} \varphi = 1 \), \( f \sim III_3 \) if \( \text{ord} \varphi = 0 \).

5.3. Lemma. All the singularities listed in Table 3 are pairwise inequivalent.

Proof. Since the ideals \( m \) and \( n \) are invariant with respect to automorphism of \( \mathcal{O} \), then the singularities from different lines of the table and different \( n, m_1, m_2, l \) are pairwise inequivalent. Let us show that two singularities \( f \) and \( g \) of type III_a \((n, m_1, m_2, l)\) with different sets of parameters \( a, a_1, \ldots, a_k \), and \( b, b_1, \ldots, b_k \) are inequivalent.

On the contrary, let \( f \sim g \). Any change of variables sending \( f \) into \( g \) preserves \( f^{(1)} + f^{(2)} = g^{(1)} + g^{(2)} \). Any such change is of the form \( x \rightarrow x + \tau \), where \( \tau \in n^2 \), \( \xi_i \leftarrow \xi_i + \sum a_{ij}(x) \xi_j \), where \( \deg a_{ij}(x) \geq n - m_2 - 1 \).

Under this change \( f^{(4)} \) turns into

\[
(f^{(4)} + (m_2 \xi_1 \xi_2 \tau x^{m_2 - 1} + m_1 \xi_1 \xi_2 \tau x^{m_1 - 1}) \cdot \xi_1 \xi_2 \xi_3 \xi_4 (1 + \text{tr}(a_{ij})) = f^{(4)} + \varphi, \]

where \( \text{ord}_{x} \varphi \geq \min (n - m_1 - 1, m_2 - 1) \).

Therefore \( a = b \), \( a_i = b_i \) for \( i < \min (n - m_1 - 2, m_2 - 2 - b) \).

It remains to consider the case \( n - m_1 - m_2 = l \), \( a = m_2/n \). To preserve \( f^{(1)} + f^{(2)} \) we need that \( a_{ij}(x)(\omega) = -\tau_n x^{n - 1} \), where \( \omega = x^{m_1} \xi_1 \xi_2 + x^{m_2} \xi_3 \xi_4 \), and \( a_{ij}(x) \) act on \( \omega \) as the derivative \( \sum a_{ij}(x) \xi_i(\partial/\partial \xi_j) \).

This implies that \( \text{tr} a_{ij} = -nx^{n-m_1-1} + 0(x^{n-m_1}) \), and therefore \( \text{ord}_{x} \varphi \geq m_2 \) for \( a = m_2/n, n - m_1 = m_2 - l \) as required.

5.4. Lemma. Let cork \( f = (1/5) \) and \( f = x^4 + x^2 \xi_1 \xi_2 + x^2 \xi_3 \xi_4 + f^{(4)} \), where \( f^{(4)} = a \xi_1 \xi_2 \xi_3 \xi_4 \mod (x) \). Then \( f \sim IV' \) for \( a \neq 2, 1/2 \).

Proof. Consider the change

\[
x \mapsto x + d \xi_1 \xi_5 + \beta \xi_2 \xi_5 + \gamma \xi_3 \xi_5 + \xi_4 \xi_5,
\]
\[
\xi_1 \mapsto \xi_1 - 4\beta x \xi_5, \xi_2 \mapsto \xi_2 - 4\alpha x \xi_5, \xi_3 \mapsto \xi_3 - 4\delta x \xi_5, \xi_4 \mapsto \xi_4 - 4\gamma x \xi_5.
\]
Then $f$ turns into
\[ g = f + 2x(2 - a)(x_1 x_5 x_3 x_4 + \beta x_5 x_2 x_3 x_4 + \gamma x_1 x_2 x_3 x_5 + \delta x_1 x_2 x_5 x_4) + x^2 \varphi, \text{ where } \varphi \in \mathbb{n}^4. \]

If $a = 1/2$ we may select $\alpha, \beta, \gamma$ and $\delta$ so that
\[ g = x^4 + x^2 \xi_1 \xi_2 + x^2 \xi_3 \xi_4 + (a + \mu x) \xi_1 \xi_2 \xi_3 \xi_4 + x^2 \varphi. \]

Now with the help of Morse's lemma we get rid of the term $\varphi$ and the term $\mu x \xi_1 \xi_2 \xi_3 \xi_4$ for $a = 1/2$ may be killed thanks to Lemma 5.2.

5.4.1. Corollary. Let cork $f = 1/5$ and $f = x^n + ax^{m_1} \xi_1 \xi_2 + bx^{m_2} \xi_3 \xi_4 + f^{(4)}$, where $n > 3, m_1, m_2 > 2$. Then $f$ is adjacent to IV'.

5.4.2. Corollary. Let cork $f = 1/5$ and $f = x^n + x^2 \xi_1 \xi_2 + bx^m \xi_3 \xi_4 + f^{(4)}$, where $n > 3, m > 2, f^{(4)} \in (\xi_1 \xi_2)n^2 + (x)$. Then $f$ is adjacent to IV'.

5.4.3. Corollary. Let cork $f = 1/5$ and
\[ f = x^n + x^2 \xi_1 \xi_2 + x^2 \xi_3 \xi_4 + a \xi_1 \xi_2 \xi_3 \xi_4 + \varphi, \]
where $n > 3, \varphi \in \mathbb{n}^4$. Then $f$ is adjacent to IV'.

5.5. Lemma. Let cork $f = 1/5, f = x^n + x^2 \xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}$, where $n > 3, m \leq n - 1, f^{(4)} \not\equiv (x) + \xi_1 \xi_2 n^2$.

Then $f \sim IV_4(n, m)$. If $m = 2$ and $f^{(4)} \not\equiv (\xi_1 \xi_2 \xi_3 \xi_4) + (x)$, then $f \sim IV_4(n, 2), f \sim IV_4(n, m)$.

Proof. Making a linear change of variables $\xi_1, \xi_2$ and multiplying $\xi_5$ by a constant we may reduce $f^{(4)}$ to the form $f^{(4)} = \xi_1 \xi_2 \xi_3 \xi_4 + \varphi$, where $\varphi \in n^4(x)$. Let us show that $f \sim f - \varphi$. Let for $\gamma \in \mathbb{n}^4$
\[ \varphi = ax \xi_1 \xi_2 \xi_3 \xi_4 + bx \xi_1 \xi_2 \xi_3 \xi_5 + cx \xi_1 \xi_2 \xi_4 \xi_5 + dx \xi_1 \xi_3 \xi_4 \xi_5 + ex \xi_2 \xi_3 \xi_4 \xi_5 + \gamma. \]

With the change $\xi_5 \mapsto \xi_5 - a \xi_2$ equate $a$ to zero and with the change $x \mapsto x - (b \xi_3 \xi_5 + c \xi_4 \xi_5)/2, \xi_3 \mapsto \xi_3 + \frac{1}{an} \xi_5 x^{n-1} - m, \xi_4 \mapsto \xi_4 + \frac{1}{an} \xi_5 x^{n-1} - m$ equate $b$ to $c$. After that with the change $\xi_2 \rightarrow \xi_2 - \frac{d}{e} \xi_1$ let us equate $d$ to $0$. Finally, with the change $\xi_1 \rightarrow \xi_1 - x \xi_2$ we equate $e$ to $0$, i.e. reduce $f = \varphi$ to the form $f - t$, with $t \in (x^2)n^4$. Further with Morse's lemma we get $f \sim f - t$.

The case $m = 2$ is similar.

5.6. Lemma. Let cork $f = 1/5$ and $f = x^n + x^2 \xi_1 \xi_2 + x^m \xi_3 \xi_4 + f^{(4)}$, where $m < n - 1$. 

a) If \( f^{(4)} \in (\xi_1, \xi_2) + (x) \) or \( m = 1 \) then \( f \sim IV_2(n, m) \)

b) If \( f \) fails to satisfy a) then \( f \sim IV_1(n, m) \).

**Proof.** a) If \( m \neq 1 \) then the change \( x \mapsto x - v \), where \( v \) is found from the condition \( \xi_1 \xi_2 v = f^{(4)} \) mod \( (x) \), reduces \( f \) to the form \( f = x^n + x \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \phi \), with \( \phi \in \mathbb{n}^2(x) + \mathbb{n}^4(x) \) and if \( m = 1 \) then \( f^{(4)} = v_1 \xi_1 \xi_2 + v_2 \xi_4 + \phi_1 \), with \( \phi_1 \in \mathbb{n}^4(x) \). The change \( x \mapsto x - v_1 v_2 \) also reduces \( f \) to the same form.

Let \( m > 1 \) and \( f = x^n + x \xi_1 \xi_2 + x^m \xi_3 \xi_4 + \phi_1 \). Then \( \phi = \xi_1 \Psi_1 + \xi_2 \Psi_2 \) and the change \( \xi_1 \mapsto \xi_1 + \Psi_1, \xi_2 \mapsto \xi_2 + \Psi_2 \) establishes the equivalence of \( f \) and \( f - \phi \). This yields \( f \sim IV_2(n, m) \). The case \( m = 1 \) follows from Morse’s lemma.

b) If \( f \) fails to satisfy the condition a) then \( f^{(4)} = a \xi_1 \xi_3 \xi_4 \xi_5 + b \xi_2 \xi_3 \xi_4 \xi_5 + \xi_1 \xi_2 v + w \), where \( v \in A^2(\xi), w \in \mathbb{n}^4(x) \). The linear change \( \xi_1 \mapsto a \xi_1 - b \xi_2, \xi_2 \mapsto a^{-1} \xi_1 \) reduces the germ of \( f \) to the form where \( b = 0, a = 1 \). Applying the same changes of coordinates as in the case a) we can reduce \( f \) to the form \( IV_4(n, m) \).

5.7. **Lemma.** Let cork \( f = 1/5 \) and \( f = x^n + x \xi_1 \xi_2 + f^{(4)} \)

a) If \( f^{(4)} \in (x) \) then \( f \sim IV_3(n) \)

b) If \( f^{(4)} \notin (x) \) but \( f^{(5)} \in (\xi_1, \xi_2) + (x) \), then \( f \sim IV_2(n, n - 1) \).

c) If \( f^{(4)} \notin (x) \) and \( f^{(5)} \notin (\xi_1, \xi_2) \), then \( f \sim IV_1(n, n - 1) \).

**Proof.** a) follows from Morse’s lemma and b) and c) are proved as Lemma 5.6.

5.8. **Lemma.** Let cork \( f = 1/5 \) and \( f = x^3 + f^{(4)} \).

a) If \( \text{ord}_x f^{(4)} \geq 2 \), then \( f \sim IV_6 \)

b) If \( \text{ord}_x f^{(4)} = 1 \), then \( f \sim IV_5 \)

If \( \text{ord}_x f^{(4)} = 0 \), then \( f \sim IV_4(3, 2) \)

**Proof.** a) The change \( x \mapsto x - f^{(4)}/3x^2 \) turns \( f \) into \( x^3 \).

b) Let \( \text{ord}_x f^{(4)} = 1 \). A linear change of variables \( \xi_1, \ldots, \xi_5 \) reduces \( f \) to the form \( f = x^3 + x \xi_1 \xi_2 \xi_3 \xi_4 + \phi \), where \( \phi \in (x^3) \mathbb{n}^4 \). Further apply the change of variables \( x \mapsto x - \phi/3x^2 \)

c) Let \( \text{ord}_x f^{(4)} = 0 \). A linear change in \( \langle \xi_1, \ldots, \xi_5 \rangle \) reduces \( f \) to the form \( f = x^3 + \xi_1 \xi_3 \xi_4 \xi_5 + \phi \) with \( \phi \in (x) \mathbb{n}^4 \); then kill \( \phi \) by a change of the form \( \xi_i \mapsto \sum a_{ij}(x) \xi_j \).

It is not difficult to verify that under the change

\[
x \mapsto x + \xi_1 \xi_2/3 + \xi_3 \xi_4/3, \xi_5 \mapsto \xi_5 - \xi_2/3
\]

\( x^3 + \xi_1 \xi_3 \xi_4 \xi_5 \) turns into \( IV_4(3, 2) \).
5.9. **Lemma.** Let cork $f = 1/6$ and
\[ f = x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x \xi_5 \xi_6 + f^{(4)} + f^{(6)}. \]
Then $f \sim V(n)$.

**Proof.** Let $f^{(4)} + f^{(6)} = v + w + tx$, where
\[ v \in \Lambda^4(\xi_1, \ldots, \xi_6), w \in \Lambda^6(\xi_1, \ldots, \xi_6). \]
It is easy to see that
\[ v = (\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6)v_0, \quad w = (\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6)w_0. \]
The change $x \mapsto x + v_0 + w_0$ reduces $f$ to the form
\[ f = x^n + x(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) + \Psi \]
for some $\Psi \in (x^2)n^2 + (x)n^4$, which thanks to Morse’s lemma yields $f \sim V(n)$.

5.10. **Lemma.** Let cork $f = 1/6$ and
\[ f = x^n + ax^{m_1} + bx^{m_3} + f^{(4)} + f^{(6)}, \]
where $m_i > 1$ for at least one $i$. Then $f$ is adjacent to $V'$.

**Proof.** Let $m_1 < m_2 < m_3$,
\[ g = x^3 + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x^2 \xi_5 \xi_6 + a \xi_5 \xi_6 (\xi_1 \xi_2 - \xi_3 \xi_4), \]
and $h_\lambda = f + \lambda g$. Then
\[ h_\lambda \sim x^3 + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x^2 \xi_5 \xi_6 + v_0 + \varphi, \]
where $\varphi \in (x)n^4 + n^6$. Express $v_0$ in the form
\[ v_0 = \xi_5 \xi_6 w_1 + \xi_5 w_2 + \xi_6 w_3 + \xi_1 \xi_2 \xi_3 \xi_4. \]
The changes of the form $x \mapsto x + \alpha$ reduce $h_\lambda$ to the form with $v_0 = \xi_5 \xi_6 w_1$ and $w_1(\xi_1 \xi_2 + \xi_3 \xi_4) = 0$. Notice that $w_1^2 \neq 0$ for almost all $\lambda$ and therefore the linear changes in $\xi_1 \xi_2 \xi_3 \xi_4$ send $w_1$ into $\mu(\xi_1 \xi_2 - \xi_3 \xi_4)$. Thus,
\[ h_\lambda = x^3 + (x \xi_1 \xi_2 + \xi_3 \xi_4) + x^2 \xi_5 \xi_6 + a \xi_5 \xi_6 (\xi_1 \xi_2 + \xi_3 \xi_4) + \varphi, \]
where $\varphi \in (x)n^4 + n^6$. Killing $\varphi$ as usual (Morse’s lemma) we get $h_\lambda \sim V'$.

5.11. **Lemma.** Let $V = \langle \xi_1, \ldots, \xi_6 \rangle$.
Then in $\Lambda^3(V)$ there is exactly five orbits of $\text{GL}(V)$ and any element $w \in \Lambda^3 V$ reduces to one of the following:
- a) $0$,
- b) $\xi_1 \xi_3 \xi_5$,
- c) $\xi_1(\xi_3 \xi_5 + \xi_4 \xi_6)$,
d) \( \xi_1 \xi_3 \xi_5 + \xi_2 \xi_4 \xi_6 \),
e) \( \xi_1 \xi_3 \xi_5 + \xi_1 \xi_4 \xi_6 + \xi_2 \xi_4 \xi_5 \).

5.12. Lemma. Let \( \omega \in \Lambda^2(V) \), with \( \text{rk } \omega = 6 \) and \( w \in \Lambda^3(V) \) such that \( \omega \wedge w = 0 \). Then by linear transformations of \( V \) we may reduce \( w \) to one of the forms a) - e) and \( \omega \) to the form: \( \xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6 \).

Proof. Consider all the cases. For a 3-dimensional subspace \( L \subset V \), \( L = \langle \xi_i, \xi_j, \xi_k \rangle \), set \( \Delta_L = \xi_i \xi_j \xi_k \) and \( L^1 = \{ f \in V^* : f|_L = 0 \} \) and for \( f \in V^* \) denote by \( f^c \) an element of \( \Lambda^5 V \) such that if \( f(v) = 0 \) then \( f^c \wedge v = 0 \) (\( f^c \) is defined up to a factor). It is not difficult to see that \( \omega \Delta_L = 0 \) if \( L^1 \) is Lagrangean with respect to \( \omega \) and \( \omega \Delta_L \in \text{Ker } \omega \mid L^1 \) and nonzero otherwise. Let \( t_1, \ldots, t_6 \) be the dual basis of \( \xi_1, \ldots, \xi_6 \).

a) Trivially.

b) \( L^1 = \langle \xi_1, \xi_3, \xi_5 \rangle^\perp \) is Lagrangean with respect to \( \omega \) and therefore \( \omega \) reduces to the indicated form.

c) Let \( L_1 = \langle \xi_1, \xi_3, \xi_5 \rangle, L_2 = \langle \xi_1, \xi_4, \xi_6 \rangle \).

Since \( (\Delta_{L_1} + \Delta_{L_2}) \omega = 0 \), then \( \text{Ker } \omega |_{L_1} = \text{Ker } \omega |_{L_2} \) or, in other words, \( L_1^\perp \) and \( L_2^\perp \) are Langrangean. In any case \( t_2 \in \text{Ker } \omega |_{L_1^\perp + L_2^\perp} \). This makes it clear that \( t_2 \perp \langle t_3, t_4, t_5, t_6 \rangle \) and therefore \( \omega \) is expressed in the form \( a \xi_1 \xi_2 + \omega' \), where \( \omega' \in \Lambda^2(\xi_3, \xi_4, \xi_5, \xi_6) \), \( \omega'(\xi_3 \xi_5 + \xi_4 \xi_6) = 0 \). But the changes in \( \langle \xi_3, \xi_4, \xi_5, \xi_6 \rangle \) preserving \( \xi_3 \xi_5 + \xi_4 \xi_6 \) send \( \omega' \) to the form \( \lambda(\xi_3 \xi_4 + \xi_5 \xi_6) \). After that the change

\[ \xi_3 \rightarrow \xi_3/\lambda, \xi_6 \rightarrow \xi_6/\lambda, \xi_1 \rightarrow \lambda \xi_1, \xi_2 \rightarrow \xi_2/a \lambda \]

reduces \( \omega \) to the form: \( \omega = \xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6 \).

d) Let \( L_1 = \langle \xi_1, \xi_3, \xi_5 \rangle, L_2 = \langle \xi_2, \xi_4, \xi_6 \rangle \). Since \( L_1 \cap L_2 = 0 \) and \( (\Delta_{L_1} + \Delta_{L_2}) \omega = 0 \), then \( L_1^\perp \) and \( L_2^\perp \) are Lagrangean with respect to \( \omega \) and therefore \( \omega \) reduces to the form \( \xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6 \) by the changes of variables in \( \langle \xi_2, \xi_4, \xi_6 \rangle \).

e) Let \( v = \xi_1 \xi_3 \xi_5 + \xi_1 \xi_4 \xi_6 + \xi_2 \xi_4 \xi_5, \quad L_1 = \langle \xi_1, \xi_3, \xi_5 \rangle, \quad L_2 = \langle \xi_1, \xi_4, \xi_6 \rangle, \quad L_3 = \langle \xi_2, \xi_4, \xi_5 \rangle \). Then \( \dim L_i \cap L_j = 1 \) if \( i \neq j \) and \( L_1 \cap L_2 \cap L_3 = 0 \). Since \( (\Delta_{L_1} + \Delta_{L_2} + \Delta_{L_3}) \omega = 0 \) then at least one \( L_i^\perp \) is Langrean with respect to \( \omega \). Without loss of generality we may assume \( i = 1 \). Then since \( (\Delta_{L_2} + \Delta_{L_3}) \omega = 0 \) the by c) \( L_2^\perp \) and \( L_3^\perp \) are also Lagrangean. But then \( \langle \xi_1, \xi_4, \xi_5 \rangle^\perp \) is Lagrangean and \( \omega \) is of the form

\[ \omega = a \xi_1 \xi_2 + b \xi_3 \xi_4 + c \xi_5 \xi_6. \]

The change of variables

\[ \xi_1 \mapsto \sqrt{bc} \xi_1, \quad \xi_2 \mapsto \xi_2/a \sqrt{bc} \]
\[ \xi_4 \mapsto \sqrt{ac} \xi_4, \quad \xi_3 \mapsto \xi_3/b \sqrt{ac} \]
\[ \xi_5 \mapsto \sqrt{ab} \xi_5, \quad \xi_6 \mapsto \xi_4/c \sqrt{ab} \]

reduces \( \omega \) to the form: \( \omega = \xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6 \).
5.13. Lemma. Let cork \( f = 1/7 \) and
\[
f = x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x \xi_5 \xi_6 + f^{(4)} + f^{(6)}.
\]
Then
\[
f \sim x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x \xi_5 \xi_6 + \xi_7 w,
\]
where
\[
w \in A^3(\xi_1, \ldots, \xi_6), \quad w(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) = 0.
\]

Proof. Let \( f^{(4)} = \xi_7 w_1 + w_2 \mod (x) n^4 \). The change of variables of the form
\[
x \mapsto x + \xi_7 v_1 + v_2 + v_3 + v_4,
\]
where \( v_j \in A^4(\xi_1, \ldots, \xi_6) \), reduces \( f \) to the form
\[
f = x^n + x(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) + \xi_7 w + \varphi,
\]
where \( \varphi \in (x^2) n^2 + (x) n^4 \),
\[
w \in A^3(\xi_1, \ldots, \xi_6), \quad w(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) = 0.
\]
Morse's lemma implies that \( f \sim f - \varphi \).

5.14. Lemma. If cork \( f = 1/7 \) and
\[
f = x^n + x(\xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6) + f^{(4)} + f^{(6)}
\]
then \( f \) is equivalent to one of the singularities \( VI_1(n) - VI_5(n) \).

Proof follows from Lemmas 5.11–5.13.

5.15. Lemma. Let cork \( f = 1/7 \) and
\[
f = x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x \xi_5 \xi_6 + f^{(4)} + f^{(6)}, \quad m_1, m_2 > 1.
\]
Then \( f \) is adjacent to \( VI_2 \).

Proof. Set \( f + \lambda VI_2 = g_\lambda \). Then \( g_\lambda \sim x^3 + x \xi_1 \xi_2 + f^{(4)} + f^{(6)} \) for a sufficiently small \( \lambda \). Let \( q = f^{(4)} \mod n^4(x) + n^6 \). Present \( q \) in the form
\[
q = v_0 + \xi_1 v_1 + \xi_2 v_2 + \xi_1 \xi_2 v_3,
\]
where \( v_0 \in A^4(\xi_7, \xi_6, \xi_5, \xi_4, \xi_3), \quad v_1, v_2 \in A^3(\xi_7, \xi_6, \xi_5, \xi_4, \xi_3) \) and \( v_3 \in A^2(\xi_7, \xi_6, \xi_5, \xi_4, \xi_3) \). For a sufficiently small \( \lambda \) the vector \( q \) is generic in \( A^4(\xi_1, \ldots, \xi_7) \).
Without loss of generality we may set \( v_0 = \xi_7 \xi_6 \xi_5 \xi_4 \).

The changes of the form \( \xi_i \mapsto \xi_i + a \xi_1 + b \xi_2, i \geq 4 \), enable us to reduce \( q \) to the form
\[
q = \xi_7 \xi_6 \xi_5 \xi_4 + \xi_1 \xi_3 w_1 + \xi_2 \xi_3 w_2 + \xi_1 \xi_2 w_3,
\]
without affecting \( g^{(2)}_\lambda \). The changes
\[
\xi_1 \mapsto a \xi_1 + b \xi_2, \xi_2 \mapsto c \xi_1 - a \xi_1, \xi_3 \mapsto d_1 \xi_1 + d_2 \xi_2
\]
reduce \( q \) to the form

\[
q = \xi_7, \xi_6, \xi_5, \xi_4 + \xi_1, \xi_3, \xi_6, + \xi_2, \xi_3, \xi_5, \xi_4 + \xi_1 \xi_2 \xi_3 z_0 + \\
+ \xi_1 \xi_2 (\gamma_1 \xi_5 \xi_6 + \gamma_2 \xi_5 \xi_7 + \gamma_3 \xi_4 \xi_6 + \gamma_4 \xi_4 \xi_7) + \\
+ \xi_1 \xi_2 \xi_3 (\delta_1 \xi_7 + \delta_2 \xi_6 + \delta_3 \xi_5 + \delta_4 \xi_4).
\]

The composition of the changes from the group \( \text{SL}(\xi_5, \xi_4) \times \text{SL}(\xi_7, \xi_6) \) with \( \xi_3 \mapsto \lambda \xi_3 \) reduces \( q \) to the form

\[
\xi_7 \xi_6 \xi_5 \xi_4 + \xi_3 \xi_1 \xi_7 \xi_6 + \xi_3 \xi_2 \xi_5 \xi_4 + \xi_1 \xi_2 \xi_7 \xi_5 + \xi_1 \xi_2 \xi_4 \xi_6 + a \xi_1 \xi_2 \xi_3 \xi_4.
\]

Therefore, \( g_\lambda \sim \text{VI}_2 + \varphi \), where \( \varphi \in n^4(x) + n^6 \). Let us show that \( g_\lambda \sim g \text{VI}_2 \).

First, by Morse’s lemma

\[
g_\lambda \sim x^3 + \xi_7 \xi_6 \xi_5 \xi_4 + \xi_1 \xi_3 \xi_7 \xi_6 + \xi_2 \xi_3 \xi_5 \xi_4 + \\
+ a \xi_1 \xi_2 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_7 \xi_5 + \xi_1 \xi_2 \xi_4 \xi_6 + x \cdot \Psi(x, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7).
\]

Since \( \xi_7 \xi_6 \xi_5 \xi_4 \) belongs to the open orbit of \( \text{GL}(5) \) in \( A^4(\xi_7, \ldots, \xi_3) \) then there exists \( g \in \text{gl}(5) \cdot x \) such that

\[
g(\xi_7 \xi_6 \xi_5 \xi_4) = x \cdot \Psi(0, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7).
\]

Therefore \( g_\lambda \sim \text{VI}_2 \cdot \Psi(\xi_3, \ldots, \xi_5, x) \cdot x^2 \), but then \( g_\lambda \sim \text{VI}_2 \).

**5.16. Lemma.** Let cork \( f = 1/7 \) and

\[
f = x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x^2 \xi_5 \xi_6 + f^{(4)} + f^{(6)},
\]

where \( n > 3 \), \( t = f^{(4)} \mod (\xi_1 \xi_2 + \xi_3 \xi_4) + (x) + (A^4(\xi_1, \ldots, \xi_6)) \).

a) If \( \text{rk} t = 3 \), then \( f \sim \text{VI}_6(n) \).

b) If \( \text{rk} t < 3 \), then \( f \) is adjacent to \( \text{IV}' \).

**Proof.** Let us express \( t \) in the form

\[
t = \xi_7 \xi_5 \xi_6 t_0 + \xi_7 \xi_5 w_1 + \xi_7 \xi_6 w_2,
\]

where \( t_0(\xi_1, \ldots, \xi_4), w_1, w_2 \in \Lambda^2(\xi_1, \ldots, \xi_4) \)

and \( w_2(\xi_1 \xi_2 + \xi_3 \xi_4) = w_2(\xi_1 \xi_2 + \xi_3 \xi_4) = 0 \).

Let \( t_0 \neq 0 \). Then by the changes of variables in \( \langle \xi_1, \ldots, \xi_4 \rangle \) we may reduce \( t_0 \) to \( \xi_1 \) without affecting \( f^{(2)} \).

The change \( \xi_5 \mapsto \xi_5 + \sum_{1 \leq i \leq 4} a_i \xi_i, \xi_6 \mapsto \xi_6 + \sum_{1 \leq i \leq 4} b_i \xi_i \), reduces \( t \) to the form

\[
t = \xi_7 \xi_5 \xi_6 t_1 + \xi_7 \xi_5 (a \xi_2 \xi_3 + b \xi_2 \xi_4) + \xi_7 \xi_6 (c \xi_2 \xi_3 + d \xi_2 \xi_4).
\]

Then the composition of the changes from \( \text{SL}(\xi_5, \xi_6) \) and the transformation

\[
\xi_7 \mapsto \lambda \xi_7, \xi_1 \mapsto \lambda^{-1} \xi_1, \xi_2 \mapsto \lambda \xi_2
\]
reduces t to the form \( t = \xi_7 \xi_5 \xi_6 \xi_1 + \xi_7 \xi_5 \xi_2 \xi_3 + \xi_7 \xi_6 \xi_2 \xi_4. \)

Let \( f^{(4)} = t + t_1 \mod (x). \) Since \( t_1 \in (\xi_1 \xi_2 + \xi_3 \xi_4) + A^4(\xi_1, \xi_2, \ldots, \xi_6), \) then \( t_1 \) splits into the sum \( t_1 = (\xi_1 \xi_2 + \xi_3 \xi_4)v_0 + \xi_5 \xi_6 v_1, \) where \( v_1 \in A^2(\xi_1, \ldots, \xi_4), \) and \( v_1 \cdot \omega = 0. \)

The first summand may be put to zero with the change \( x \mapsto x = v_0, \) and the second one with the change \( \xi_7 \mapsto \xi_7 + \sum_{i \leq 4} a_i \xi_i, \xi_5 \mapsto \xi_5 + \sum_{i \leq 4} b_i \xi_i, \xi_6 \mapsto \xi_6 + \sum_{i \leq 4} c_i \xi_i. \)

Now making use of Morse's lemma let us kill all terms that belong to \( n^2(x^2) + n^4(x) + n^6, \) and get \( f \sim V_{16}(n). \)

Let \( \text{rk } t < 3. \) Then \( t \) reduces to the form \( t = \xi_7 \xi_5 \xi_6 \xi_1 + k \xi_7 \xi_5 \xi_2 \xi_3. \) Consider the singularity \( h_\lambda = f + \lambda \xi_1 \xi_3. \) For \( \lambda \neq 0 \) it satisfies the conditions of Corollary 5.2.4 and therefore it is adjacent to \( IV'. \)

Let \( t_0 = 0. \) Then the composition of the changes from \( \text{SL}_2(\xi_5, \xi_6) \) with \( \xi_7 \mapsto \lambda \xi_7, \) reduces to the form \( t = \xi_7 \xi_5 \xi_1 \xi_3 + \xi_7 \xi_6 \xi_2 \xi_4, \) therefore \( \text{rk } t = 0. \) Let \( h_\lambda = f + \lambda \xi_1 \xi_4. \) Then, by the reasons similar to the preceding case \( h_\lambda \) is adjacent to \( IV'. \)

5.16.1. Corollary. Let

\[
    f = x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x^m \xi_5 \xi_6 + f^{(4)} + f^{(6)}.
\]

If \( m > 2, \) then \( f \) is adjacent to \( IV'. \)

Proof. Let \( t \) be the same as in Lemma. It suffices to prove Corollary for the case \( \text{rk } t = 3. \) As had been proved in the proof of Lemma in this case

\[
    f \sim x^n + x \xi_1 \xi_2 + x \xi_3 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 \xi_5 \xi_6 \xi_1 + \xi_7 \xi_5 \xi_2 \xi_3 + \xi_7 \xi_6 \xi_2 \xi_4.
\]

Let \( g_\lambda = f + \lambda \xi_1 \xi_3, \) where \( \lambda \neq 0. \) Then

\[
    g_\lambda \sim x^n + x^2 \xi_2 \xi_4 + x^m \xi_5 \xi_6 + \xi_7 \xi_6 \cdot \xi_2 \xi_4.
\]

Since \( m > 2, \) then \( g_\lambda \) falls into the conditions of Corollary 5.2.2 and therefore is adjacent to \( IV'. \)

5.17. Lemma. Let \( \text{cork } f = 1/7 \) and

\[
    f = x^3 + x \xi_1 \xi_2 + x \xi_3 \xi_4 + f^{(4)} + f^{(6)}
\]

with \( f^{(4)} = t_0 \mod (A^2(\xi_1, \ldots, \xi_4)) + (x). \)

a) If \( t_0 = 0, \) then \( f \) is adjacent to \( VI'. \)

b) If \( t_0 \neq 0, \) then
\[
f \sim x^3 + x \xi_1 \xi_2 + x \xi_3 \xi_4 + \xi_7 \xi_6 \xi_5 \xi_1 + a \xi_7 \xi_6 \xi_2 \xi_3 + b \xi_7 \xi_5 \xi_2 \xi_4 + c \xi_6 \xi_5 \xi_3 \xi_4.
\]

**Proof.** Let
\[
f = x^3 + x \xi_1 \xi_2 + x \xi_3 \xi_4 + f^{(4)} + f^{(6)}.
\]
Similarly to the case of Lemma 4.4.16, \( f \sim f + \phi \), if \( \phi \in (A^3(\xi_1, \ldots, \xi_4)) + (x) \).
Therefore without affecting \( f^{(0)}, f^{(2)}, \) we may reduce \( f \) to the form with \( f^{(4)} + f^{(6)} \in L = \langle \xi_1, \ldots, \xi_4 \rangle \otimes A^3(\xi_7, \xi_5, \xi_6) \oplus A^2(\xi_1, \ldots, \xi_4) \otimes A^2(\xi_7, \xi_5, \xi_6) \).

On the other hand, the two reduced singularities \( f_1 \) and \( f_2 \) are equivalent only if so are \( f_2^{(4)} \) and \( f_1^{(4)} \) with respect to the subgroup \( G \) of linear transformations in \( \langle \xi_1, \ldots, \xi_7 \rangle \) preversing \( \xi_1 \xi_2 + \xi_3 \xi_4 \). Let
\[
f^{(4)} = \xi_7 \xi_6 \xi_5 v_0 + \xi_7 \xi_6 w_1 + \xi_7 \xi_5 w + \xi_6 \xi_5 w_3
\]
where \( w_i \in A^2(\xi_1, \ldots, \xi_4) \), \( v_0 \in \langle \xi_1, \ldots, \xi_4 \rangle \).

a) Let \( v_0 = 0 \). Then a generic element \( f^{(4)} \) reduces with the help of transformations from \( G \) to the form
\[
\xi^{(4)} = \xi_7 \xi_5 \xi_2 \xi_3 + \xi_7 \xi_6 \xi_1 \xi_4 + \xi_7 \xi_6 (\xi_1 \xi_2 / a + a \xi_3 \xi_4).
\]
Implies that \( f \) is adjacent to \( VI_7 \).

b) Let \( v_0 \neq 0 \). Then \( v_0 \) reduces to \( \xi_1 \) with the help of transformations from \( SP(\xi_1, \ldots, \xi_4) \) and \( w_1, w_2, w_3 \) could be forced to belong to \( A^2(\xi_2, \xi_3, \xi_4) \) with the help of transitions \( \xi_i \mapsto \xi_i + \sum_{1 \leq j \leq 4} a_{ij} \xi_j \) for \( i = 5, 6, 7 \).

Therefore
\[
f^{(4)} \sim \xi_7 \xi_6 \xi_5 \xi_1 + \xi_7 \xi_6 (a_1 \xi_2 \xi_3 + b_1 \xi_2 \xi_4 + c_1 \xi_3 \xi_4) + \\
+ \xi_7 \xi_5 (a_2 \xi_2 \xi_3 + b_2 \xi_2 \xi_4 + c_2 \xi_3 \xi_4) + \xi_6 \xi_5 (a_3 \xi_2 \xi_3 + b_3 \xi_2 \xi_4 + c_3 \xi_3 \xi_4).
\]
The linear changes from \( GL(\xi_5, \xi_6, \xi_7) \) enable us to kill \( b_1, c_1, a_2, c_2; a_3 \) and \( b_3 \).

**5.18. Lemma.** If \( f \) corresponds to case 23 of Classifier and \( a, b, c \neq 0 \), then \( f \sim VI_4(3) \). If \( a = 0 \) and either \( b \) or \( c \) are nonzero then \( f \sim VI_7 \). If \( a = 0, b, c \neq 0 \), then \( f \sim VI_8 \). If \( a = 0, b = c = 0 \), then \( f \sim VI_9 \). If \( a = 0, \) and either \( b \) or \( c \) are nonzero, then \( f \sim VI_{10} \). If \( a = b = c = 0 \), then \( f \sim VI_{11} \).

**Proof.** is straightforward.

**5.19. Lemma.** Let cork \( f = (2/2 k) \). Then \( f \) is adjacent to \( VII_3 \).

**Proof.** Consider
\[
g_\lambda = \lambda (x^3 + xy^2 + x \xi_1 \xi_2 + ay \xi_1 \xi_2 + \xi_3 \xi_4 + \ldots + \xi_{2k-1} \xi_{2k}) + f.
\]
Then $g_\lambda$ is a singularity of corank $(2, 2)$. The change of coordinates $(x, y)$ reduces $g_\lambda$ for a sufficiently small $\lambda$ to the form

$$g_\lambda = x^3 + xy^2 + (bx + dy)\xi_1\xi_2 + \varphi(x, y)\xi_1\xi_2,$$

where $\varphi(x, y) \in m^2$ (see [A]). Let $\varphi(x, y) = (bx + dy)\Psi(x, y) + xy f(x, y)$. The change

$$\xi_\lambda \mapsto \xi_2/b, \; \xi_1 \mapsto \xi_1(1 + \Psi(x, y)), \; y \mapsto y + f(x, y)\xi_1\xi_2,$$

reduces $g_\lambda$ to the form $\text{VII}'_3$.

**5.20. Lemma.** Let cork $f = (2/2 k + 1)$. If $k > 1$, then $f$ is adjacent to $\text{VII}'_2$.

**Proof.** It suffices to prove the case $k = 2$. Let

$$f^{(2)} = \varphi x + \Psi y \mod m^4,$$

where $\varphi, \Psi \in A^2(\xi_1, \ldots, \xi_5)$.

Reduce $\varphi$ to the form $\varphi = \xi_1\xi_3 + a\xi_2\xi_4$ by linear changes in $\langle \xi_1, \ldots, \xi_5 \rangle$. Let

$$g_\lambda = \lambda(x^3 + xy^2 + ay\xi_1\xi_2 + \xi_3\xi_4) + f.$$

Then

$$g_\lambda \sim x^3 + xy^2 + x\xi_1\xi_2 + \varphi,$$

where $\varphi \in n^2 m^2$.

The term $\varphi$ is killed in the same way as in Lemma 5.19. Adding $\mu y\xi_1\xi_2$ to $g_\lambda$ we see that $g_\lambda$ is adjacent to $\text{VII}'_3$.

**5.21. Lemma.** Let cork $f = (2/3)$, and $t = f^{(2)} \mod m^4$.

a) If $\text{rk} \; t = 1$, then $f$ is adjacent to $\text{VII}'_3$.

b) If $\text{rk} \; t = 2$, then $f^{(0)}$ is nonsimple, and $f$ is adjacent to either $\text{VII}_1'$ or $\text{VII}'_2$.

c) If $\text{rk} \; t = 2$ and $f^{(0)}$ is simple then $f$ is equivalent to one of $\text{VII}_1' - \text{VII}_4'$ depending on the equivalence class of $f^{(0)}$.

**Proof.** If $\text{rk} \; t = 1$, then $t$ reduces by a linear change of coordinates $\xi_1, \xi_2, \xi_3$ to the form $t = (ax + by)\xi_1\xi_2$. By the reasons similar to those from Lemma 5.20, $f$ is adjacent to $\text{VII}'_2$.

If $\text{rk} \; t = 2$, then $t$ reduces to the form $t = x\xi_1\xi_2 + y\xi_2\xi_3$ by a linear change of coordinates $\xi_1, \xi_2, \xi_3$. Let $f = f^{(0)} + x\xi_1\xi_2 + y\xi_2\xi_3 + \varphi$, where $\varphi \in n^2 m^2$. Morse's lemma implies that we may kill $\varphi$ without affecting $f^{(0)}$. Therefore adjacency and equivalence for singularities of the form $f$ are equivalent to adjacency and equivalence for the singularities of the form $f^{(0)}$. Now the statements b) and c) follow from the classifical results for the germs in even variables (see [A]).
5.22. Lemma. Let $cork f = 0/q$, $q \leq 7$. Then

a) if $f^{(4)} = 0$ then $f = f^{(6)} \sim I_4$ if $f^{(6)} \neq 0$ (or $I_{11}$ if $f^{(6)} = 0$);
b) if $f^{(4)} = 0$ then $f \sim f^{(4)}$.

Proof. a) If $q = 6$ the statement is obvious. If $q = 7$ then in $\Lambda^6 \langle \xi_1, \ldots, \xi_7 \rangle$ there is only one nonzero GL(7)-orbit: that of $\xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6$.

b) If $f^{(6)} = 0$ we have nothing to prove. Let $f^{(6)} \neq 0$. Then reduce $f^{(6)}$ to the form $I_4 = \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6$ and represent $f^{(4)}$ in the form $f^{(1)} = \xi_7 \varphi + r$, where $\varphi \in \Lambda^3 \langle e_1, \ldots, e_6 \rangle$, $r \in \Lambda^4 \langle e_1, \ldots, e_6 \rangle$. If $\varphi \neq 0$ then there exists $\psi \in \Lambda^3 \langle \xi_1, \ldots, \xi_6 \rangle$ such that $\varphi\psi = f^{(6)}$ and the change of variables $\xi_7 \mapsto \xi_7 + \psi$ kills $f^{(6)}$.

If $\varphi = 0$ then for at least one of $\xi_1, \ldots, \xi_6$ (say for $\xi_1$) we have the decomposition $f^{(4)} = \xi_1 \varphi + r$ for $r \in \Lambda^4 \langle \xi_2, \xi_3, \ldots, \xi_6 \rangle$ with a nonzero $\varphi$. Taking now $\psi$ such that $\varphi\psi = f^{(6)}$ we reduce $f$ to $f^{(4)}$ by the change $\xi_1 \mapsto \xi_1 + \psi$.

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REFERENCES


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