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AN F. AND M. RIESZ THEOREM FOR COMPACT LIE GROUPS

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1. Introduction.

This note is a sequel to [B1], where the author proved an F. and M. Riesz-type theorem for compact groups whose center contains a circle group $\mathsf{T} = \{e^{i\theta}: \theta \in (-\pi, \pi]\}$.

Here we will use the techniques of [B1] to prove an F. and M. Riesz theorem for compact Lie groups G. The main idea is to let the rôle of the circle group in [B1] be taken by an abelian connected subgroup of G, for example a Cartan subgroup.

The main difference between the resulting F. and M. Riesz theorem, theorem I below, and the one of [B1] is the following. In the present situation it no longer suffices to impose conditions only on the support of the Fourier transform of a measure μ to be able to conclude that μ is absolutely continuous with respect to the Haar measure on G. If we denote the Fourier transform by $\hat{\mu}$ then one also needs a condition on the orthogonal complement of $\operatorname{Ker} \hat{\mu}(\tau)$ for τ in $\operatorname{supp} \hat{\mu}$. Something similar occurs in an F. and M. Riesz theorem for the Heisenberg group proved by the author in [B2]. Nevertheless, the F. and M. Riesz theorem of [B1] can be considered as a special case of the one proved here.

An interesting corollary of theorem I is the following. If G is semisimple and if μ is a measure on G such that for all irreducible unitary representations τ of G, $\hat{\mu}(\tau) = 0$ on the orthogonal complement of the highest weight subspace of τ, then μ is absolutely continuous with respect to Haar measure. See theorem II below.

2.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be an abelian subalgebra of $\mathfrak{g}$ and let T be the analytic subgroup of G associated with $\mathfrak{h}$. For

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example, $\mathfrak{h}$ could be the center of $\mathfrak{g}$ or $\mathfrak{h}$ could be a Cartan subalgebra of $\mathfrak{g}$ in which case T is a maximal torus in G.

Let $\hat{G}$, the dual of G, be a maximal set of pairwise inequivalent irreducible unitary representations of G. If $\tau \in \hat{G}$ we let $V(\tau)$ denote the representation space of τ; $N(\tau) := \dim_{\mathbf{C}} V(\tau)$.

Let $\tau \in G$ and let $d\tau: \mathfrak{g} \to \operatorname{End}(V(\tau))$ be the differential of τ at e. Write $V = V(\tau)$ and let

$$V = \sum_{\lambda \in (\mathfrak{h}^{\mathbf{C}})'} V_\lambda \tag{1}$$

be the *weight space decomposition* of τ with respect to $\mathfrak{h}$. Here $\mathfrak{h}^{\mathbf{C}} = \mathfrak{h} \otimes_{\mathbf{R}} \mathbf{C}$, the complexified algebra and $(\mathfrak{h}^{\mathbf{C}})' =$ complex dual of $\mathfrak{h}^{\mathbf{C}}$. Recall that (1) means that if $v \in V_\lambda$ then for all $H \in \mathfrak{h}$:

$$d\tau(H)v = \lambda(H)v. \tag{2}$$

Let $W(\tau) := \{\lambda \in (\mathfrak{h}^{\mathbf{C}})' : V_\lambda \neq 0\}$, $V = V(\tau)$, be the set of weights of τ with respect to $\mathfrak{h}$. The weight space V_λ, $\lambda \in W(\tau)$, are pairwise orthogonal. Since τ is unitary each $\lambda(H)$ is imaginary for all $H \in \mathfrak{h}$, $\lambda \in W(\tau)$. It follows from (2) that $\tau(\exp H)v = e^{\lambda(H)}v$. In particular, each $\lambda \in W(\tau)$ is *analytically integral*: $\exp H = 1$ implies that $\lambda(H) \in 2\pi i\mathbf{Z}$.

Let $\Lambda_a \subseteq i\mathfrak{h}'$, $\mathfrak{h}' =$ real dual of $\mathfrak{h}$, denote the lattice of analytically integral functionals. $\hat{T}$, the dual of T, can be identified with Λ_a via the map

$$\Lambda_a \in \lambda \to \chi_\lambda \in \hat{T}, \quad \chi_\lambda(\exp H) := e^{\lambda(H)}.$$

Denote by $M(G)$ the space of finite Borel measures on G; dg and dt will be the normalized Haar measures on G and T, respectively. If $\mu \in M(G)$, $\hat{\mu}$ will denote the Fourier transform of μ:

$$\hat{\mu}(\tau) = \int_G {}^t\tau(g^{-1})\,dg, \quad \tau \in \hat{G};$$

the upper index t means taking the transpose. If $f \in L^1(G, dg) = L^1(G)$, $\hat{f} := (f\,dg)^\wedge$. Let $\mathfrak{T}(G)$ denote the space of trigonometric polynomials on G: $\mathfrak{T}(G) = \{f \in L^1(G) : \hat{f}(\tau) = 0$ for all but finitely many $\tau \in \hat{G}\}$. We use similar notions and notations for T.

3.

We will consider subsets E of Λ_a which satisfy the following condition:

For some $p < 1$ the linear functionals

$$f \to \hat{f}(\lambda) = \int_T f(t)\chi_\lambda(t^{-1})\,dt, \quad \lambda \in E, \tag{*}$$

are L^p-continuous on $\mathfrak{T}_E(T) := \{f \in \mathfrak{T}(T) : \hat{f}(\lambda) = 0$ if $\lambda \notin E\}$.

EXAMPLE. Choose a basis of Λ_a over $\mathbf{Z}$ so that Λ_a is identified with $\mathbf{Z}^r$, $r = \dim_{\mathbf{R}} \mathfrak{h}$, and T with $\mathbf{T}^r$. Let $\pi_j: Z^r \to \mathbf{Z}$ denote the projection onto the j-th coordinate. Suppose that $E \subseteq \Lambda_a = \mathbf{Z}^r$ satisfies the following condition:

$$\forall k, 0 \leq k < r: \quad \forall (m_1, \ldots, m_k) \in \mathbf{Z}^k:$$

$$(**) \qquad \pi_{k+1}((\pi_1 \times \ldots \times \pi_k)^{-1}(m_1, \ldots, m_k) \cap E) \subseteq \mathbf{Z}$$

is bounded from above or from below.

Then E satisfies condition (*). For $r = 2$ this was noted by Aleksandrov [A, appendix, theorem 2]. It is a matter of routine to extend Aleksandrov's argument to larger r; cf. [B2, section 3] where this is done for $\mathbf{R}^n$.

4.

Let $P_{\tau,\lambda}$ denote the orthogonal projection of $V = V(\tau)$ onto V_λ, $\lambda \in W(\tau)$. Our F. and M. Riesz theorem for G is the following result:

THEOREM I. *Let $E \subseteq \Lambda_a$ satisfy condition (*) in section 3 for some $p < 1$. Let $\Delta \subseteq \hat{G}$ and suppose that to each $\tau \in \Delta$ there is associated a subset $\Gamma(\tau) \subseteq W(\tau)$ such that*

(i) $$\forall \lambda \in E: \text{ the set } \{\tau \in \Delta: \lambda \in \Gamma(\tau)\} \text{ is finite,}$$

(ii) $$\forall \tau \in \Delta: \quad \Gamma(\tau) \subseteq E.$$

Let μ in $M(G)$ be such that $\hat{\mu}(\tau) = 0$ for $\tau \notin \Delta$ and such that for $\tau \in \Delta$,

(iii) $$\hat{\mu}(\tau) \circ P_{\tau,\lambda} = 0 \quad \text{if } \lambda \notin \Gamma(\tau).$$

Then μ is absolutely continuous with respect to the Haar measure dg.

REMARKS. (1) Condition (iii) may be reformulated in the following way: for all $\tau \in \Delta$,

$$\sum_{\lambda \notin \Gamma(\tau)} V_\lambda \subseteq \operatorname{Ker} \hat{\mu}(\tau) \quad (V = V(\tau) \text{ again})$$

or equivalently, if $W^\perp$ denotes the orthogonal complement of a subspace W of V,

$$(\operatorname{Ker} \hat{\mu}(\tau))^\perp \subseteq \sum_{\lambda \in \Gamma(\tau)} V_\lambda.$$

(2) Theorem 3.2 of [B1] is contained in theorem I: take $\mathfrak{h}$ equal to a one-dimensional subalgebra of the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. Then $T = \mathbf{T} \subseteq Z(G)$ and by Schur's lemma there exists for each $\tau \in \hat{G}$ an $n(\tau) \in \mathbf{Z}$ such that $W(\tau) = \{\operatorname{in}(\tau)\}$. Now take for E a subset of $\mathbf{Z}$ which is bounded from below or from above.

PROOF OF THEOREM I. Let $X_\mu := \{f * \mu: f \in \mathfrak{T}(G)\}$ and let X_μ^p denote the closure of X_μ in $L^p(G) = L^p(G, dg)$, $p < 1$ as in theorem I. To prove that μ is absolutely

continuous with respect to dg it suffices, by [B1, theorem 2.7] to show that $X_\mu^p \cap \mathfrak{T}(G)$ has sufficiently many L^p-continuous linear functionals to separate its points.

Introduce the auxiliary spaces

$$\mathfrak{T}_\tau(G) = \operatorname{span}\{(\tau(\cdot)v, w): v, w \in V(\tau)\}, \tag{3}$$

$$\mathfrak{T}_{\tau,\lambda}(G) = \operatorname{span}\{(\tau(\cdot)v, w): v \in V_\lambda, w \in V = V(\tau)\}, \quad \tau \in \hat{G}, \lambda \in W(\tau).$$

The spaces $\mathfrak{T}_{\tau,\lambda}(G)$, $\lambda \in W(\tau)$, are pairwise orthogonal in $L^2(G, dg)$ and together they span $\mathfrak{T}_\tau(G)$. It is easily seen that

$$\mathfrak{T}_{\tau,\lambda}(G) = \{f \in \mathfrak{T}_\tau(G): f(gt) = \chi_\lambda(t) f(g), \forall g \in G, t \in T\}. \tag{4}$$

We claim that the hypotheses on μ in theorem I imply that

$$X_\mu \subseteq \sum_{\tau \in \Delta} \sum_{\lambda \in \Gamma(\tau)} \mathfrak{T}_{\tau,\lambda}(G). \tag{5}$$

To prove (5), let $\phi := f * \mu \in X_\mu$. Then $\hat{\phi}(\tau) = 0$ if $\tau \notin \Delta$ and $\hat{\phi}(\tau) \circ P_{\tau,\lambda} = 0$ if $\tau \in \Delta$, $\lambda \notin \Gamma(\tau)$. For each $\tau \in \Delta$ let $\{e_1^\tau, \ldots, e_{N(\tau)}^\tau\}$ be an orthonormal basis of $V = V(\tau)$ such that each e_j^τ is in some V_λ. Denote by $\tau_{jk}(g)$ and $\hat{\phi}(\tau)_{jk}$ the matrix elements of $\tau(g)$ and $\hat{\phi}(\tau)$ with respect to this basis. Then

$$\phi(g) = \sum_{\tau \in \Delta} N(\tau) \cdot \sum_{j,k} \hat{\phi}(\tau)))_{jk} \tau_{jk}(g)$$

and $\hat{\phi}(\tau)_{jk} = (\hat{\phi}(\tau) e_k^\tau, e_j^\tau) = 0$ unless $e_k^\tau \in V_\lambda$ with $\lambda \in \Gamma(\tau)$. This implies (5), by definition of $\mathfrak{T}_{\tau,\lambda}(G)$.

Denote the space on the right hand side of (5) by Y. It now suffices to show that the following two statements hold:

$$Y^p \cap \mathfrak{T}(G) = Y, \text{ where } Y^p = \text{closure of } Y \text{ in } L^p(G, dg); \tag{6}$$

(7) Y has sufficiently many L^p-continuous linear functionals to separate it points.

Let $Q_{\tau,\lambda}$ denote the orthogonal projection of $L^2(G)$ onto $\mathfrak{T}_{\tau,\lambda}(G)$. For $\lambda \in \Lambda_a$ define $\Pi_\lambda: \mathfrak{T}(G) \to \mathfrak{T}(G)$ by

$$\Pi_\lambda f(g) := \int_T f(gt) \chi_\lambda(t^{-1})\, dt. \tag{8}$$

Note that for f in $\mathfrak{T}(G)$:

$$\Pi_\lambda f = \sum_{\tau \in \hat{G}: \lambda \in W(\tau)} Q_{\tau,\lambda} f. \tag{9}$$

If f is in Y the sum in (9) is over $\{\tau \in \Delta: \lambda \in \Gamma(\tau)\}$.

Let $f \in Y$. Since $\Gamma(\tau) \subset E$ for all $\tau \in \Delta$, the functions $t \to f(gt)$, $g \in G$ fixed, are in

$\mathfrak{T}_E(T)$. Since E satisfies condition (*) in section 3 for $p < 1$,

$$|\Pi_\lambda f(g)|^p \leqq C(\lambda, p) \int_T |f(gt)|^p \, dt, \quad \forall g \in G.$$

Integration over G yields the inequality

$$\|\Pi_\lambda f\|_p \leqq C(\lambda, p) \|f\|_p, \quad \lambda \in E, f \in Y.$$

Furthermore, Π_λ projects Y onto the subspace

$$\sum_{\tau \in \Delta:\, \lambda \in \Gamma(\tau)} \mathfrak{T}_{\tau,\lambda}(G)$$

of $\mathfrak{T}(G)$, which is finite dimensional by condition (i).

As in [B1] one now easily shows that (6) holds and that the linear functionals

$$f \to Q_{\tau,\lambda} f(g), \quad \tau \in \Delta, \lambda \in \Gamma(\tau), g \in G,$$

are all L^p-continuous on Y, which proves (7) and thereby the theorem.

5. Example.

We take a new look at the example of the unit sphere $S \subseteq \mathbf{C}^2$ which was also considered in [B1]. In [B1] S was identified with $\mathscr{U}(2)/\mathscr{U}(1)$, $\mathscr{U}(n)$ being the unitary group and the F. and M. Riesz theorem of [B1], applied to $\mathscr{U}(2)$, gave an F. and M. Riesz theorem for S; cf. [B1, theorem 1.1].

Here we identify S with $\mathscr{SU}(2)$ via

$$(\alpha, \beta) \in S \to \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathscr{SU}(2)$$

One can take $\mathscr{SU}(2)^\wedge = \{\tau^l : l = 0, 1/2, 1, 3/2, \ldots\}$, where the representation τ^l acts on the space $V(l) := \{F(X, Y)\colon F$ homogeneous polynomial in X, Y of degree $2l\}$ by

$$\left(\tau^l \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} F\right)(X, Y) = F(\bar{\alpha} X + \bar{\beta} Y, -\beta X + \alpha Y).$$

Take $T = \{t_\theta = \operatorname{diag}(e^{-i\theta}, e^{i\theta})\colon \ \theta \in (-\pi, \pi]\}$. A basis of $V(l)$ is given by

$$e_j^l(X, Y) = X^{l+j} Y^{l-j}, \quad j = -l, -l+1, \ldots, l-1, l.$$

Since $\tau^l(t_\theta) e_j^l = e^{2ij\theta} e_j^l$, we can identify $W(\tau^l)$ with $\{-2l, -2l+2, \ldots, 2l-2, 2l\}$. One can now apply theorem I with $\Delta = \mathscr{SU}(2)^\wedge$, $E \subset \hat{T} = \mathbf{Z}$ a subset which is bounded from above or from below and $\Gamma(\tau^l) \subset W(\tau^l)$ such that $\forall k \in \mathbf{Z}$: $\{l\colon k \in \Gamma(\tau^l)\}$ is finite. The reader may check that such an application of theorem

I again yields theorem 1.1 of [B1], in view of the equality

$$H(l-k, l+k) = \mathfrak{T}_{\tau^l, k}(\mathcal{S}\mathcal{U}(2))$$

(see [B1] or [R] for the definition of the spaces $H(p, q)$).

6.

Note that in the above example the special choice $E = \mathbf{N}$, $\Gamma(\tau^l) = \{2l\}$ is allowed. This is a special instance of a more general theorem.

Let G be a semisimple compact connected linear Lie group. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Introduce a lexicographic ordering on $i\mathfrak{h}'$ with respect to some arbitrary but fixed basis of $i\mathfrak{h}'$ (cf. [K, chapter IV, section 3]). Let Δ^+ denote the set of positive roots with respect to this ordering. Finally, let $\langle , \rangle$ denote the real inner product on $i\mathfrak{h}'$ induced by the trace form.

THEOREM II. *Let $\mu \in M(G)$ be such that for all $\tau \in \hat{G}$, $\hat{\mu}(\tau)v = 0$ if v is orthogonal to the highest weight subspace of τ. Then μ is absolutely continuous with respect to the Haar measure.*

PROOF. Recall the theorem of the highest weight (cf. [K, theorem 4.28]): The map which sends $\tau \in \hat{G}$ to the highest weight in $W(\tau)$ is a bijection from G onto the set of dominant analytically integral linear functionals. Here $\lambda \in i\mathfrak{h}'$ is called *dominant* if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$. Actually, we will only need to know that this map is into the set of dominant functionals in Λ_a and injective.

First suppose that G is simply connected. Then Λ_a is equal to the lattice L_a of algebraically integral forms ([K, theorem 4.28]; see [K, chapter IV, section 5] for the definition of L_a). We now apply theorem I with $\Delta = \hat{G}$, $\Gamma(\tau) = \{$heighest weight in $W(\tau)\}$ and $E = \{\lambda \in \Lambda_a\colon \lambda$ dominant$\}$. Then (i) and (ii) of theorem I are satisfied. We now show that E satisfies condition (*) in section 3.

Let $\{\alpha_1, \ldots, \alpha_l\}$, $l = \dim \mathfrak{h}$, be the set of simple roots in Δ^+. Let $\{\lambda_1, \ldots, \lambda_l\}$ be the dual basis defined by $2 < \lambda_i, \alpha_j > /|\alpha_j|^2 = \delta_{ij}$. Then $\{\lambda_1, \ldots, \lambda_l\}$ is a $\mathbf{Z}$-basis of L_a and if $\lambda = \Sigma\, m_i \lambda_i \in L_a$ is dominant, then $m_i = 2 < \lambda, \alpha_i > /|\alpha_i|^2 \geq 0$. This shows that $\Lambda_a = L_a$ has a $\mathbf{Z}$-basis $\{\lambda_1, \ldots, \lambda_l\}$ such that E satisfies condition (**) of the example in section 3 and we are done.

The general case can be reduced to the previous one: the universal covering group of a semisimple compact Lie group is again compact and semisimple by Weyl's theorem, cf. [K, theorem 4.26].

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