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THE CHARACTERISTIC ALGEBRA OF A POLYNOMIAL COVERING MAP

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Throughout the paper, X denotes a compact (Hausdorff) space, or more generally, a realcompact space in the sense of Hewitt ([8], Chapter 8). We shall consider finite covering maps $\pi: E \to X$ onto X. By $C(X)$ and $C(E)$ we denote the rings of complex-valued continuous functions on X and E. The map $\pi: E \to X$ induces a monomorphism of rings $\pi^*: C(X) \to C(E)$, along which we can consider $C(E)$ as a commutative $C(X)$-algebra with identity.

As is well known from Gelfand-Kolmogoroff theory [8], the algebraic properties of the ring of continuous functions on a compact space determines the topological type of the space. Using the classical theorem of Gelfand and Kolmogoroff we prove in Theorem 1, that the $C(X)$-algebra $C(E)$ of a covering map $\pi: E \to X$ determines the topological equivalence class of the covering map. Therefore we call $C(E)$ the characteristic $C(X)$-algebra of the covering map. Particular attention will be given to the polynomial covering maps onto X introduced by th author in [9], [10]. The total space E in a polynomial covering map $\pi: E \to X$ is defined as the zero set for a separable (in earlier papers called simple) Weierstrass polynomial over X, i.e. a continuous family of complex polynomials without multiple roots $P(x, z): X \times \mathbf{C} \to \mathbf{C}$ of the form $P(x, z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i}$. We think of $P(x, z)$ as an element in the polynomial ring $C(X)[z]$ in one complex variable z over $C(X)$. As our main result, we identify in Theorem 2 the characteristic algebra of the polynomial covering map $\pi: E \to X$, defined by the separable Weierstrass polynomial $P(x, z) \in C(X)[z]$, with the quotient algebra $C(X)[z]/(P(x, z))$. Here $(P(x, z))$ denotes the principal ideal in $C(X)[z]$ generated by $P(x, z)$. In Corollaries 1 and 2 we reword Theorem 1 into statemens about equivalence of polynomial covering maps using this algebra, and in Propositions 1 and 2 we relate properties of $P(x, z)$ to properties of the algebra $C(X)[z]/(P(x, z))$. In a final section we give a simple application of characteristic algebras.

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In the theory of commutative complex Banach algebras, the $C(X)$-algebra $C(X)[z]/(P(x,z))$ is called an extension of the algebra $C(X)$ of algebraic type, or of Arens-Hoffman type, since such algebras were introduced in [1]. See also [16] for references to the literature on these algebras. In Magid [13], it is shown that extensions of commutative regular Banach algebras without radicals, which are finitely generated, projective and separable in the sense of [5], correspond to finite coverings of the carrier spaces. Earlier these coverings were described for the case of extensions of Arens-Hoffman type by Lindberg [12] and Brown [4]. By inspection we see that they are polynomial coverings. This clarifies the role of extensions of Arens-Hoffman type among all separable extensions of commutative complex Banach algebras from the covering space point of view of Magid.

1. Function algebras and equivalence of finite covering maps.

In this section we shall prove the following easy generalization of a classical theorem of Gelfand and Kolmogoroff.

THEOREM 1. *Let $\pi_i: E_i \to X$, $i = 1, 2$, be finite covering maps onto the compact space X. Then we have:*

An equivalence of covering maps,

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

induces an isomorphism of $C(X)$-algebras,

$$\varphi = h^*: C(E_2) \to C(E_1),$$

and conversely.

Theorem 1 shows that the $C(X)$-algebra $C(E)$ for a finite covering map $\pi: E \to X$ determines the topological equivalence class of π. Hence we call it *the characteristic algebra* over $C(X)$ of the covering map π.

The first part of the proof of Theorem 1 is immediate. Suppose then conversely, that we have an isomorphism of $C(X)$-algebras $\varphi: C(E_2) \to C(E_1)$. By a classical theorem of Gelfand and Kolmogoroff, see e.g. ([8], 4.9, 8.3), ([15], Theorem 3.2) or ([7], Theorem 6.5, p. 289), there exists a homeomorphism $h: E_1 \to E_2$, such that $\varphi = h^*$ when considered as ring isomorphisms.

ASSERTION. The homeomorphism $h: E_1 \to E_2$ commutes with projections onto X and hence it is an equivalence of covering maps.

PROOF. Suppose that h does not preserve fibres, and let $e_1 \in E_1$ be a point such

that $\pi_2(h(e_1)) \neq \pi_1(e_1)$. Since X is compact, in particular a normal space, we can choose a function $g \in C(X)$ such that $g(\pi_1(e_1)) = 1$ and $g(\pi_2(h(e_1))) = 0$. Since $h^* = \varphi$ is an isomorphism of $C(X)$-algebras, we have

$$\begin{aligned} g \circ (\pi_2 \circ h) &= (g \circ \pi_2) \circ h = h^*(g \circ \pi_2) \\ &= h^*(g \cdot (1 \circ \pi_2)) = g \cdot h^*(1 \circ \pi_2) \\ &= (g \circ \pi_1) \cdot (1 \circ \pi_2 \circ h) = g \circ \pi_1, \end{aligned}$$

where $1 \in C(X)$ denotes the constant function on X with value 1 and the dot is multiplication in $C(X)$-algebras. Applying this to $e_1 \in E_1$, we get $g(\pi_2(h(e_1))) = g(\pi_1(e_1))$, which is obviously a contradiction. Hence the assertion, and thereby Theorem 1, is proved.

2. The characteristic algebra of a polynomial covering map.

Assume now that $\pi: E \to X$ is an n-fold polynomial covering map associated with the separable Weierstrass polynomial $P(x, z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i}$ over the compact space X, ([9], [10]). We shall prove that the characteristic algebra of $\pi: E \to X$ can be determined directly from $P(x, z)$.

Following Duchamp and Hain ([6], §1), the algebra $C(E)$ has a *primitive* as a module over $C(X)$, i.e. there exists a continuous function $f: E \to \mathbf{C}$ such that the set of functions $1, f, \ldots, f^{n-1}$ is a basis for $C(E)$ as a $C(X)$-module. In particular, by examining Remark 1.2 in the paper by Duchamp and Hain, we see that a primitive $f: E \to \mathbf{C}$ can be obtained from the Weierstrass polynomial $P(x, z)$ by the definition $f(x, z) = z$ whenever $(x, z) \in E$, i.e. $P(x, z) = 0$.

Let $(P(x, z))$ denote the principal ideal in $C(X)[z]$ defined by $P(x, z)$. Then we can define a homomorphism of $C(X)$-algebras, $\psi: C(X)[z]/(Px, z)) \to C(E)$, by associating to the residue class of the polynomial $S(x, z) \in C(X)[z]$, the function $S|E: E \to \mathbf{C}$ defined by restriction of S to E.

We note, that by ψ the polynomial $z \in C(X)[z]$ is mapped onto the primitive $f: E \to \mathbf{C}$ for $C(E)$ as a $C(X)$-module.

By polynomial division it is easily seen that the $C(X)$-module $C(X)[z]/(P(x, z))$ is freely generated by the polynomials $1, z, \ldots, z^{n-1}$.

Collecting facts, it follows that by the $C(X)$-algebra homomorphism ψ the basis $1, z \ldots,, z^{n-1}$ for the $C(X)$-module $C(X)[z]/(P(x, z))$ is mapped onto the basis $1, f, \ldots, f^{n-1}$ for the $C(X)$-module $C(E)$. As a result we get

THEOREM 2. *Let $\pi: E \to X$ be a polynomial covering map associated with the separable Weierstrass polynomial $P(x, z)$. Then there is an isomorphism of $C(X)$-algebras,*

$$\psi: C(X)[z]/(P(x, z)) \to C(E),$$

defined by mapping the residue class of $S(x,z) \in C(X)[z]$ onto the restriction $S|E : E \to \mathbf{C}$ *of* S *to* E.

Combining Theorem 1 and Theorem 2 we get the following corollaries.

COROLLARY 1. *Let* $\pi_i : E_i \to X$, $i = 1, 2$, *be n-fold polynomial covering maps associated with the separable Weierstrass polynomials* $P_i(x,z)$ *over the compact space* X. *Then we have:*

An equivalence of covering maps,

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

induces an isomorphism of $C(X)$*-algebras,*

$$\varphi = h^* : C(X)[z]/(P_2(x,z)) \to C(X)[z]/(P_1(x,z)),$$

and conversely.

COROLLARY 2. *A finite covering map* $\pi : E \to X$ *onto a compact space* X *is equivalent to a polynomial covering map if and only if the characteristic* $C(X)$*-algebra* $C(E)$ *of* π *is isomorphic to a* $C(X)$*-algebra of the form* $C(X)[z]/(P(x,z))$ *for a separable Weierstrass polynomial* $P(x,z)$ *over* X.

Using Corollary 1 we can give a quick proof of ([11], Theorem 1.2) that the term of degree $n-1$ in a separable Weierstrass polynomial $P(x,z)$ of degree n can always be eliminated without changing the topological equivalence class of the associated polynomial covering map. The proof goes by observing that the Tschirnhausen substitution $w = z + a_1(x)/n$ transforms $P(x,z)$ into a separable Weierstrass polynomial $\tilde{P}(x,w)$, whose term of degree $n-1$ is zero, and induces an isomorphism $C(X)[z]/(P(x,z)) \simeq C(X)[w]/(\tilde{P}(x,w))$ of $C(X)$-algebras. Hence the associated polynomial covering maps are equivalent.

3. Weierstrass polynomials and characteristic algebras.

For a Weierstrass polynomial $P(x,z)$ over a topological space X, we can form the quotient algebra $C(X)[z]/(P(x,z))$, which is a commutative algebra over the commutative ring $C(X)$. If $P(x,z)$ is separable, this algebra is the characteristic algebra for the associated polynomial covering map. As the next result shows, the characteristic algebra of a polynomial covering map is separable in the sense of [2] and [5].

PROPOSITION 1. *A Weierstrass polynomial* $P(x,z)$ *over a compact space* X *is separable if and only if the algebra* $C(X)[z]/(P(x,z))$ *is a separable* $C(X)$-algebra.

PROOF. The algebra $C(X)[z]/(P(x,z))$ over $C(X)$ is clearly finitely generated. By Gelfand-Kolmogoroff theory we also know that any maximal ideal in $C(X)$ has the form $m_{x_0} = \{f \in C(X) \mid f(x_0) = 0\}$ for a point $x_0 \in X$, and that $C(X)/m_{x_0} \cong \mathbf{C}$. Then it follows by ([5], Theorem 7.1, p. 72) that $C(X)[z]/(P(x,z))$ is a separable $C(X)$-algebra if and only if $(C(X)/m_{x_0})[z]/(P(x_0,z))$ is a separable $(C(X)/m_{x_0})$-algebra for every $x_0 \in X$, or equivalently, if and only if $\mathbf{C}[z]/(P(x_0,z)$ is a separable $\mathbf{C}$-algebra for every $x_0 \in X$. By ([5], Theorem 2.5, p. 50) the latter is equivalent to that for every $x_0 \in X$, the polynomial $P(x_0,z)$ has no multiple roots, i.e. $P(x,z)$ is a separable Weierstrass polynomial. This proves the proposition.

Irreducibility of a Weierstrass polynomial $P(x,z)$ in the sense of [11] is also reflected in the characteristic algebra.

PROPOSITION 2. *Let $P(x,z)$ be a separable Weierstrass polynomial over the compact, connected topological space X. Then $P(x,z)$ is irreducible over X if and only if the algebra $C(X)[z]/(P(x,z))$ has no other idempotents than* 0 *and* 1.

PROOF. Let $\pi: E \to X$ be the polynomial covering map associated with $P(x,z)$. Then idempotents in the algebra $C(X)[z]/(P(x,z))$ correspond to idempotents in the algebra $C(E)$, since these algebras are isomorphic as $C(X)$-algebras by Theorem 2. It is easy to prove that the Weierstrass polynomial $P(x,z)$ is irreducible over X if and only if the space E is connected, ([11], Theorem 5.2). Proposition 2 now follows, since it is well known – and not difficult to prove directly – that E is connected if and only if $C(E)$ has no other idempotents than 0 and 1. See ([5], Proposition 4.7, p. 28) or ([3], Chapter II, § 4.3, Corollary 2, p. 104).

If $P(x,z)$ is an irreducible, separable Weierstrass polynomial, Propositions 1 and 2 together with ([5], Chapter III) imply that there is a Galois type theory for its characteristic algebra $C(X)[z]/(P(x,z))$, since this algebra is free, and hence projective, as a module over $C(X)$.

4. A simple application of characteristic algebras.

Let S^1 denote the unit circle considered as the space of complex numbers of modulus 1. For each $n \geqq 1$ there is an n-fold covering map $p_n: S^1 \to S^1$ defined by $p_n(z) = z^n$ for $z \in S^1$. The covering map p_n is equivalent to the polynomial covering map $\pi_n: E_n \to S^1$ associated with the separable Weierstrass polynomial $P_n(x,w) = w^n - x, x \in S^1, w \in \mathbf{C}$. A specific equivalence $h: S^1 \to E_n$ can be defined by $h(z) = (z^n, z)$.

LEMMA. *Let $q: X \to \mathbf{C}^*$ be an arbitrary continuous map of the topological space X into the nonzero complex numbers $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Then $P(x,z) = z^n - q(x)$ is a separable Weierstrass polynomial over X, and the associated polynomial covering*

map $\pi: E \to X$ *is equivalent to the pull-back of* $p_n: S^1 \to S^1$ *along the normalized map* $\bar{q}(x) = \dfrac{q(x)}{|q(x)|}$ *for all* $x \in X$.

A specific equivalence $h: E \to \bar{q}^*(S^1)$ as asserted in the lemma can be defined by $h(x, z) = \left(x, \dfrac{z}{|z|}\right)$ for $(x, z) \in E$.

PROPOSITION 3. *Let* $P(x, z) = z^2 + a_1(x)z + a_2(x)$ *be an arbitrary separable Weierstrass polynomial of degree* 2 *over the compact space* X. *Then the associated polynomial covering map* $\pi: E \to X$ *is equivalent to the pull-back of* $p_2: S^1 \to S^1$ *along the normalized discriminant map* $\bar{D}: X \to S^1$ where $D(x) = a_1(x)^2 - 4a_2(x)$ *and* $\bar{D}(x) = \dfrac{D(x)}{|D(x)|}$ *for all* $x \in X$.

PROOF. By completion of the square we can rewrite $P(x, z)$ in the form

$$P(x, z) = (z + a_1(x)/2)^2 - D(x)/4.$$

The Tschirnhausen substitution $w = z + a_1(x)/2$ induces an isomorphism of $C(X)$-algebras

$$C(X)[z]/(P(x, z)) \simeq C(X)[w]/(w^2 - D(x)/4).$$

According to Corollary 1, the polynomial covering map $\pi: E \to X$ associated with $P(x, z)$ is therefore equivalent to the polynomial covering map associated with the separable Weierstrass polynomial $w^2 - D(x)/4$ over X, which in turn by the preceding lemma is equivalent to the pull-back of $p_2: S^1 \to S^1$ along the normalized discriminant map $\bar{D}: X \to S^1$. Proposition 3 is proved.

Proposition 3 is a sharpening of ([9], Remark 7.5), and the case $n = 2$ in ([14], Example 3.4), by explicitly constructing a map $\bar{D}: X \to S^1$ out of the separable Weierstrass polynomial $P(x, z) = z^2 + a_1(x)z + a_2(x)$, such that the associated 2-fold polynomial covering map $\pi: E \to X$ is equivalent to the pull-back of $p_2: S^1 \to S^1$ along $\bar{D}: X \to S^1$. See also ([11], Theorem 3.1).

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